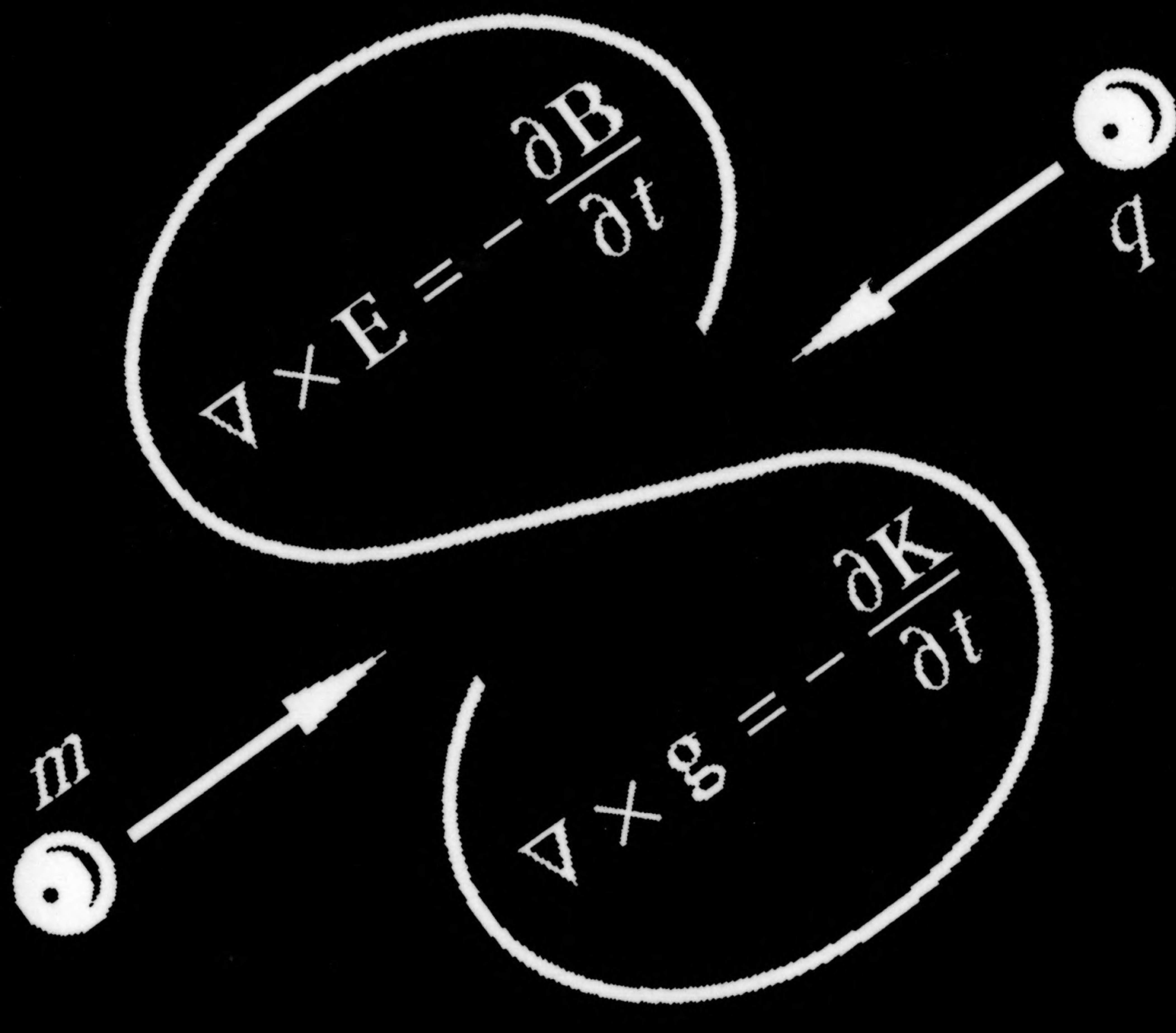
CAUSALITY ELECTROMAGNETIC INDUCTION AND GRAVITATION

Oleg D. Jefimenko



Second Edition

A DIFFERENT APPROACH TO THE THEORY OF ELECTROMAGNETIC AND GRAVITATIONAL FIELDS

Oleg D. Jefimenko
WEST VIRGINIA UNIVERSITY

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SECOND EDITION

Electret Scientific Company
Star City

Library of Congress Catalog Card Number: 99-76935

ISBN 0-917406-23-0

Printed in the United States of America

To the memory
of
my mother

PREFACE

There are two important theories in classical physics, which have not been properly developed to their logical and mathematical conclusion. They are the Faraday-Maxwell theory of electromagnetic induction and Newton's theory of gravitation. Electromagnetic induction is one of the most important physical phenomena. Any misinterpretation or misrepresentation of this phenomenon may weaken the entire electromagnetic theory and may have undesirable practical consequences. Newton's theory of gravitation is the basic working theory of astronomers and other scientists dealing with space exploration and celestial mechanics. Therefore this theory must also be as accurate and complete as possible. What is more, one cannot really judge the significance and value of alternative theories of gravitation without a thorough understanding of all the peculiarities and consequences of Newton's gravitational theory in its most general form.

But why would one discuss electromagnetic induction and gravitation in the same book? What is the connection between the two theories? As far as I know, there is no direct connection. But, as I have found, and as the readers will see, there is a very strong indirect one. First, neither electromagnetic induction as it is now understood, nor Newtonian gravitation is compatible with the principle of causality. Second, Newtonian gravitation is also incompatible with the law of conservation of momentum. When Newton's gravitational law is modified to satisfy the conservation of momentum, two gravitational equations, similar to Maxwell's curl equations of electromagnetic theory (which are customarily interpreted as representing the phenomenon of electromagnetic induction), are obtained. And when the electromagnetic and gravitational curl equations are transformed into equations compatible with the principle of causality, the resulting electromagnetic and gravitational equations turn out to be almost identical. Thus, both electromagnetism and gravitation are subject to very similar basic laws, to the same mathematical formalism, and, unfortunately, to very similar misconceptions.

These findings have very important consequences for our understanding of electromagnetic and gravitational phenomena.

For electromagnetism, one of the consequences is that, contrary to the presently accepted view, time-variable electric and magnetic fields cannot cause each other. Another consequence is that Lenz's law of electromagnetic induction is a manifestation of the previously ignored electric force produced by the time-dependent electric current. It is this force, rather than changing magnetic or electric fields, that is responsible for the electromagnetic induction.

For gravitation, one of the consequences is that there must exist a second ("cogravitational") field, similar to the magnetic field of electromagnetism. Other consequences are that the "gravitational drag" and "Coriolis-like" gravitational forces (heretofore considered to be exclusive consequences of Einstein's general relativity theory) have a simple explanation in terms of Newtonian gravitational theory generalized to time-dependent systems.

I suppose that some readers will find it difficult at first to accept the new ideas on electromagnetic induction and gravitation presented in this book. Naturally, it is difficult to abandon the familiar and very comfortable concept of electromagnetic induction as a phenomenon where one of the two fields creates the other. It is difficult to accept that this concept is illusory. It is equally difficult to accept the existence of a second gravitational field and to accept new nonrelativistic gravitational equations, especially since the conventional Newton's theory of gravitation has worked so well. But the analysis and conclusions presented in this book cannot be questioned without questioning the very foundation of physics and its mathematical methods: Maxwell's electromagnetic equations, the principle of causality, the law of conservation of momentum, and the general vector field theory.

Now and then we find a new path for exploring the world around us. Viewed from this path, familiar objects look different, and new, previously unknown objects, become visible. This book is an account of such an exploration.

Although the book presents results of an original research (it is based on several research articles that at first were intended for publication in scientific journals), it is written in the style and format of a textbook and can be considered a sequel to my *Electricity and Magnetism*, 2nd ed., (Electret Scientific, Star City, 1989). Its mathematical apparatus is the same as that developed in *Electricity and Magnetism*, it uses the same terminology, it provides derivations of all new equations, and it contains a number of illustrative examples similar in style and format to those included in *Electricity and Magnetism*. Therefore it can be used not only for independent reading, but also as a supplementary textbook in courses on electromagnetic theory.

I am pleased to express my gratitude to Dr. David K. Walker of Marshall University for reading the manuscript of this book and to my wife Valentina for helping me to make the manuscript ready for publication.

Oleg D. Jefimenko
September 1, 1991

PREFACE TO THE SECOND EDITION

The second edition of this book is intended to update the subject matter and literature references presented in the first edition. Most of the new material included in the second edition deals with Newton's gravitational theory generalized to time-dependent gravitational systems. In the seven years since the publication of the first edition of the book this theory has shown itself to be a sound, functional, and viable theory of gravitation deserving the utmost attention. New material pertaining to this theory has been added in Chapters 5, 7, and 8 as well as in Appendixes.

The number of Appendixes has been increased from four to eight. A novel graphical representation of electric and gravitational fields is discussed in Appendixes 5 and 6. Based on "dynamic field maps," this graphical representation provides new insights into the properties and effects of electric and gravitational fields of fast moving charges and masses. An analysis of gravitational forces according to Newton's gravitational theory generalized to time-dependent gravitational systems is presented in Appendix 7. An important addition to the book is the Heaviside's 1893 article on the theory of gravitation. It is reproduced in its entirety in Appendix 8. To facilitate the understanding of the mathematical expressions appearing in the Heaviside's article, Heaviside's original mathematical notation has been converted to the conventional modern form. This pioneering article is the foundation of the generalized Newton's theory of gravitation developed in this book.

I am pleased to express my thanks to Dr. Yu. G. Kosarev of the Novosibirsk Institute of Mathematics for sending to me his comments and a list of misprints that he had noticed in the first edition of the book. In his comments, Prof. Kosarev has pointed out that, since all causal interdependencies involve a time delay (retardation), a new science, which he proposes to call "retardics," should be created. This science should investigate cause and effect relationships (not necessarily restricted to the domain of physical phenomena) specifically in terms of the temporal characteristics of these relationships. I welcome his idea and I hope that it will be implemented before long.

I am also pleased to express my gratitude to my wife Valentina for her assistance in the preparation of this edition of the book.

Oleg D. Jefimenko
November 30, 1999

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CAUSALITY
ELECTROMAGNETIC INDUCTION
AND
GRAVITATION

1

MAXWELL'S EQUATIONS AND CAUSALITY IN ELECTROMAGNETIC PHENOMENA

In this chapter Maxwell's electromagnetic equations are analyzed on the basis of the principle of causality. It is shown that Maxwell's equations do not depict cause and effect relations between electromagnetic phenomena and between time-variable electric and magnetic fields in particular. It is concluded that causal dependencies in electromagnetic phenomena are described by solutions of Maxwell's equations involving integrals over retarded charges and currents. A discussion of these solutions from the viewpoint of causality is presented.

1-1. Basic Physical Laws and Causal Relations Between Physical Phenomena

One of the most important tasks of physics is to establish causal relations between physical phenomena. No physical theory can be complete unless it provides a clear statement and description of causal links involved in the phenomena encompassed by that theory. In establishing and describing causal relations it is important not to confuse equations which we call "basic laws" with "causal equations." A "basic law" is an equation (or a system of equations) from which we can derive most (hopefully all) possible correlations between the various quantities involved in a particular group of phenomena subject to the "basic law." A "causal equation," on the other hand, is an equation that unambiguously relates a quantity representing an effect to one or more quantities representing the cause of this effect. Clearly, a "basic law" need not constitute a causal relation, and an equation depicting a causal relation may not necessarily be among the "basic laws" in the above sense.

Causal relations between phenomena are governed by the *principle of causality*. According to this principle, all present phenomena are exclusively determined by past events. Therefore equations depicting causal relations between physical phenomena must, in general, be equations where a present-time quantity (the effect) relates to one or more quantities (causes) that existed at some previous time. An exception to this rule are equations constituting causal relations by definition; for example, if force is defined as the cause of acceleration, then the equation $\mathbf{F} = m\mathbf{a}$, where $\mathbf{F}$ is the force and $\mathbf{a}$ is the acceleration, is a causal equation by definition.

In general, then, according to the principle of causality, an equation between two or more quantities simultaneous in time but separated in space cannot represent a causal relation between these quantities. In fact, even an equation between quantities simultaneous in time and not separated in space cannot represent a causal relation between these quantities because, according to this principle, the cause *must precede* its effect. Therefore the only kind of equations representing causal relations between physical quantities, other than equations representing cause and effect by definition, must be equations involving "retarded" (previous-time) quantities.

Let us apply these considerations to the basic electromagnetic field laws. These laws are represented by the four Maxwell's equations, which, in their differential form, are

$$\nabla \cdot \mathbf{D} = \rho \tag{1-1.1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1-1.2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1-1.3}$$

and

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \tag{1-1.4}$$

where **E** is the electric field vector, **D** is the displacement vector, **H** is the magnetic field vector, **B** is the magnetic flux density vector, **J** is the current density vector, and ρ is the electric charge density. For fields in a vacuum, Maxwell's equations are supplemented by the two constitutive equations

$$\mathbf{D} = \varepsilon_0 \mathbf{E} \tag{1-1.5}$$

and

$$\mathbf{B} = \mu_0 \mathbf{H}. \tag{1-1.6}$$

(The names and designations of electromagnetic quantities used in this book are the same as those used in the author's textbook *Electricity and Magnetism.*[1])

Since none of the four Maxwell's equations is defined to be a causal relation, and since each of these equations connects quantities simultaneous in time, none of these equations can represent a causal relation. That is, $\nabla\cdot\mathbf{D}$ cannot be a consequence of ρ (and vice versa), $\nabla\times\mathbf{E}$ cannot be a consequence of $\partial\mathbf{B}/\partial t$ (and vice versa), and $\nabla\times\mathbf{H}$ cannot be a consequence of $\mathbf{J} + \partial\mathbf{D}/\partial t$ (and vice versa). But is it possible that **D** rather than $\nabla\cdot\mathbf{D}$ is a consequence of ρ, that **E** rather than $\nabla\times\mathbf{E}$ is a consequence of $\partial\mathbf{B}/\partial t$, and that **H** rather than $\nabla\times\mathbf{H}$ is a consequence of $\mathbf{J} + \partial\mathbf{D}/\partial t$? We shall provide detailed answers to these questions in the following two sections.

1-2. The Third Maxwell's Equation and the Principle of Causality

Can the electric field **E** be caused by (be a consequence of) the time-variable magnetic field $\partial\mathbf{B}/\partial t$? To answer this question we shall make use of the following equation representing Helmholtz's theorem of vector analysis applicable to any vector field **V** regular at infinity[2]

$$\mathbf{V} = -\frac{1}{4\pi}\int_{\text{All space}} \frac{\nabla'(\nabla'\cdot\mathbf{V}) - \nabla'\times(\nabla'\times\mathbf{V})}{r}\,dv', \quad (1\text{-}2.1)$$

where r is the distance between the field point x, y, z (point for which **V** is evaluated) and the source point x', y', z' (volume element dv'), and where the primed operator ∇' operates on the source-point coordinates only (here and in most other integrals appearing in this book the integration is over all space).

Let us apply Helmholtz's theorem, Eq. (1-2.1), to the electric field **E**. Since in a vacuum,[3] by Eqs. (1-1.1) and (1-1.5), $\nabla\cdot\mathbf{E} = \rho/\varepsilon_0$, and since we are not interested in **E** due to ρ, we can ignore the gradient term in Eq. (1-2.1). We then obtain from Eqs. (1-2.1) and (1-1.3)

$$\mathbf{E} = -\frac{1}{4\pi}\int \frac{\nabla' \times \partial \mathbf{B}/\partial t}{r} dv'. \qquad (1\text{-}2.2)$$

The integrand in this equation can be transformed by using vector identity (V-24) (see Appendix 1 for a list of vector identities; letter "m" after the number means "modified")

$$\frac{\nabla' \times \mathbf{V}}{r} = \nabla' \times \frac{\mathbf{V}}{r} + \frac{\mathbf{V} \times \mathbf{r}_u}{r^2}, \qquad (\text{V-24m})$$

where $\mathbf{r}_u$ is a unit vector directed from the point x', y', z' to the point x, y, z. We then have

$$\mathbf{E} = -\frac{1}{4\pi}\int \nabla' \times \frac{\partial \mathbf{B}/\partial t}{r} dv' - \frac{1}{4\pi}\int \frac{\partial \mathbf{B}/\partial t \times \mathbf{r}_u}{r^2} dv'. \qquad (1\text{-}2.3)$$

The first integral can now be transformed into a surface integral by means of vector identity (V-21)

$$\int \nabla' \times \mathbf{A} dv = -\oint \mathbf{A} \times d\mathbf{S}', \qquad (\text{V-21m})$$

which gives

$$-\frac{1}{4\pi}\int \nabla' \times \frac{\partial \mathbf{B}/\partial t}{r} dv' = \frac{1}{4\pi}\oint \frac{\partial \mathbf{B}/\partial t}{r} \times d\mathbf{S}'. \qquad (1\text{-}2.4)$$

Since $\partial \mathbf{B}/\partial t$ must be confined to a finite region of space, and since the surface of integration in the surface integral is at infinity, the surface integral is zero. Therefore we finally obtain

$$\mathbf{E} = -\frac{1}{4\pi}\int \frac{\partial \mathbf{B}/\partial t \times \mathbf{r}_u}{r^2} dv'. \qquad (1\text{-}2.5)$$

In this equation, too, $\mathbf{E}$ and $\partial \mathbf{B}/\partial t$ are evaluated for the same instant of time. Hence, according to the causality principle, there is no causal relation between $\partial \mathbf{B}/\partial t$ and $\mathbf{E}$ either. Therefore

Maxwell's Eq. (1-1.3), although a basic law, is not a causal equation.

But if there is no causal relation between **E** or $\nabla\times\mathbf{E}$ and $\partial\mathbf{B}/\partial t$, what is the meaning of the relation represented by Maxwell's Eq. (1-1.3)? A plausible interpretation of Eq. (1-1.3) is that both $\partial\mathbf{B}/\partial t$ and $\nabla\times\mathbf{E}$ are coupled together because they are always simultaneously caused by some third quantity, not revealed by Eq. (1-1.3). Let us explore this interpretation.

To do so, we shall make use of the following two equations, which constitute solutions of Maxwell's equations for fields in a vacuum [4]

$$\mathbf{E} = -\frac{1}{4\pi\varepsilon_0}\int\frac{\left[\nabla'\rho + \frac{1}{c^2}\frac{\partial\mathbf{J}}{\partial t}\right]}{r}dv' \qquad (1\text{-}2.6)$$

and

$$\mathbf{H} = \frac{1}{4\pi}\int\frac{[\nabla'\times\mathbf{J}]}{r}dv'. \qquad (1\text{-}2.7)$$

The square brackets in these equations are the retardation symbol indicating that the quantities between the brackets are to be evaluated for the time $t' = t - r/c$, where t is the time for which **E** and **H** are evaluated, ρ is the electric charge density, **J** is the current density, r is the distance between the field point x, y, z (point for which **E** and **H** are evaluated) and the source point x', y', z' (volume element dv'), and c is the velocity of light. As usual, the integrals in both equations are extended over all space.

Evaluating $\nabla\times\mathbf{E}$ by taking the curl of Eq. (1-2.6), we have

$$\nabla\times\mathbf{E} = -\frac{1}{4\pi\varepsilon_0}\int\nabla\times\frac{\left[\nabla'\rho + \frac{1}{c^2}\frac{\partial\mathbf{J}}{\partial t}\right]}{r}dv', \qquad (1\text{-}2.8)$$

where we have placed the operator ∇ under the integral sign, since ∇ operates on unprimed coordinates, while the integration is over the primed coordinates. Separating the integral of Eq. (1-2.8) into two integrals, we have

$$\nabla\times\mathbf{E} = -\frac{1}{4\pi\varepsilon_0}\int\nabla\times\frac{[\nabla'\rho]}{r}dv' - \frac{1}{4\pi\varepsilon_0 c^2}\int\nabla\times\frac{1}{r}\left[\frac{\partial\mathbf{J}}{\partial t}\right]dv'. \quad (1\text{-}2.9)$$

Since the first integral does not explicitly depend on time, we may ignore it for this particular calculation (if we did not ignore it, we would find that after transformations it vanishes anyway). We then obtain

$$\nabla\times\mathbf{E} = -\frac{1}{4\pi\varepsilon_0 c^2}\int\nabla\times\frac{1}{r}\left[\frac{\partial\mathbf{J}}{\partial t}\right]dv'. \qquad (1\text{-}2.10)$$

Let us now multiply Eq. (1-2.7) by μ_0 and differentiate it with respect to time. Since in a vacuum, by Eq. (1-1.6), $\mathbf{B} = \mu_0\mathbf{H}$, we have

$$\frac{\partial\mathbf{B}}{\partial t} = \mu_0\frac{\partial\mathbf{H}}{\partial t} = \frac{\mu_0}{4\pi}\frac{\partial}{\partial t}\int\frac{[\nabla'\times\mathbf{J}]}{r}dv'. \qquad (1\text{-}2.11)$$

The integrand in Eq. (1-2.11) can be transformed by means of the vector identity (V-32)[5]

$$\frac{[\nabla'\times\mathbf{V}]}{r} = \nabla\times\frac{[\mathbf{V}]}{r} + \nabla'\times\frac{[\mathbf{V}]}{r}. \qquad (\text{V-32m})$$

We then have

$$\frac{\partial\mathbf{B}}{\partial t} = \frac{\mu_0}{4\pi}\frac{\partial}{\partial t}\int\nabla\times\frac{[\mathbf{J}]}{r}dv' + \frac{\mu_0}{4\pi}\frac{\partial}{\partial t}\int\nabla'\times\frac{[\mathbf{J}]}{r}dv'. \quad (1\text{-}2.12)$$

Using vector identity (V-21), we can transform the last integral of Eq. (1-2.12) into a surface integral, just as we transformed the first integral of Eq. (1-2.3). We obtain

$$\frac{\mu_0}{4\pi}\frac{\partial}{\partial t}\int \nabla' \times \frac{[\mathbf{J}]}{r}dv' = -\frac{\mu_0}{4\pi}\frac{\partial}{\partial t}\oint \frac{[\mathbf{J}]}{r} \times d\mathbf{S}'. \quad (1\text{-}2.13)$$

But these integrals vanish, because **J** is confined to a finite region of space, while the surface of integration is at infinity. Thus only the first integral of Eq. (1-2.12) remains. Differentiating this integral with respect to time, taking into account that $\partial[\mathbf{J}]/\partial t = [\partial \mathbf{J}/\partial t]$ and that $\mu_0 = 1/\varepsilon_0 c^2$, we finally obtain

$$\frac{\partial \mathbf{B}}{\partial t} = \frac{1}{4\pi\varepsilon_0 c^2}\int \nabla \times \frac{1}{r}\left[\frac{\partial \mathbf{J}}{\partial t}\right]dv', \quad (1\text{-}2.14)$$

the right side of which, except for the minus sign, is the same as in Eq. (1-2.10).

It is now clear that the two terms of Maxwell's Eq. (1-1.3), $\nabla \times \mathbf{E}$ and $\partial \mathbf{B}/\partial t$, do indeed have the same common cause: the changing electric current density **J**.

1-3. The Fourth Maxwell's Equation and the Principle of Causality

Let us now consider Maxwell's Eq. (1-1.4). Let us see if there is a causal relation between **H** and $\partial \mathbf{D}/\partial t$. Since at this time we are not interested in the effect of **J** on **H**, we shall set in Eq. (1-1.4) $\mathbf{J} = 0$. Substituting Eq. (1-1.4) into Eq. (1-2.1), and taking into account that $\nabla \cdot \mathbf{H} = 0$, we have

$$\mathbf{H} = \frac{1}{4\pi}\int \frac{\nabla' \times \partial \mathbf{D}/\partial t}{r}dv'. \quad (1\text{-}3.1)$$

Transforming this integral in the same manner as the integral of Eq. (1-2.2), and taking into account that $\partial \mathbf{D}/\partial t$ is confined to a finite region of space, we obtain

$$\mathbf{H} = \frac{1}{4\pi}\int \frac{\partial \mathbf{D}/\partial t \times \mathbf{r}_u}{r^2}\, dv'. \qquad (1\text{-}3.2)$$

In this equation, too, $\mathbf{H}$ and $\partial \mathbf{D}/\partial t$ are evaluated for the same instant of time. Hence, by the causality principle, $\partial \mathbf{D}/\partial t$ cannot be a cause of $\mathbf{H}$, and, consequently, Maxwell's Eq. (1-1.4), just like Eq. (1-1.3), is not a causal equation.

But if there is no causal relation between $\mathbf{H}$ and $\partial \mathbf{D}/\partial t$, what is the meaning of the relation represented by Maxwell's Eq. (1-1.4)? A plausible interpretation of Eq. (1-1.4) is that both $\partial \mathbf{D}/\partial t$ and $\nabla \times \mathbf{H}$ are coupled together because they are always simultaneously caused by some third quantity, not revealed by Eq. (1-1.4). Let us explore this interpretation.

Evaluating $\nabla \times \mathbf{H}$ by using Eq. (1-2.7) and vector identity (V-32), we have

$$\begin{aligned}\nabla \times \mathbf{H} &= \frac{1}{4\pi}\int \nabla \times \frac{[\nabla' \times \mathbf{J}]}{r}\, dv' \\ &= \frac{1}{4\pi}\int \frac{[\nabla' \times (\nabla' \times \mathbf{J})]}{r}\, dv' - \frac{1}{4\pi}\int \nabla' \times \frac{[\nabla' \times \mathbf{J}]}{r}\, dv'.\end{aligned} \qquad (1\text{-}3.3)$$

Making use of vector identity (V-21), we can write

$$\nabla \times \mathbf{H} = \frac{1}{4\pi}\int \frac{[\nabla' \times (\nabla' \times \mathbf{J})]}{r}\, dv' + \frac{1}{4\pi}\oint \frac{[\nabla' \times \mathbf{J}]}{r} \times d\mathbf{S}' \qquad (1\text{-}3.4)$$

and, since the surface integral vanishes (because $\mathbf{J}$ is confined to a finite region of space), we obtain

$$\nabla \times \mathbf{H} = \frac{1}{4\pi}\int \frac{[\nabla' \times (\nabla' \times \mathbf{J})]}{r}\, dv'. \qquad (1\text{-}3.5)$$

Let us now find the time derivative of $\mathbf{D}$ by using Eq. (1-2.6). Since the fields under consideration are in a vacuum, so that $\mathbf{D} = \varepsilon_0 \mathbf{E}$, we have

$$\frac{\partial \mathbf{D}}{\partial t} = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} = -\frac{1}{4\pi}\int \frac{\partial}{\partial t}\frac{\left[\nabla'\rho + \dfrac{1}{c^2}\dfrac{\partial \mathbf{J}}{\partial t}\right]}{r}dv'$$

$$= -\frac{1}{4\pi}\int \frac{\left[\nabla'\dfrac{\partial\rho}{\partial t} + \dfrac{1}{c^2}\dfrac{\partial^2 \mathbf{J}}{\partial t^2}\right]}{r}dv', \tag{1-3.6}$$

and, making use of the continuity equation $\nabla'\cdot\mathbf{J} = -\partial\rho/\partial t$, we obtain

$$\frac{\partial \mathbf{D}}{\partial t} = \frac{1}{4\pi}\int \frac{\left[\nabla'(\nabla'\cdot\mathbf{J}) - \dfrac{1}{c^2}\dfrac{\partial^2 \mathbf{J}}{\partial t^2}\right]}{r}dv'. \tag{1-3.7}$$

Next, let us subtract Eq. (1-3.7) from Eq. (1-3.5). We obtain

$$\nabla\times\mathbf{H} - \frac{\partial \mathbf{D}}{\partial t}$$
$$= -\frac{1}{4\pi}\int \frac{\left[\nabla'(\nabla'\cdot\mathbf{J}) - \nabla'\times(\nabla'\times\mathbf{J}) - \dfrac{1}{c^2}\dfrac{\partial^2 \mathbf{J}}{\partial t^2}\right]}{r}dv'. \tag{1-3.8}$$

But, according to vector identity (V-27) (see Appendixes 1 and 2),

$$\mathbf{V} = -\frac{1}{4\pi}\int \frac{\left[\nabla'(\nabla'\cdot\mathbf{V}) - \nabla'\times(\nabla'\times\mathbf{V}) - \dfrac{1}{c^2}\dfrac{\partial^2 \mathbf{V}}{\partial t^2}\right]}{r}dv', \tag{V-27}$$

the integral in Eq. (1-3.8) is simply the time-variable current density $\mathbf{J}$. Thus, $\nabla\times\mathbf{H}$ and $\partial\mathbf{D}/\partial t$ are indeed simultaneously created by the changing $\mathbf{J}$ (more accurately, by changing $\mathbf{J}$ and changing ρ, since $\nabla'\cdot\mathbf{J} = -\partial\rho/\partial t$). Consequently, Maxwell's Eq. (1-1.4) may be regarded as a causal equation, if it is written as

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}. \tag{1-3.9}$$

Of course, the causal relation depicted by Eq. (1-3.9) is between the current density **J** and the entire left side of the equation, rather than between **H** and **D** (observe that this equation is an exception to the normal causal equations discussed on p. 4).

1-4. Causal Equations for Electric and Magnetic Fields

The analysis of Maxwell's Eqs. (1-1.3) and (1-1.4) presented in the two preceding sections shows that, according to the causality principle, neither Eq. (1-1.3) nor Eq. (1-1.4) is a causal equation. It also shows that electric and magnetic fields cannot cause one another, and that Maxwell's Eqs. (1-1.3) and (1-1.4) correlate quantities simultaneously created by time-variable electric charges and currents. But then what are the fundamental causal equations of electromagnetic theory? For fields in a vacuum, Eqs. (1-2.6) and (1-2.7) can be regarded as such equations. However, the spatial derivatives contained in these equations make them awkward and not particularly clear.

If we eliminate the spatial derivatives from Eqs. (1-2.6) and (1-2.7), we obtain two much more lucid alternative causal equations in which the causative sources of time-dependent electric and magnetic fields are clearly and explicitly revealed.[4,6] These equations are

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\int\left\{\frac{[\rho]}{r^2} + \frac{1}{rc}\frac{\partial[\rho]}{\partial t}\right\}\mathbf{r}_u\,dv' - \frac{1}{4\pi\varepsilon_0 c^2}\int\frac{1}{r}\left[\frac{\partial\mathbf{J}}{\partial t}\right]dv' \tag{1-4.1}$$

and

$$\mathbf{H} = \frac{1}{4\pi}\int\left\{\frac{[\mathbf{J}]}{r^2} + \frac{1}{rc}\frac{\partial[\mathbf{J}]}{\partial t}\right\}\times\mathbf{r}_u\,dv', \tag{1-4.2}$$

where $\mathbf{r}_u$ is the unit vector directed from dv' to the field point. We now see from these equations that the electric field has three causative sources: the charge density ρ, the time derivative of ρ, and the time derivative of $\mathbf{J}$. And we see that the magnetic field has two causative sources: the electric current density $\mathbf{J}$ and the time derivative of $\mathbf{J}$.

Equations (1-4.1) and (1-4.2) clearly show that, since both equations contain the time derivative of $\mathbf{J}$, an electric and a magnetic field are always simultaneously created by the same time-variable electric current.

Equation (1-4.1) also reveals the explicit causal relation between the electric field and the electric charge, thus reinforcing our previous conclusion that Maxwell's Eq. (1-1.1) does not represent a causal relation between the electric displacement vector $\mathbf{D}$ (which, in a vacuum, is proportional to $\mathbf{E}$) and the charge density ρ.

For a single point charge q moving in a vacuum with velocity v and acceleration a, Eqs. (1-4.1) and (1-4.2) become[7]

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0 s^3}\left\{\mathbf{R}\left[1 - \frac{v^2}{c^2}\right] + \frac{1}{c^2}[\mathbf{r}] \times (\mathbf{R} \times [\mathbf{a}])\right\}, \quad (1\text{-}4.3)$$

and

$$\mathbf{H} = \frac{q}{4\pi s^3}\left\{[\mathbf{v}]\left[1 - \frac{v^2}{c^2}\right] - \frac{1}{c[r]}[\mathbf{r}] \times (\mathbf{R} \times [\mathbf{a}])\right\} \times [\mathbf{r}], \quad (1\text{-}4.4)$$

with

$$\mathbf{H} = \frac{[\mathbf{r}] \times \mathbf{E}}{\mu_0 c[r]}, \quad (1\text{-}4.5)$$

where $[\mathbf{r}]$ is the retarded position vector of the moving point charge given by $t' = t - [r]/c$ and directed from the charge to the point of observation (Fig. 1.1), $\mathbf{R} = [\mathbf{r} - r\mathbf{v}/c]$ is the

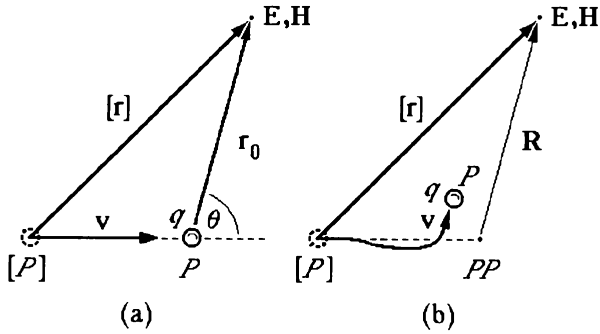

Fig. 1.1 Geometrical relations between the point of observation for the fields **E** *and* **H** *and the retarded position [P], present position P, and "projected" present position PP of a moving point charge q. (a) The charge moves with constant velocity* **v**. *(b) The charge moves with variable velocity* **v**. *(The present projected position is the position that q would have had, if* [**v**] *were constant.)*

"projected" present position vector of the point charge (also directed toward the point of observation), $s = [\, r - \mathbf{r} \cdot \mathbf{v}/c]$, and where the square brackets denote retarded values.

For a point charge moving without acceleration, Eq. (1-4.3) can be expressed in terms of the present position vector $\mathbf{r}_0$ as[7,8]

$$\mathbf{E} = \frac{q(1 - v^2/c^2)}{4\pi\varepsilon_0 r_0^3[1 - (v^2/c^2)\sin^2\theta]^{3/2}}\mathbf{r}_0. \qquad (1\text{-}4.6)$$

where θ is the angle between $\mathbf{v}$ and $\mathbf{r}_0$, with

$$\mathbf{H} = \frac{\mathbf{v} \times \mathbf{E}}{\mu_0 c^2}. \qquad (1\text{-}4.7)$$

Note that for time-independent systems, Eq. (1-4.1) reduces to the ordinary Coulomb field equation expressing **E** in terms of

ρ, and Eq. (1-4.2) reduces to the Biot-Savart law expressing **H** in terms of **J**. Therefore Eqs. (1-4.1) and (1-4.2) constitute the most general equations for electric and magnetic fields in a vacuum within the framework of Maxwellian electromagnetic theory.

1-5. Can Time-Variable Electric and Magnetic Fields Create Each Other?

There is a widespread belief that time-variable electric and magnetic fields can cause each other. The analysis of Maxwell's equations presented above does not support this belief. It is true that whenever there exists a time-variable electric field, there also exists a time-variable magnetic field. This follows from Maxwell's Eqs. (1-1.3) and (1-1.4) as well as from Eqs. (1-4.1) and (1-4.2). But, as we have seen, neither Maxwell's equations nor their solutions indicate an existence of causal links between electric and magnetic fields. Therefore we must conclude that an electromagnetic field is a dual entity always having an electric and a magnetic component simultaneously created by their common sources: time-variable electric charges and currents.[9]

This conclusion must hold for all electromagnetic fields, including electric and magnetic fields in electromagnetic waves.

However, concerning electromagnetic waves, an interesting connotation emerges from our analysis. According to vector identity (V-28), a time-variable vector field can always be represented as

$$\mathbf{V} = -\frac{1}{4\pi}\int \frac{\left[\nabla'^2\mathbf{V} - \dfrac{1}{c^2}\dfrac{\partial^2\mathbf{V}}{\partial t^2}\right]}{r}dv'. \qquad \text{(V-28)}$$

On the other hand, the equation

$$\nabla^2 \mathbf{V} - \frac{1}{c^2}\frac{\partial^2 \mathbf{V}}{\partial t^2} = 0 \qquad (1\text{-}5.1)$$

with $\mathbf{V} = \mathbf{E}$ or $\mathbf{V} = \mathbf{H}$ is the equation for electromagnetic waves in free space. Hence there appears to be a contradiction between Eq. (V-28) and Eq. (1-5.1): Eq. (V-28) states that if the integrand is zero, then also **V** is zero; but Eq. (1-5.1) states that, on the contrary, **V** in this case should be a wave field and not at all zero. In other words, it appears that, according to Eq. (1-5.1), there can exist "sourceless" electromagnetic fields (sourceless electromagnetic waves), whereas, according to Eq. (V-28), such fields are impossible. Since there is no doubt that Eq. (1-5.1) does indeed represent a wave field, it appears that Eq. (V-28) may yield wrong results. If this is true, then our causal equations, Eqs. (1-2.6), (1-2.7), (1-4.1), and (1-4.2), may be wrong, because, as it is shown in Appendix 2, these equations are merely special cases of Eq. (V-28) for electric and magnetic fields.

Actually, however, there is no conflict between Eqs. (V-28) and (1-5.1). The seeming contradiction disappears as soon as we realize that the integration in Eq. (V-28) is over all space, and that **V** on the left of this equation will not be zero if Eq. (1-5.1) applies only to a limited region of space. Thus, Eq. (V-28) provides important information about wave fields, which is not provided by Eq. (1-5.1): there can be no wave field unless Eq. (1-5.1) holds, but Eq. (1-5.1) must apply to a limited region of space only. If $\mathbf{V} \neq 0$ somewhere in space, then there must be a region of space where Eq. (1-5.1) does not hold. This region contains (or did contain in the past) the sources of **V**, which are the retarded quantities that appear in Eqs. (V-28), (1-2.6), (1-2.7), (1-4.1) and (1-4.2).

References and Remarks for Chapter 1

1. O. D. Jefimenko, *Electricity and Magnetism*, 2nd ed., (Electret Scientific, Star City, 1989).
2. See Ref. 1, pp. 42-43. Note that because of the finite speed of propagation of electric and magnetic fields, these fields must be zero at infinity unless they have been created infinitely long ago.
3. We shall limit our discussion to fields in a vacuum, since the presence of material media cannot affect basic causality relations.
4. For derivation see Appendix 2 and Ref. 5(a) pp. 1-18; Ref. 1 pp. 47, 514-516.
5. For operations with retarded quantities see (a) O. D. Jefimenko, *Electromagnetic Retardation and Theory of Relativity* (Electret Scientific, Star City, 1997) pp. 6-10 and (b) Ref. 1, pp. 46-52.
6. For alternative derivations see J. D. Jackson, *Classical Electrodynamics*, 3rd ed., (Wiley, New York, 1999) pp. 246-247; J. Griffiths and M. A. Heald, "Time-dependent generalizations of the Biot-Savart and Coulomb laws," Am. J. Phys. **59**, 111-117 (1991); and Ton Tran-Cong, "On the time-dependent, generalized Coulomb and Biot-Savart laws," Am. J. Phys. **59**, 520-528 (1991).
7. (a) Ref. 5(a) pp. 62-95; (b) W. G. V. Rosser, *Classical Electromagnetism Via Relativity* (Plenum, New York, 1968) pp. 38-41.
8. For more information about the electric field of a uniformly moving point charge see Appendix 5.
9. The conclusion that electric and magnetic fields do not create each other is not entirely new. On p. 517 of the 1st edition of Ref. 1 (Appleton-Century-Crofts, New York, 1966) the author stated: "although a displacement current is associated with a magnetic field, this does not constitute a cause and effect relationship." See also Ref. 7(b), pp. 81, 95, 107, 122, 234-236, 242, 280, 285, and W. G. V. Rosser, "Does the displacement current in empty space produce a magnetic field?" Am. J. Phys. **44**, 1221-1223 (1976).

2

THE NATURE OF ELECTROMAGNETIC INDUCTION

Electromagnetic induction is one of the most important physical phenomenon. It is traditionally attributed to generation of electric fields by changing magnetic fields and to generation of magnetic fields by changing electric fields. But, as we found in Chapter 1, there is no causal relation between electric and magnetic fields. We must look therefore for a different explanation of electromagnetic induction. In this chapter we shall unveil its true nature and shall establish its correct theoretical foundation.

2.1. What is Electromagnetic Induction?

Electromagnetic induction is frequently explained as a phenomenon in which a changing magnetic field produces an electric field ("Faraday induction") and a changing electric field produces a magnetic field ("Maxwell induction"). The very useful and successful method of calculating induced voltage (emf) in terms of changing magnetic flux appears to support the reality of Faraday induction. And the existence of electromagnetic waves appears to support the reality of both Faraday induction and Maxwell induction.

Maxwell's equations do not provide a conclusive answer to whether or not the two phenomena are real. In Maxwell's equations electric and magnetic fields are linked together in an intricate manner, and neither field is explicitly represented in terms of its sources. However, an examination of the causal relations in time-dependent electric and magnetic fields presented in Chapter 1 shows that Maxwell's equations are not at all causal equations, and that neither of the two fields can create the other.

As it was shown in Chapter 1, the causal equations for electric and magnetic fields in a vacuum are

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int\left\{\frac{[\rho]}{r^2} + \frac{1}{rc}\frac{\partial[\rho]}{\partial t}\right\}\mathbf{r}_u dv' - \frac{1}{4\pi\epsilon_0 c^2}\int\frac{1}{r}\left[\frac{\partial \mathbf{J}}{\partial t}\right]dv' \quad (1\text{-}4.1)$$

and

$$\mathbf{H} = \frac{1}{4\pi}\int\left\{\frac{[\mathbf{J}]}{r^2} + \frac{1}{rc}\frac{\partial[\mathbf{J}]}{\partial t}\right\}\times\mathbf{r}_u dv'. \quad (1\text{-}4.2)$$

According to these equations, in time-variable systems electric and magnetic fields are always created simultaneously, because they have a common causative source: the changing electric current [the last term of Eq. (1-4.1) and the last term in the integral of Eq. (1-4.2)]. Once created, the two fields coexist from then on without any effect upon each other. Therefore electromagnetic induction as a phenomenon in which one of the fields creates the other is an illusion. The illusion of the "mutual creation" arises from the facts that in time-dependent systems the two fields always appear prominently together, while their causative sources (the time-variable current in particular) remain in the background.

But if the two fields are created simultaneously and coexist from then on as a dual entity, then our concept of electromagnetic induction requires a thorough reexamination. We shall start such a reexamination by reviewing the history of this concept.

2-2. Faraday Induction in Historical Perspective

In 1820 Oersted discovered the fundamental electromagnetic phenomenon, the fact that an electric current was accompanied by a magnetic field encircling that current.[1]

In the same year Ampère discovered that current-carrying conductors attracted each other if the currents were in the same direction, and repelled each other if they were in opposite directions; he named the forces between the current-carrying conductors "electro-dynamic" forces. Later he identified these forces as magnetic, and suggested that magnetism was really an electrical phenomenon: magnetized bodies owed their magnetic effects to circular electric currents within the bodies.

Also in 1820 Biot and Savart provided a mathematical description of Oersted's discovery, and Davy discovered that a current-carrying wire attracted iron filings to itself.

These discoveries inspired Faraday to start his celebrated researches in electricity and magnetism. At first he was mainly interested in finding answers to two questions: can a current induce a secondary current in neighboring bodies, and can magnetism be converted into electricity? By 1831 Faraday had the answers to both questions. In a letter of November 29, 1831, addressed to his friend Richard Phillips, Faraday wrote:[2]

§I. When an electric current is passed through one of two parallel wires it causes at first a current in the same direction through the other, but this induced current does not last a moment notwithstanding the inducing current (from the Voltaic battery) is continued . . . , but when the current is stopped then a return current occurs in the wire under induction of about the same intensity and momentary duration but in the opposite direction to that first found. Electricity in currents therefore exerts an inductive action like ordinary electricity but subject to peculiar laws: the effects are a current in the same direction when the induction is established, a reverse current when the induction ceases and a peculiar state in the interim . . .

§II. Then I found that magnets would induce just like voltaic currents and by bringing helices and wires and jackets up to the poles of magnets, electrical currents were produced in them, these currents being able to deflect the galvanometer, or to make, by means of the helix, magnetic needles, or in one case even to give a spark. Hence the evolution of *electricity from magnetism*. The currents were not permanent, they ceased the moment the wires ceased to approach the magnet because the new and apparently quiescent state was assumed just as in the case of the induction of currents. But when the magnet was removed, and its induction therefore ceases, the return currents appeared as before. These two kinds of induction I have distinguished by the terms *Volta-electric* and *Magneto-electric* induction. Their identity of action and results is, I think, a very powerful proof of the truth of M. Ampère's theory of magnetism.

§III. The new electrical condition which intervenes by induction between the beginning and end of the inducing current gives rise to some very curious results. It explains why chemical action or other results of electricity have never been as yet obtained in trials with the magnet. In fact, the currents have no sensible duration . . . The condition of matter I have dignified by the term *Electrotonic*, The Electrotonic State.

A detailed account of the various and numerous experiments that had led Faraday to his discovery and to the study of electromagnetic induction was communicated to the Royal Society and was published in *Philosophical Transactions* and later in his famous *Experimental Researches in Electricity.*[3] The experiments that he conducted in 1831 comprised the first 75 pages of the first volume of *Experimental Researches.*

Just as Faraday wrote in his letter to Phillips, he discovered two basic effects of electromagnetic induction: the induction of electric current in a circuit due to a changing current of the inducing circuit, and the induction of a current in a circuit due to a relative motion of this circuit with respect to a current-carrying circuit or magnet (he also discovered "self-induction" of current in a single circuit). It is important to note that although Faraday was the originator of the concept of the magnetic field (which he described in terms of the "magnetic curves" – our present day "magnetic lines of force"), he never as much as suggested that the induced currents were a result of changing magnetic fields. On the contrary, he clearly associated the phenomenon of electromagnetic induction with changing electric currents, even when the induction was caused by a motion of a permanent magnet (reference to Ampère in the letter to Phillips).

The mathematical formulation of the phenomenon of electromagnetic induction was due to Maxwell. In Chapter III (entitled "On the Induction of Electric Currents") of Vol. 2 of his famous *Treatise on Electricity and Magnetism* he wrote:[4]

It was perhaps for the advantage of science that Faraday, though thoroughly conscious of the fundamental forms of space, time, and force, was not a professed mathematician . . . He was thus left at leisure to do his proper work, to coordinate his ideas with the facts, and to express them in natural, untechnical language. It is mainly with the hope of making these ideas the basis of a mathematical method that I have undertaken this treatise.

Maxwell then reviewed Faraday's observations in four sections entitled "Induction by Variation of the Primary Current," "Induction by Motion of the Primary Circuit," "Induction by Motion of the Secondary Circuit," and "Induction by the Relative Motion of a Magnet and the Secondary Circuit." Next he analyzed Faraday's concept of the electrotonic state. Finally he formulated his law of electromagnetic induction:

The number of the lines of force which at any instant pass through the circuit is mathematically equivalent to Faraday's earlier conception of the electrotonic state of that circuit . . .

It is only since the definitions of electromotive force . . . and its measurement have been made more precise, that we can enunciate completely the true law of magneto-electric induction in the following terms:

The total electromotive force acting round a circuit at any instant is measured by the rate of decrease of the number of lines of magnetic force which pass through it . . . Instead of speaking of the number of lines of magnetic force, we may speak of the magnetic induction through the circuit, or the surface-integral of magnetic induction extended over any surface bounded by the circuit.

As we see, Maxwell, too, considered the electromagnetic induction as a phenomenon in which a current (or electromotive force) is induced in a circuit, but not as a phenomenon in which a changing magnetic field causes an electric field. He clearly said that the induced electromotive force is *measured* by, not *caused* by, the changing magnetic field. Just as Faraday, he made no allusion to any causal link between magnetic and electric fields.

2-3. Maxwell Induction in Historical Perspective

The expression "Maxwell induction" is of a relatively recent origin. The expression refers to an alleged phenomenon where a magnetic field is created by a changing electric field. Its theoretical basis is presumed to be in the fourth Maxwell's Eq. (1-1.4), specifically in the last term of this equation – the time-variable displacement current density $\partial \mathbf{D}/\partial t$. The reality of this phenomenon has never been demonstrated experimentally.

There is just one, frequently used, "theoretical" illustration of Maxwell induction: the computation of the magnetic field between the plates of a thin parallel-plate capacitor with circular plates in a circuit with a slowly varying current.[5] However, this illustration actually demonstrates the utility of the displacement current concept and does not really manifest an induction phenomenon. What is more, it has been repeatedly shown that the same result can be obtained from the Biot-Savart law applied to the conduction current in the lead wires and in the capacitor plates, without using the displacement current at all.[6-9] Attempts have been made by some investigators to observe displacement current experimentally. However, according to an authoritative article by one of these investigators, these attempts were futile.[9]

The concept of the displacement current was introduced in the electromagnetic theory by Maxwell. Let us see how Maxwell defined displacement current and what properties he attributed to it. In Chapter I of Vol. 1 of his *Treatise* Maxwell wrote:[10]

The electric polarization of an elementary portion of a dielectric is a forced state into which the medium is thrown by the action of electromotive force, and which disappears when that force is removed. We may conceive it to consist in what we may call an electric displacement, produced by the electromotive intensity . . . whatever electricity may be, and whatever we may understand by the movement of electricity, the phenomenon which we have called electric displacement is a movement of electricity in the same sense as the transference of a definite quantity of electricity through a wire is a movement of electricity.

In Chapter IX of Vol. 2 of his *Treatise* he wrote:[11]

One of the peculiarities of this treatise is the doctrine which it asserts, that the true electric current, that on which the electromagnetic phenomena depend, is not the same thing as **K**, the current of conduction, but that the time-variation of **D**, the electric displacement, must be taken into account in estimating the total movement of electricity.

Thus, according to Maxwell, the displacement current was not a changing electric field, as we interpret it today, but a displacement of actual electric charges residing inside dielectric media. In this respect we should remember that in Maxwell's times it was believed that all space was occupied by ether, a dielectric medium. On the last page of his *Treatise*, speaking about the propagation of light, Maxwell wrote:[12]

Hence all these theories lead to the conception of a medium in which the propagation takes place, and if we admit this medium as an hypothesis, I think it ought to occupy a prominent place in our investigations, and that we ought to endeavor to construct a mental representation of all the details of its action, and this has been my constant aim in this treatise.

It is therefore completely groundless to associate Maxwell's name with the idea that a changing electric field can cause a magnetic field. What is more, since this idea is without any experimental or theoretical support, it must be completely discarded. Faraday induction, as a phenomenon where an electric current is generated in a conductor by a changing electric current in another (or in the same) conductor, is a true physical effect. "Maxwell induction," on the other hand, is an inappropriately named illusion.

2-4. What is Electromagnetic Induction? (The Answer)

What is then the true nature and cause of "electromagnetic induction?" Before answering this question let us emphasize the fact that there is only one phenomenon that can be called electromagnetic induction in systems at rest. Since, as we have seen, electric and magnetic fields cannot cause one another, the only electromagnetic induction in systems at rest is Faraday induction of an electric current in a conductor due to a changing electric current in some other (or in the same) conductor.

As far as the electromagnetic induction in conductors moving with respect to other conductors (or magnets) is concerned, this induction is reducible to the induction in systems at rest (see Sections 2-5, 2-6, and Example 3-3.6), and, as an alternative, can be considered in terms of Lorentz's fields and forces [13] without invoking any induction effect at all.

And now, once again, what is the true nature and cause of electromagnetic induction? The answer is quite simple. According to Eq. (1-4.1),

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int\left\{\frac{[\rho]}{r^2} + \frac{1}{rc}\frac{\partial[\rho]}{\partial t}\right\}\mathbf{r}_u\,dv' - \frac{1}{4\pi\epsilon_0 c^2}\int\frac{1}{r}\left[\frac{\partial\mathbf{J}}{\partial t}\right]dv', \qquad (1\text{-}4.1)$$

a time-variable electric current creates an electric field parallel to that current [the last term of Eq. (1-4.1)]. This field exerts an electric force on the charges in nearby conductors thereby creating induced electric currents in them. Thus, the term "electromagnetic induction" is actually a misnomer, since no magnetic effect is involved in the phenomenon, and since the induced current is caused solely by the time-variable electric current and by the electric field produced by that current.

Observe that the electric field produced by a time-variable current differs in two important respects from the ordinary electric field produced by electric charges at rest: the field is directed along the current rather than along a radius vector, and it exists only as long as the current is changing in time. Therefore the electric force caused by this field is also different from the ordinary electric (electrostatic) force: it is directed along the current and it lasts only as long as the current is changing. Unlike the electrostatic force, which is always an attraction or repulsion between electric charges, the electric force due to a time-variable current is a *dragging* force: it causes electric charges to move parallel (or antiparallel) relative to the direction of the current. If the time-variable current is a convection current, then the force that this current exerts on neighboring charges causes them to move parallel to the convection current, rather than toward or away from the charges forming the convection current [the total electric force is, of course, given by all three terms of Eq. (1-4.1)].

The electric field created by time-variable currents is very different from all other fields encountered in electromagnetic phenomena. Therefore a special name should be given to it. Taking into account that the cause of this field is a motion of electric charges (current), we may call it the *electrokinetic field*, and we may call the force which this field exerts on an electric charge the *electrokinetic force*. Of course, we could simply call this field the "induced field." However, such a designation would not reflect the special nature and properties of this field.[14,15] We shall designate the electrokinetic field by the vector $\mathbf{E}_k$. From Eq. (1-4.1) we thus have

$$\mathbf{E}_k = - \frac{1}{4\pi\epsilon_0 c^2} \int \frac{1}{r} \left[\frac{\partial \mathbf{J}}{\partial t} \right] dv'. \qquad (2\text{-}4.1)$$

The electrokinetic field provides a precise and clear explanation of one of the most remarkable properties of electromagnetic induction: the Lenz's law. Consider a straight current-carrying conductor parallel to another conductor. According to Lenz's law, the current induced in the second conductor is opposite to the inducing current in the first conductor when the inducing current is increasing, and is in the same direction as the inducing current when the inducing current is decreasing. In the past no convincing explanation of this effect was known. But the electrokinetic field provides the definitive explanation of Lenz's law: by Eq. (2-4.1), the sign (direction) of the electrokinetic field is opposite to the sign of the time derivative of the *inducing* current. When the derivative is positive, the electrokinetic field is opposite to the inducing current, when the derivative is negative, the electrokinetic field is in the same direction as the inducing current. Since the *induced* current is caused by the electrokinetic field, the direction of this field determines the direction of the induced current: opposite to the inducing current when that current increases (positive derivative), the same as the inducing current when the inducing current decreases (negative derivative).

Of course, since the direction of inducing current usually varies from point to point in space, the ultimate direction of the electrokinetic field and of the current that it produces is determined, in general, by the combined effect of all the current elements of the inducing current in the integral of Eq. (2-4.1).

The electrokinetic field also gives a simple explanation of the fact (first noted by Faraday) that the strongest induced current is produced between parallel conductors, whereas no induction takes place between conductors at right angles to each other. This phenomenon is now easily understood from the fact that the electrokinetic field due to a straight conductor carrying an inducing current is always parallel to the conductor.

Although we have been discussing the electrokinetic field as the cause of induced currents in conductors, its significance is much more general. This field can exist anywhere in space and can manifest itself as a pure force field by its action on free electric charges. Of course, because of the c^2 in the denominator in Eq. (2-4.1), the electrokinetic field cannot be particularly strong except when the current changes very fast. This is probably the main reason why this field was ignored in the past. Another reason is the temporal (transient) nature of this field.

But even a weak electric field can produce strong currents in conductors, and that is why the current-producing effect of the electrokinetic field is much more prominent than its force effect on electric charges in free space.

If we compare Eq. (2-4.1) with the expression for the retarded magnetic vector potential **A*** produced by a current **J**,[16]

$$\mathbf{A}^* = \frac{\mu_0}{4\pi}\int \frac{[\mathbf{J}]}{r}\,dv', \qquad (2\text{-}4.2)$$

we recognize that the electrokinetic field is equal to the negative time derivative of **A*** (observe that $\mu_0 = 1/\varepsilon_0 c^2$):

$$\mathbf{E}_k = -\frac{\partial \mathbf{A}^*}{\partial t}. \qquad (2\text{-}4.3)$$

It is interesting to note that Eq. (2-4.3) points out to a possibility of a new definition and interpretation of the magnetic vector potential. Let us integrate Eq. (2-4.3). We obtain

$$\mathbf{A}^* = -\int \mathbf{E}_k dt + \text{const.} \tag{2-4.4}$$

Let us call the time integral of $\mathbf{E}_k$ the *electrokinetic impulse*. We can say then that the magnetic vector potential created by a current at a point in space is equal to the negative of the electrokinetic impulse produced by this current at that point when the current is switched on. Since the electrokinetic impulse is, in principle, a measurable quantity, we thus have an operational definition and a physical interpretation of the magnetic vector potential.[17]

It may be useful to mention that although Eqs. (2-4.3) and (2-4.4) correlate the electrokinetic field with the magnetic vector potential, there is no causal link between the two: the correlation merely reflects the fact that both the electrokinetic field and the magnetic vector potential are simultaneously caused by the same electric current.[18]

Important as it is, the electrokinetic field has not been studied (or even recognized as a special force field) until now, although the fact that the time derivative of the retarded vector potential is associated with an electric field has been known for a long time.

▼

Example 2-4.1 Show that if $\mathbf{E}_k$ is linear in time, $\mathbf{E}_k = \mathbf{a} + \mathbf{b}t$, then, for a vacuum, the retarded magnetic vector potential in Eq. (2-4.3) can be replaced by the ordinary (unretarded) vector potential.

We shall solve this problem by using Helmholtz's theorem of vector analysis, vector identity (V-23):

$$\mathbf{V} = -\frac{1}{4\pi}\int \frac{\nabla'(\nabla'\cdot\mathbf{V}) - \nabla'\times(\nabla'\times\mathbf{V})}{r}dv'. \qquad \text{(V-23)}$$

As it is known,[19] the divergence of the retarded magnetic vector potential satisfies the Lorentz's condition

$$\nabla'\cdot\mathbf{A}^* = -\frac{1}{c^2}\frac{\partial\varphi^*}{\partial t}, \qquad (2\text{-}4.5)$$

where φ^* is the retarded scalar potential of **E**. Therefore, by Eqs. (2-4.3) and (2-4.5), we have

$$\nabla'\cdot\mathbf{E}_k = \frac{1}{c^2}\frac{\partial^2\varphi^*}{\partial t^2}. \qquad (2\text{-}4.6)$$

For the curl of $\mathbf{E}_k$ we have, by Eq. (2-4.3) and by the definition of the magnetic vector potential,

$$\nabla'\times\mathbf{E}_k = -\frac{\partial\mathbf{B}}{\partial t}. \qquad (2\text{-}4.7)$$

For $\nabla'(\nabla'\cdot\mathbf{E}_k)$ we then have, by Eq. (2-4.6), by the definition of the retarded scalar potential,[16] and by Eq. (2-4.3),

$$\nabla'(\nabla'\cdot\mathbf{E}_k) = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}(\mathbf{E} + \frac{\partial\mathbf{A}^*}{\partial t}) = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}(\mathbf{E} - \mathbf{E}_k), \qquad (2\text{-}4.8)$$

where **E** is the total electric field given by Eq. (1-4.1). For $\nabla'\times(\nabla'\times\mathbf{E}_k)$ we have, by Eqs. (2-4.7), (1-1.4), (1-1.5), and (1-1.6), noting that $\mu_0\varepsilon_0 = 1/c^2$,

$$\nabla'\times(\nabla'\times\mathbf{E}_k) = -\mu_0\frac{\partial\mathbf{J}}{\partial t} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}. \qquad (2\text{-}4.9)$$

Substituting Eqs. (2-4.8) and (2-4.9) into Eq. (V-23), canceling $(1/c^2)(\partial^2\mathbf{E}/\partial t^2)$, noting that $\partial^2\mathbf{E}_k/\partial t^2 = 0$ (because $\mathbf{E}_k$ is linear in t), and comparing the result with Eq. (2-4.2), we finally obtain

$$\mathbf{E}_k = -\frac{\mu_0}{4\pi}\int \frac{1}{r}\frac{\partial \mathbf{J}}{\partial t}dv' = -\frac{\partial \mathbf{A}}{\partial t}. \tag{2-4.10}$$

▲

2-5. Induction by Moving Currents

Let us now assume that an initially stationary current $\mathbf{J}(x',y',z')$ moves as a whole with a constant velocity v relative to a stationary observer. The current is then a function of $(x'-v_xt)$, $(y'-v_yt)$, and $(z'-v_zt)$, or

$$\mathbf{J} = \mathbf{J}(x'-v_xt, y'-v_yt, z'-v_zt). \tag{2-5.1}$$

The time derivative of the current is

$$\frac{\partial \mathbf{J}}{\partial t} = -\frac{\partial \mathbf{J}}{\partial x'}v_x - \frac{\partial \mathbf{J}}{\partial y'}v_y - \frac{\partial \mathbf{J}}{\partial z'}v_z = -(\mathbf{v}\cdot\nabla')\mathbf{J}. \tag{2-5.2}$$

The electrokinetic field caused by the moving current is then, by Eqs. (2-4.1) and (2-5.2),

$$\mathbf{E}_k = \frac{\mu_0}{4\pi}\int \frac{[(\mathbf{v}\cdot\nabla')\mathbf{J}]}{r}dv'. \tag{2-5.3}$$

The spatial derivative appearing in Eq. (2-5.3) can be eliminated as follows. Using vector identity (V-6),

$$\nabla'(\mathbf{v}\cdot\mathbf{J}) = (\mathbf{v}\cdot\nabla')\mathbf{J} + \mathbf{v}\times(\nabla'\times\mathbf{J}) + (\mathbf{J}\cdot\nabla')\mathbf{v} + \mathbf{J}\times(\nabla'\times\mathbf{v}), \tag{V-6m}$$

and taking into account that $\mathbf{v}$ is a constant vector, we obtain

$$\mathbf{E}_k = \frac{\mu_0}{4\pi}\int \frac{[\nabla'(\mathbf{v}\cdot\mathbf{J})]}{r}dv' - \frac{\mu_0}{4\pi}\int \frac{[\mathbf{v}\times(\nabla'\times\mathbf{J})]}{r}dv'. \tag{2-5.4}$$

If we compare Eq. (2-5.4) with the equation representing the magnetic flux density field produced by a time-variable current [which is given by Eq. (1-2.7) multiplied by μ_0],

$$\mathbf{B} = \frac{\mu_0}{4\pi}\int\frac{[\nabla'\times\mathbf{J}]}{r}dv', \tag{2-5.5}$$

we find that Eq. (2-5.4) can be written as

$$\mathbf{E}_k = \frac{\mu_0}{4\pi}\int\frac{[\nabla'(\mathbf{v}\cdot\mathbf{J})]}{r}dv' - \mathbf{v}\times\mathbf{B}, \tag{2-5.6}$$

where **B** is the magnetic flux density field created by the moving current **J**.

Like any electric field, the electrokinetic field of a moving current exerts forces on electric charges located in this field. However, a moving electric current can create not only an electrokinetic field, but also an "ordinary" electric field given by the first integral of Eq. (1-4.1).[20,21] This is because a current-carrying conductor moving in the direction of the current or in the direction opposite to the current appears to acquire additional electric charges in consequence of its motion. In the literature this is erroneously considered to be a relativistic effect.[22] Actually, however, this effect is a consequence of retardation and is explainable on the basis of Eq. (1-4.1) (see Appendix 3).

A neutral conductor carrying a current I and moving with velocity **v** in the direction of the current (+) or opposite to it (−) appears to acquire a line charge of density

$$\lambda = \pm\frac{Iv}{c^2}. \tag{2-5.7}$$

A magnetic dipole of dipole moment $\mathbf{m} = \mu_0 I\mathbf{S}$ moving with velocity **v** appears to acquire an electric dipole moment

$$\mathbf{p} = \frac{\mathbf{v} \times \mathbf{m}}{\mu_0 c^2} = \frac{I(\mathbf{v} \times \mathbf{S})}{c^2}, \tag{2-5.8}$$

where **S** is the surface area of the current loop forming the magnetic dipole, right-handed relative to the current of the loop.

Thus the "electromagnetic induction" by a moving current, just as the induction by a stationary current, is a result of the creation of an electrokinetic field (with a possible creation of an additional "ordinary" electric field) by this current. However, also in this case the induction has no causal link with any magnetic field. Its cause is not a changing magnetic field, but the electric field (or fields) produced by the moving electric current.[23]

2-6. Induction by Moving Magnets

As far as it is presently known, all magnetic fields are created by electric currents. Therefore we may represent a moving magnet by moving microscopic electric currents forming elementary magnetic dipoles.

Let us assume that Eq. (2-5.6) represents the electrokinetic field of a single elementary microscopic current of a magnet moving with a velocity $v \ll c$, and let us assume that at the point of observation outside the magnet there is a stationary point charge q. Since $v \ll c$, the retardation in Eq. (2-5.6) can be ignored. We then have

$$\mathbf{E}_k = \frac{\mu_0}{4\pi} \int \frac{\nabla'(\mathbf{v} \cdot \mathbf{J})}{r} dv' - \mathbf{v} \times \mathbf{B}. \tag{2-6.1}$$

We can transform Eq. (2-6.1) by using vector identity (V-26),

$$\frac{\nabla'(\mathbf{v} \cdot \mathbf{J})}{r} = \nabla \frac{\mathbf{v} \cdot \mathbf{J}}{r} + \nabla' \frac{\mathbf{v} \cdot \mathbf{J}}{r}, \tag{V-26m}$$

where the unprimed ∇ operates upon field-point coordinates only. We obtain

$$\mathbf{E}_k = \frac{\mu_0}{4\pi}\int \nabla \frac{\mathbf{v}\cdot\mathbf{J}}{r} dv' + \frac{\mu_0}{4\pi}\int \nabla' \frac{\mathbf{v}\cdot\mathbf{J}}{r} dv' - \mathbf{v}\times\mathbf{B}. \qquad (2\text{-}6.2)$$

Using vector identity (V-20), we can transform the second volume integral in Eq. (2-6.2) into a surface integral. But, because there are no currents at infinity, the surface integral vanishes, and so does the volume integral. Since ∇ in the first integral does not operate on source-point coordinates, it can be factored out from under the integral sign. This gives

$$\mathbf{E}_k = \frac{\mu_0}{4\pi}\nabla\int \frac{\mathbf{v}\cdot\mathbf{J}}{r} dv' - \mathbf{v}\times\mathbf{B}. \qquad (2\text{-}6.3)$$

The currents forming the microscopic dipoles can be considered filamentary. Therefore we can write

$$\mathbf{E}_k = \frac{\mu_0 I}{4\pi}\nabla\oint \frac{\mathbf{v}\cdot d\mathbf{l}'}{r} - \mathbf{v}\times\mathbf{B}, \qquad (2\text{-}6.4)$$

where $d\mathbf{l}'$ is a length element vector in the direction of the microscopic current I. Factoring out $\mathbf{v}\,\cdot$, using vector identity (V-18), and taking into account that the linear dimensions of the dipoles are much smaller than r, we then have

$$\begin{aligned} \mathbf{E}_k &= -\frac{\mu_0 I}{4\pi}\nabla\left(\mathbf{v}\cdot\int \frac{\mathbf{r}_u}{r^2}\times d\mathbf{S}'\right) - \mathbf{v}\times\mathbf{B} \\ &= -\frac{\mu_0 I}{4\pi}\nabla\left(\mathbf{v}\cdot\frac{\mathbf{r}_u}{r^2}\times\mathbf{S}'\right) - \mathbf{v}\times\mathbf{B}, \end{aligned} \qquad (2\text{-}6.5)$$

where $\mathbf{S}'$ is the surface area of the current loop forming the microscopic dipole. Transposing $\mathbf{v}$ and $\mathbf{r}_u$ in Eq. (2-6.5), we can write

$$\mathbf{E}_k = \nabla\left(\frac{\mu_0 I}{4\pi}\frac{\mathbf{r}_u}{r^2}\cdot\mathbf{v}\times\mathbf{S}'\right) - \mathbf{v}\times\mathbf{B}. \qquad (2\text{-}6.6)$$

As we know from Section 2-5, a moving current generates not only an electrokinetic field, but also an "ordinary" electric field. By Eq. (2-5.8), the "ordinary" field generated by a current constituting a moving magnetic dipole is an electric dipole field. The field of an electric dipole **p** is given by[24]

$$\mathbf{E}_{\text{dipole}} = -\nabla \frac{\mathbf{p} \cdot \mathbf{r}_u}{4\pi\varepsilon_0 r^2}. \qquad (2\text{-}6.7)$$

Substituting **p** from Eq. (2-5.8) into Eq. (2-6.7), we obtain

$$\mathbf{E}_{\text{dipole}} = -\nabla\left(\frac{\mu_0 I}{4\pi}\frac{\mathbf{r}_u}{r^2} \cdot \mathbf{v} \times \mathbf{S}'\right). \qquad (2\text{-}6.8)$$

Adding Eqs. (2-6.8) and (2-6.6), we finally obtain for the total electric field induced by each elementary microscopic current of the moving magnet

$$\mathbf{E} = \mathbf{E}_{\text{dipole}} + \mathbf{E}_k = -\mathbf{v} \times \mathbf{B}. \qquad (2\text{-}6.9)$$

The force exerted by this field on the point charge q outside the magnet is

$$\mathbf{F} = -q\mathbf{v} \times \mathbf{B}. \qquad (2\text{-}6.10)$$

If q were within a conductor, then the force given by Eq. (2-6.10) would create a conduction current in this conductor.

Thus each elementary current of a moving magnet exerts on an external charge a force given by Eq. (2-6.10), and therefore the total force exerted by a moving magnet on an external charge is also given by Eq. (2-6.10) with **B** being the external magnetic field of the magnet.

However, the expression "force exerted by a moving magnet" is actually a misnomer, since, as we have seen, this force has no causal link with the magnetic field of the magnet. The phenomenon of "induced electric force" or "induced current" by a moving magnet is simply the effect of the electric field caused by the collective translational motion of microscopic currents participating in the motion of the magnet.

References and Remarks for Chapter 2

1. The term "magnetic field" was not yet known then. Oersted described his discovery as follows: "To the effect which takes place in this conductor and in the surrounding space, we shall give the name *conflict of electricity* . . . The electric conflict acts only on the magnetic particles of matter in a revolving manner;" see H. C. Oersted, Ann. of Phil. **16**, 273-276 (1820). Later he described his discovery more clearly: ". . . the magnetic effect of electrical current has a circular motion around it;" see Kirstine Meyer, *H. C. Oersted, Scientific Papers* (Copenhagen, 1920), Vol. II, pp. 356-358.

2. Silvanus P. Thompson, *Michael Faraday, His Life and Work* (Macmillan, New York, 1898) pp. 115, 116.

3. Michael Faraday, *Experimental Researches in Electricity* (Richard and John Edward Taylor, London, 1839).

4. James Clerk Maxwell, *A Treatise on Electricity and Magnetism* (Dover reprint of the 3rd edition of 1891) Vol. 2, pp. 176-177.

5. See, for example, O. D. Jefimenko, *Electricity and Magnetism*, 2nd ed., (Electret Scientific, Star City, 1989) pp. 503-504.

6. W. G. V. Rosser, "Does the displacement current in empty space produce a magnetic field?" Am. J. Phys. **44**, 1221-1223 (1976).

7. A. P. French and J. R. Tessman, "Displacement currents and magnetic fields," Am. J. Phys. **31**, 201-204 (1963).

8. A. J. Dahm, "Calculation of the displacement current using the integral form of Ampere's law," Am. J. Phys. **46**, 1227 (1978).
9. D. F. Bartlett, "Conduction current and the magnetic field in a circular capacitor," Am. J. Phys. **58**, 1168-1172 (1990).
10. Ref. 4, Vol. 1, pp. 65-69.
11. Ref. 4, Vol. 2, p. 253.
12. Ref. 4, Vol. 2, p. 493.
13. Ref. 5, pp. 388, 392, and 417.
14. The term "electrokinetic" is also used in reference to phenomena associated with the movement of charged particles through a continuous medium or with the movement of a continuous medium over a charged surface. These phenomena have no connection with the electrokinetic field defined in this book.
15. Another appropriate name for this field is the "Faraday field." This name is due to P. Beckmann who introduced it in his book *Einstein Plus Two* (Golem Press, Boulder, 1987) pp. 108-118.
16. Ref. 5, p. 519.
17. A related interpretation of the magnetic vector potential is given in Emil J. Konopinski, *Electromagnetic Fields and Relativistic Particles* (McGraw-Hill, New York, 1981) pp. 158-160.
18. There is, of course, no causal relation between the magnetic vector potential and the magnetic field, since they are simultaneously produced by the same electric current.
19. See, for example, (a) O. D. Jefimenko, *Electromagnetic Retardation and Theory of Relativity* (Electret Scientific, Star City, 1997) pp. 35-36 or (b) Ref. 5, pp. 519, 520.
20. Note that the derivative $\partial[\rho]/\partial t$ is associated with the electric current via the "continuity relation" (see Chapter 1, p. 12).
21. Observe that the first integral in Eq. (1-4.1) can be expressed in terms of the retarded scalar potential φ^* as

$$\frac{1}{4\pi\varepsilon_0}\int\left\{\frac{[\rho]}{r^2}+\frac{1}{rc}\frac{\partial[\rho]}{\partial t}\right\}\mathbf{r}_u dv' = -\nabla\varphi^*$$

[see Ref. 5, pp. 47, 48, 52, 514-516, and 519; Ref. 19(a), pp. 34-35; D. J. Griffiths and M. A. Heald, "Time-dependent generalizations of the Biot-Savart and Coulomb laws," Am. J. Phys. **59**, 111-117 (1991)]. For a point charge q moving with velocity v, the potential becomes [see, for example, Ref. 19(a), pp. 95-96]

$$\varphi^* = \frac{q}{4\pi\varepsilon_0 r[1-(v^2/c^2)\sin^2\theta]^{1/2}},$$

where r is the distance between q and the point of observation, θ is the angle between the direction of v and the direction of r, and c is the velocity of light.

22. See, for example, W. K. H. Panofsky and M. Phillips, *Classical Electricity and Magnetism*, 2nd ed., (Addison Wesley, Reading, Massachusetts, 1962) pp. 332-334.

23. In Appendix 3, Eq. (2-5.8) is derived for a conductor whose current is formed by positive and negative charges of equal magnitude moving with equal velocities in opposite directions. However, the equation does not depend on the nature of the current or on the velocities of the charge carriers. In Appendix 4, a similar derivation is presented for a *mass current* (see Section 5-4). This derivation can be easily converted into a derivation for electric convection current by replacing the mass m by the charge q. Observe that, in general, a moving current appears to acquire additional charges that are the electric counterparts of Eqs. (A-4.6) and (A-4.11). These charges are negligible, except when the current is produced by charge carriers moving with extremely high velocities.

24. Ref. 5, p. 130.

3

ELECTROKINETIC FIELDS AND FORCES

In this chapter we shall present illustrative examples on the use of electrokinetic fields and forces. We shall compute electrokinetic forces acting on charge distributions in the vicinity of conductors carrying time-variable currents. We shall establish a connection between the change of the mechanical momentum of a charge distribution subjected to an electrokinetic field and the magnetic vector potential associated with the time-variable current that produces the electrokinetic field. And we shall demonstrate how electrokinetic fields can be used for direct calculation of induced voltages (emf's) in conductors.

3-1. Electrokinetic Fields

We shall now present examples on calculation of electrokinetic fields, on calculation of electrokinetic forces, and on calculation of currents and voltages induced by electrokinetic fields. We shall use examples requiring only very simple calculations. It is not the complexity of the examples that is important for our purpose. Our purpose is to provide an unambiguous demonstration of the effects and actions of electrokinetic fields; this can be best achieved with uncomplicated examples. As far as the induced voltages are concerned, the purpose of our calculations will be merely to demonstrate that conventional results can be obtained solely by using electrokinetic fields, without invoking any causal linkage between electric and magnetic fields.

For simplicity, we shall limit our calculations to relatively small systems and relatively slow variations of electric quantities. In such systems retardation effects are negligible, so that Eq. (2-4.1) can be written without brackets as

$$\mathbf{E}_k = -\frac{\mu_0}{4\pi}\int \frac{1}{r}\frac{\partial \mathbf{J}}{\partial t}dv', \tag{3-1.1}$$

where we have replaced $1/c^2$ by $\varepsilon_0\mu_0$ and have cancelled ε_0. If the current is confined to a filament (wire), Eq. (3-1.1) can be written as

$$\mathbf{E}_k = -\frac{\partial I}{\partial t}\frac{\mu_0}{4\pi}\int \frac{d\mathbf{l}'}{r}, \tag{3-1.2}$$

where I is the current in the filament and $d\mathbf{l}'$ is a length element of the filament in the direction of the current. Finally, if the retardation is neglected, the electrokinetic field of a current $\mathbf{J}$ can be found, according to Eq. (2-4.3), from

$$\mathbf{E}_k = -\frac{\partial \mathbf{A}}{\partial t}, \tag{3-1.3}$$

where $\mathbf{A}$ is the ordinary (not retarded) magnetic vector potential associated with $\mathbf{J}$.

When the electrokinetic force acts on a charge distribution ρ, it changes the mechanical momentum $\mathbf{P}$ of the charge distribution in accordance with

$$\Delta \mathbf{P} = \int \mathbf{F}dt = \int\int \rho \mathbf{E}_k dv' dt. \tag{3-1.4}$$

If $\mathbf{E}_k$ is a function of time only, the momentum change is

$$\Delta \mathbf{P} = q\int \mathbf{E}_k dt = -q\Delta \mathbf{A}, \tag{3-1.5}$$

where q is the total charge of the distribution, and $\Delta \mathbf{A}$ is the change in the vector potential during the time interval under consideration.

If a circular electrokinetic force acts on a charge distribution restricted to a circular motion, the angular momentum of the charge distribution changes. For a charge distribution and electrokinetic field of circular symmetry, the change in the angular momentum $\Delta \mathbf{L}$ is

$$\Delta \mathbf{L} = \int\int \mathbf{r} \times \mathbf{E}_k dq dt = -\int \mathbf{r} \times \mathbf{A} dq. \tag{3-1.6}$$

As already mentioned, we are using the ordinary vector potential for simplicity; for exact calculations the retarded vector potential must be used in Eqs. (3-1.3), (3-1.4), (3-1.5), and (3-1.6).

It should be pointed out that an association between the momentum change of a charged body and the change of the magnetic vector potential at the location of the body has been noted before.[1] However, heretofore this association was erroneously interpreted as an electromagnetic effect rather than as a consequence of the fact that both an electrokinetic force and a time-variable magnetic field (and its time-variable vector potential) are simultaneously created by a time-variable current. And, which is even more important, it was not generally recognized that the actual phenomenon involved the retarded vector potential rather than the ordinary one.

As it is known, a magnetic vector potential may contain an arbitrary additive function of zero curl ("gauge calibration"). However, only the vector potential given by Eq. (2-4.2) and by its unretarded version can be used for the calculation of the electrokinetic field.

An explanatory note is required concerning calculations of forces and torques exerted on charge distributions by electrokinetic fields and concerning calculations of induced currents and voltages. The force experienced by a charge distribution is determined, in general, by the total electric field given by Eq. (1-4.1), not just by the electrokinetic field, Eqs. (2-4.1), (3-1.1), or (3-1.2). Therefore a force calculated from the electrokinetic field alone may not be the true force experienced by the charge distribution under consideration. In contrast, only the electrokinetic force has an effect on the torque experienced by rings of charge and by similar objects. This is because the torque in such systems is determined by a closed line integral of the electric field, and only the electrokinetic field gives a nonvanishing contribution to such integrals [the first term of Eq. (1-4.1), being a function of r in the direction of $\mathbf{r}$, has zero curl and therefore cannot contribute to closed line integrals]. Closed line integrals of electric fields are also involved in the calculations of induced voltages. Therefore induced voltages, too, are determined by the electrokinetic fields alone.

3-2. Examples on Calculation of Electrokinetic Fields

In this section we shall present several illustrative examples on calculation of electrokinetic fields. A direct calculation of electrokinetic fields from Eq. (2-4.1) or from its unretarded versions, Eqs. (3-1.1) and (3-1.2), involves exactly the same techniques that are used for calculating magnetic vector potentials from Eq. (2-4.2) or from its unretarded versions. Therefore we shall avoid presenting examples on direct calculation of electrokinetic fields here, since such examples would basically duplicate examples on vector potential calculations provided in most textbooks on electromagnetic theory. Instead, with one exception, we shall make use of Eq. (3-1.3) and of the readily available expressions for the vector potential.

▼

Example 3-2.1 A straight wire of length $2L$ carries a time-variable current I (Fig. 3.1). Find the electrokinetic field of this current at a distance R from the wire at a point equidistant from the ends of the wire.

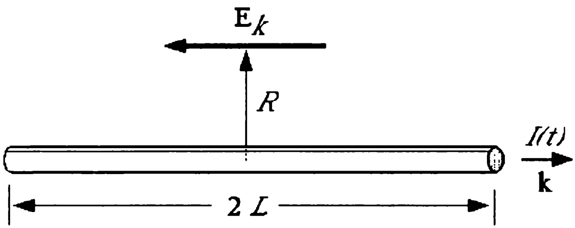

Fig. 3.1 A changing electric current produces an electrokinetic field.

The magnetic vector potential for this system is[2]

$$\mathbf{A} = \frac{\mu_0 I}{2\pi} \ln \frac{L + (L^2 + R^2)^{1/2}}{R} \mathbf{k}, \qquad (3\text{-}2.1)$$

where $\mathbf{k}$ is a unit vector in the direction of I. By Eq. (3-1.3), the electrokinetic field of the wire is then

$$\mathbf{E}_k = -\frac{\partial I}{\partial t} \frac{\mu_0}{2\pi} \ln \frac{L + (L^2 + R^2)^{1/2}}{R} \mathbf{k}. \qquad (3\text{-}2.2)$$

If the wire is long, so that $L^2 \gg R^2$, we may neglect R^2 in Eqs. (3-2.1) and (3-2.2). The electrokinetic field of the wire is then

$$\mathbf{E}_k = -\frac{\partial I}{\partial t}\frac{\mu_0}{2\pi}\ln\frac{2L}{R}\mathbf{k}. \qquad (3\text{-}2.3)$$

Example 3-2.2 A long thin-walled cylinder of radius R_0 and length $2L$ carries a time-variable current I along its length (Fig. 3.2). Find the electrokinetic field outside and inside the cylinder.

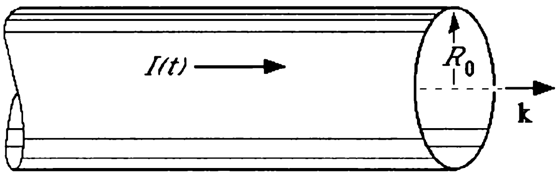

Fig. 3.2 A cylinder carries a time-variable current. There is an electrokinetic field outside and inside the cylinder.

The magnetic vector potential outside the cylinder is the same as if the current of the cylinder were confined to the axis of the cylinder.[3] This vector potential is given by Eq. (3-2.1), and the corresponding electrokinetic field is given by Eq. (3-2.3).

The vector potential inside the cylinder is constant and is equal to the vector potential just outside the cylinder. (Since there is no magnetic field inside the cylinder, this statement may appear incredulous. However, the absence of **B** inside the cylinder merely requires that **A** is the same at all points inside the cylinder. It does not require that **A** = 0.) Substituting R_0 for R in Eq. (3-2.3), we then have for the electrokinetic field inside the cylinder

$$\mathbf{E}_k = -\frac{\partial I}{\partial t}\frac{\mu_0}{2\pi}\ln\frac{2L}{R_0}\mathbf{k}. \qquad (3\text{-}2.4)$$

Example 3-2.3 Neglecting end effects, find the electrokinetic field inside a long solenoid of n turns and length L carrying a current I.

If the end effects are neglected, the magnetic field inside the solenoid is homogeneous, and the vector potential is[4]

$$\mathbf{A} = \mu_0 \frac{nI}{2L} r \boldsymbol{\theta}_u, \tag{3-2.5}$$

where r is the distance from the axis of the solenoid and $\boldsymbol{\theta}_u$ is the azimuthal unit vector whose direction is the same as that of the circulating current in the solenoid. The electrokinetic field is then, by Eq. (3-1.3),

$$\mathbf{E}_k = -\mu_0 \frac{\partial I}{\partial t} \frac{nr}{2L} \boldsymbol{\theta}_u. \tag{3-2.6}$$

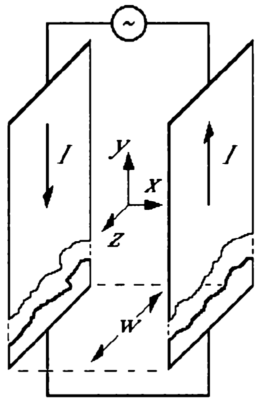

Fig. 3.3 Calculation of the electrokinetic field in the space between two current-carrying plates.

Example 3-2.4 Two long, parallel, vertical conducting plates of width w are separated by a small distance (Fig. 3.3). The plates are shorted at the bottom end, and a time-variable voltage is applied to them at the top end, so that the two plates carry equal currents I in opposite directions. Neglecting end effects, find the electrokinetic field in the space between the plates.

Let us assume that the current is downward in the left plate and upward in the right plate. The magnetic flux density field in the space between the plates is then[5] $\mathbf{B} = (\mu_0 I/w)\mathbf{k}$, where $\mathbf{k}$ is a unit vector along the width of the plates out of the page. The magnetic vector potential in the space between the plates is therefore[6]

$$\mathbf{A} = \mu_0 \frac{I}{w} x \mathbf{j}, \tag{3-2.7}$$

where $\mathbf{j}$ is a unit vector upward along the current in the right plate, and x is a perpendicular distance from the midplane of the system toward the right plate.

The electrokinetic field between the plates is therefore, by Eq. (3-1.3),

$$\mathbf{E}_k = - \mu_0 \frac{\partial I}{\partial t} \frac{x}{w} \mathbf{j}. \tag{3-2.8}$$

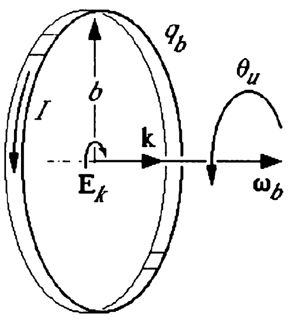

Fig. 3.4 Calculation of the electrokinetic field near the center of a rotating charged ring.

Example 3-2.5 A ring of radius b carries a uniformly distributed charge q_b and rotates with a variable angular velocity ω_b about its symmetry axis (Fig. 3.4). Find the electrokinetic field in the plane of the ring near the center of the ring.

The ring constitutes a circular current $I = \omega_b q_b / 2\pi$. The magnetic flux density field at the center of the ring is[7] $\mathbf{B} =$

$(\mu_0 I/2b)\mathbf{k}$, or $\mathbf{B} = (\mu_0 \omega_b q_b / 4\pi b)\mathbf{k}$. Within a small region near the center of the ring this field is nearly homogeneous, so that the magnetic vector potential in the plane of the ring is approximately[4]

$$\mathbf{A} = \mu_0 \frac{q_b \omega_b r}{8\pi b} \boldsymbol{\theta}_u, \tag{3-2.9}$$

where r is the distance from the center, and $\boldsymbol{\theta}_u$ is the azimuthal unit vector in the direction of the rotation of the ring. By Eq. (3-1.3), the electrokinetic field near the center of the ring is then approximately

$$\mathbf{E}_k = -\mu_0 \frac{\partial \omega_b}{\partial t} \frac{q_b r}{8\pi b} \boldsymbol{\theta}_u. \tag{3-2.10}$$

Example 3-2.6 A ring of radius a carries a uniformly distributed charge q_a and rotates with a variable angular velocity ω_a about its symmetry axis (Fig. 3.5). Find the electrokinetic field far from the ring in the plane of the ring.

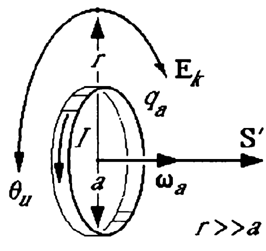

Fig. 3.5 Calculation of the electrokinetic field at a large distance from a rotating charged ring.

We shall solve this problem by direct calculation, using Eq. (3-1.2). Let us first convert the line integral of Eq. (3-1.2) into a surface integral by means of vector identity (V-18). We then have

$$\mathbf{E}_k = -\frac{\partial I}{\partial t} \frac{\mu_0}{4\pi} \oint \frac{d\mathbf{l}'}{r} = \frac{\partial I}{\partial t} \frac{\mu_0}{4\pi} \int \frac{\mathbf{r}}{r^3} \times d\mathbf{S}', \tag{3-2.11}$$

where $d\mathbf{S}'$ is right-handed relative to $d\mathbf{l}'$ (or I). For field points far from the ring, r may be considered constant over the entire surface area of the ring, and r^3 may be factored out from under the integral sign. Since we are calculating the field in the plane of the ring, $\mathbf{r}$ is perpendicular to $d\mathbf{S}'$, so that the cross product in the last integral becomes $-rdS'\boldsymbol{\theta}_u$, where the unit vector $\boldsymbol{\theta}_u$ is as shown in Fig. 3.5. Canceling r, replacing the integral by the area of the ring, πa^2, and replacing the current I by $\omega_a q_a/2\pi$, we obtain

$$\mathbf{E}_k = -\frac{\partial I}{\partial t}\frac{\mu_0 a^2}{4r^2}\boldsymbol{\theta}_u = -\mu_0\frac{\partial \omega_a}{\partial t}\frac{q_a a^2}{8\pi r^2}\boldsymbol{\theta}_u. \tag{3-2.12}$$

▲

3-3. Dynamic Effects of Electrokinetic Fields

We shall now present several examples demonstrating force effects of the electrokinetic field. For simplicity we shall use electrokinetic fields calculated in the preceding section.

▼

Example 3-3.1 The cylinder of Example 3-2.2 initially carries no current. A charged ring of charge q, radius R, and mass m is placed around the cylinder coaxially with it. The current in the cylinder is then switched on and attains a steady value I_0. The electrokinetic force causes the ring to move along the cylinder (Fig. 3.6). Assuming that no other forces act on the ring, and assuming that the ring stays near the middle of the cylinder during the time that the current changes, find the final velocity $\mathbf{v}_f$ of the ring.

According to our assumptions, the electrokinetic field through which the ring moves is a function of time only. Therefore we can use Eq. (3-1.5) for finding the final momentum and velocity of the ring. From Eqs. (3-1.5) and (3-2.3) (see Example 3-2.2), we have

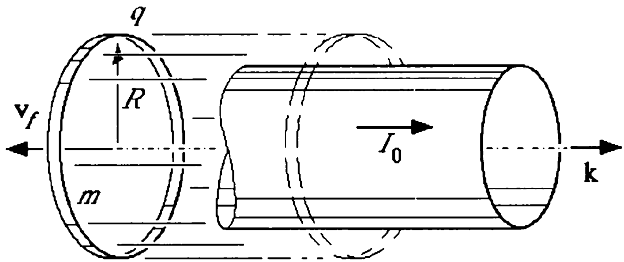

Fig. 3.6 When a current is established in the cylinder, the charged ring flies off the cylinder.

$$\Delta \mathbf{P} = m\mathbf{v}_f = -\, qI_0 \frac{\mu_0}{2\pi} \ln \frac{2L}{R} \mathbf{k}, \tag{3-3.1}$$

so that the final velocity is

$$\mathbf{v}_f = -\, \mu_0 \frac{qI_0}{2\pi m} \ln \frac{2L}{R} \mathbf{k}. \tag{3-3.2}$$

The ring moves in the direction opposite to that of the current (see, however, the explanatory note at the end of Section 3-1.).

Example 3-3.2 A thin parallel-plate capacitor of mass m and plate separation d has charges $\pm\, q$ on its plates. The capacitor is inserted between the current-carrying plates described in Example 3-2.4, so that the capacitor's plates are parallel to the current-carrying plates, and so that the midplane of the capacitor coincides with the midplane of the current-carrying plates (Fig. 3.7a). The negative plate of the capacitor is on the left, the positive plate is on the right. The current is initially I_0. When the current is switched off, the capacitor moves (Fig. 3.7b). Assuming that the capacitor stays between the current-carrying plates during the time when the current changes, find the final velocity of the capacitor.

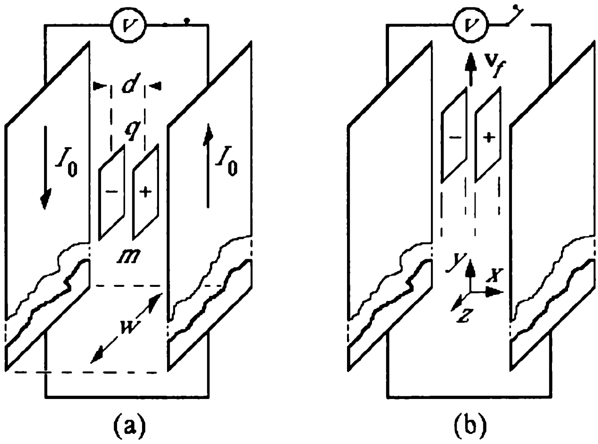

Fig. 3.7 (a) A charged parallel-plate capacitor is inserted between two current-carrying plates. (b) When the current is switched off, the capacitor moves.

By the symmetry of the system, the two plates of the capacitor experience the same force. Therefore, to solve the problem, we only need to consider the effect of the electrokinetic force on one of the plates. Since the electrokinetic field through which the capacitor moves is a function of time only, Eq. (3-1.5) applies. Considering the right plate and taking into account that the current changes from I_0 to 0, we have from Eqs. (3-1.5) and (3-2.8) [or (3-2.7)]

$$\Delta \mathbf{P}_{\text{right plate}} = \frac{m}{2}\mathbf{v}_f = q\mu_0 I_0 \frac{d}{2w}\mathbf{j}. \tag{3-3.3}$$

The final velocity of the capacitor is therefore

$$\mathbf{v}_f = \mu_0 \frac{qdI_0}{mw}\mathbf{j}. \tag{3-3.4}$$

Example 3-3.3 A polarized particle of dipole moment **p** is placed near the wire described in Example 3-2.1, so that **p** is directed toward the wire at right angles to it (Fig. 3.8a). (a) Find the torque exerted on the particle by the electrokinetic field of the wire. (b) The particle is placed so that **p** is parallel to the wire (Fig. 3.8b). Find the force exerted on the particle by the electrokinetic field of the wire.

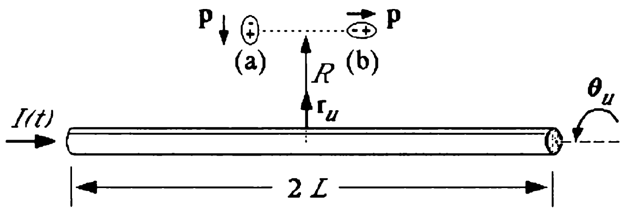

Fig. 3.8 The electrokinetic field of a current-carrying wire exerts a torque and a force on a polarized particle placed near the wire.

(a) The torque exerted on a dipole by an electric field **E** is[8]

$$\mathbf{T} = \mathbf{p} \times \mathbf{E}. \tag{3-3.5}$$

Since the particle is close to the wire, we can use Eq. (3-2.3) for the electrokinetic field. We then have

$$\mathbf{T} = -\mu_0 \frac{p}{2\pi} \ln \frac{2L}{R} \frac{\partial I}{\partial t} \boldsymbol{\theta}_u, \tag{3-3.6}$$

where $\boldsymbol{\theta}_u$ is an azimuthal unit vector right-handed relative to the direction of the current in the wire. (See the explanatory note at the end of Section 3-1.)

(b) The force exerted on a dipole by an electric field is[8]

$$\mathbf{F} = \nabla(\mathbf{p} \cdot \mathbf{E}). \tag{3-3.7}$$

Since **p** is in the direction of the current in the wire, we have

$$\mathbf{F} = \nabla(\mathbf{p}\cdot\mathbf{E}_k) = -\nabla\left(p\frac{\mu_0}{2\pi}\ln\frac{2L}{R}\frac{\partial I}{\partial t}\right)$$
$$= \mu_0\frac{p}{2\pi R}\frac{\partial I}{\partial t}\mathbf{r}_u, \qquad (3\text{-}3.8)$$

where $\mathbf{r}_u$ is a unit vector pointing away from the wire at right angles to it. (See the explanatory note at the end of Section 3-1.)

Example 3-3.4 The solenoid of Example 3-2.3 initially carries no current. A thin-walled dielectric cylinder of mass m and radius r carrying a uniformly distributed total charge q is placed inside the solenoid coaxially with it (Fig. 3.9). The current in the solenoid is then turned on and attains a final magnitude of I_0. The electrokinetic force causes the cylinder to rotate. Find the final angular velocity of the cylinder.

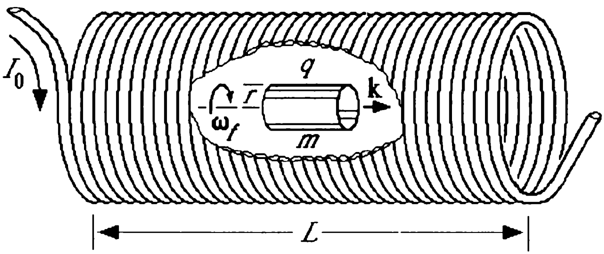

Fig. 3.9 A charged cylinder placed inside a solenoid rotates when the current is switched on. If the charge of the cylinder is positive, the rotation is against the direction of the current.

Since the electrokinetic field causing the cylinder to rotate is a function of time only, we can use Eq. (3-1.5) for finding the angular velocity of the cylinder. From Eqs. (3-1.5) and (3-2.6) we have

$$\Delta \mathbf{P} = m\mathbf{v}_f = m\boldsymbol{\omega}_f \times \mathbf{r} = -q\mu_0 \frac{nI_0}{2L} r\boldsymbol{\theta}_u, \qquad (3\text{-}3.9)$$

and since $\mathbf{r}$ is perpendicular to $\boldsymbol{\omega}_f$,

$$\boldsymbol{\omega}_f = -\mu_0 \frac{qnI_0}{2mL}\mathbf{k}, \qquad (3\text{-}3.10)$$

where $\mathbf{k}$ is a unit vector right-handed relative to the direction of the current in the solenoid. The minus sign in Eq. (3-3.10) shows that if q is positive, the rotation is against the current.

Example 3-3.5 The current in the solenoid of Example 3-2.3 is I_0. A uniformly charged disk of total charge q, radius a, and mass m is placed inside the solenoid coaxially with it (Fig. 3.10). When the current is turned off, the disk rotates. Find the final angular velocity of the disk.

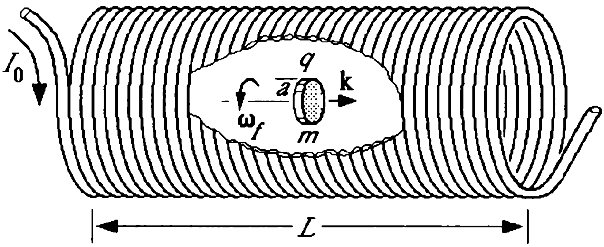

Fig. 3.10 A charged disk is placed inside a solenoid. When the current is switched off, the disk rotates. If the charge of the disk is positive, the rotation is in the direction of the original current.

The disk acquires an angular momentum that can be found from Eq. (3-1.6) and Eq. (3-2.6). Let us divide the disk into elementary rings. Consider an elementary ring of radius r and width dr. The charge of the ring is $dq = (q/\pi a^2)2\pi r dr$. Substituting dq and $\mathbf{E}_k$ from Eq. (3-2.6) in Eq. (3-1.6), integrating over dt, and taking into account that $\mathbf{r}$ is perpendicular to $\mathbf{E}_k$, we have

$$\Delta\mathbf{L} = \int_0^a r\left(\mu_0 I_0 \frac{nr}{2L}\right)\frac{q}{\pi a^2} 2\pi r dr\mathbf{k} = \frac{\mu_0 I_0 n}{2L}\frac{2q}{a^2}\int_0^a r^3 dr\mathbf{k} \qquad (3\text{-}3.11)$$

or

$$\Delta\mathbf{L} = \mu_0 \frac{qna^2 I_0}{4L}\mathbf{k}, \qquad (3\text{-}3.12)$$

where $\mathbf{k}$ is a unit vector right-handed relative to the direction of the current in the solenoid. Since the moment of inertia of the disk is $ma^2/2$, we obtain for the angular velocity

$$\boldsymbol{\omega}_f = \mu_0 \frac{qnI_0}{2mL}\mathbf{k}. \qquad (3\text{-}3.13)$$

For positive q, the rotation is in the direction of the current.

Example 3-3.6 The masses of the rings described in Examples 3-2.5 and 3-2.6 are m_b and m_a, and their radii are such that $b \gg a$. The rings are placed in the same plane, and their centers coincide (Fig. 3.11). (a) Ring b is given an angular acceleration α_b. Find the angular acceleration of ring a due to the electrokinetic field of ring b. (b) Ring a is given an angular acceleration α_a. Find the angular acceleration of ring b due to the electrokinetic field of ring a.

(a) By the definition of the electrokinetic force and by Eq. (3-2.10), the torque experienced by ring a due to the electrokinetic field of ring b is

$$\mathbf{T}_a = \mathbf{r}\times q_a\mathbf{E}_k = -aq_a\mu_0\alpha_b\frac{q_b a}{8\pi b}\mathbf{k}, \qquad (3\text{-}3.14)$$

where $\mathbf{k}$ is a unit vector along the axis of the rings right-handed

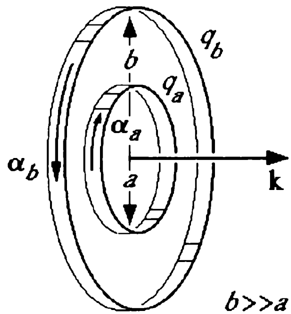

Fig. 3.11 When one of the two charged rings is rotated, the other ring starts to rotate in the opposite direction.

relative to the rotation of ring b. Since the moment of inertia of ring a is $m_a a^2$, its angular acceleration is

$$\alpha_a = -\mu_0 \frac{q_a q_b}{8\pi m_a b} \alpha_b . \qquad (3\text{-}3.15)$$

The rings rotate in opposite directions.

(b) Using Eq. ((3-2.12), we find, as above in Part (a), that the angular acceleration of ring b due to the electrokinetic field of ring a is

$$\alpha_b = -\mu_0 \frac{q_a q_b a^2}{8\pi m_b b^3} \alpha_a . \qquad (3\text{-}3.16)$$

Once again the two rings rotate in opposite directions.[9]

Example 3-3.7 Consider a circular ring of radius a carrying a current I (Fig. 3.12). Let the ring be in the x,y-plane of rectangular coordinates, and let the current in the ring be left-handed relative to the z-axis. Let the ring move with velocity $v \ll c$ along the x-axis, and let the center of the ring be momentarily at the origin of the coordinates. (a) Assuming that the ring carries no net charge, what is the force exerted by the ring on a point charge q located on the z-axis at a distance $d \gg a$ from the origin? (b) Assuming that the ring is at rest with its center at the origin, and assuming that the

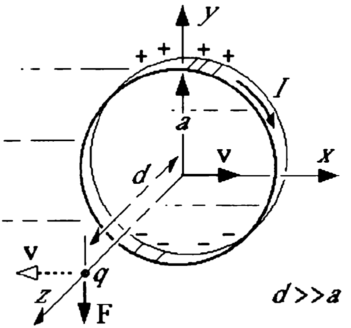

Fig. 3.12 A current-carrying ring moving past a point charge exerts a force on the point charge. The plus and minus charges on the ring are fictitious: the ring appears to acquire them as a result of its motion (see Appendix 3).

charge moves in the minus x direction with velocity $v \ll c$, what is the force experienced by the charge?

(a) Since the radius of the ring is much smaller than the distance between the ring and the point charge, the current of the ring, viewed from the location of the point charge, constitutes a magnetic dipole. As it was shown in Section 2-6, a moving magnetic dipole appears to acquire an electric dipole moment, so that the total electric field produced by the moving ring is the sum, given by Eq. (2-6.9), of the electrokinetic field and the electric dipole field. For the present example, since $v \ll c$, **B** in Eq. (2-6.9) is the flux density field produced by the ring as if it were at rest[10]

$$\mathbf{B} = -\frac{\mu_0 I a^2}{2d^3}\mathbf{k}. \tag{3-3.17}$$

Hence the force exerted by the moving current-carrying ring on the stationary charge q is, by Eqs. (2-6.9) and (3-3.17),

$$\mathbf{F} = -\frac{\mu_0 q v I a^2}{2d^3}\mathbf{j}. \tag{3-3.18}$$

(b) If the ring is at rest, but the charge is moving, the charge experiences the Lorentz's force

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} = -\frac{\mu_0 q v I a^2}{2d^3}\mathbf{j}, \qquad (3\text{-}3.19)$$

which, under the assumed conditions, is exactly the same as the force experienced by the stationary charge due to the current in the moving ring.

▲

3-4. Induction of Currents and Voltages by Electrokinetic Fields

In Chapter 1, by using the principle of causality, we found that electric and magnetic fields could not cause each other. Then we found in Chapter 2 that Faraday induction was not at all an electromagnetic phenomenon. We found that induced currents in conductors were caused not by changing magnetic fields, but by the electrokinetic field created by changing electric currents.

Convincing as our analysis of the induction phenomenon may be, it still lacks one important element: a direct demonstration that one can account for the induced voltages and currents solely by the electrokinetic field, without invoking any electromagnetic effects, and without invoking changing magnetic fields in particular. We shall now present several such demonstrations.

As before, we shall use very simple illustrative examples. It is the clarity of the examples, not their complexity, that is important. Accordingly, we shall make use of the simple electrokinetic fields computed in Section 3-2.

▼

Example 3-4.1 Consider two equal, square-shaped loops of insulated wire. (a) The loops are placed one upon the other so that their sides completely overlap (Fig. 3.13a). A current is switched on in the bottom loop. What is the direction of the induced current in the top loop? (b) The top loop is positioned to the right of the bottom loop, so that the loops are now side by side (Fig. 3.13b). A current is switched on in the left loop. What is the direction of the induced current in the right loop? (c) What is the ratio of the induced current when the loops are in the first position to the induced current when the loops are in the second position?

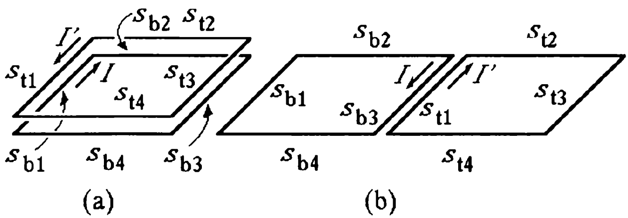

Fig. 3.13 The direction of the induced current I' is easily determined by considering the electrokinetic field of the inducing current I.

(a) Let us designate the sides of the top loop as s_{t1}, s_{t2}, s_{t3}, and s_{t4}, and let us designate the sides of the bottom loop as s_{b1}, s_{b2}, s_{b3}, and s_{b4}. Consider the side s_{t1}. Since, by Eq. (2-4.1), there can be no induction between perpendicular sides, only the electrokinetic field of sides s_{b1} and s_{b3} parallel to s_{t1} can cause a current in s_{t1}. Since s_{b1} is close to s_{t1}, while s_{b3} is relatively far from s_{t1}, the electrokinetic field of s_{b3} can be neglected. Since, when the current is switched on, its electrokinetic field is opposite to the current, the current induced in s_{t1} is opposite to the current in s_{b1}. And since the same considerations apply to all four sides of the top loop, the current induced in it circulates in a direction opposite to that of the inducing current in the bottom loop.

(b) When the loops are side by side, only the sides s_{b3} and s_{t1} are close together. Therefore the current is essentially induced only in

s_{t1}. The induced current is again opposite to the current in s_{b3}. However, since the loops are side by side, the direction of circulation of the induced current in the right loop is the same as that of the inducing current in the left loop.

(c) When the loops are one upon the other, each of the four sides of the top loop is subjected to the electrokinetic field of the side of the bottom loop directly below. When the loops are side by side, only the left side of the right loop is subjected to the electrokinetic field of only the right side of the left loop (we ignore the interaction between the distant sides). Therefore, when the loops are one upon the other, the induced current is very nearly four times stronger than when the loops are side by side.

Example 3-4.2 Find the voltage (emf) induced in a rectangular loop of wire of width b and height a by the current I of a long straight wire of length $2L$ placed in the plane of the loop parallel to its horizontal sides at a distance d from the nearest side (Fig. 3.14). Assume that a, b, and d are much smaller than L.

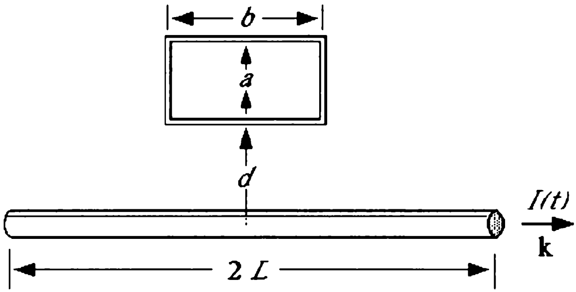

Fig. 3.14 The voltage induced in a rectangular loop by a current-carrying wire can be easily determined by using the electrokinetic field of the wire.

Let the loop and the wire be in the *yz*-plane of a rectangular system of coordinates, and let the current in the wire be in the *z*-direction. The electrokinetic field of the wire was found in Example 3-2.1 and is, for large L,

$$\mathbf{E}_k = -\frac{\partial I}{\partial t}\frac{\mu_0}{2\pi}\ln\frac{2L}{R}\mathbf{k}. \tag{3-4.1}$$

To find the induced voltage, we integrate the electrokinetic field along the rectangular loop

$$V_{\text{ind}} = \oint \mathbf{E}_k \cdot d\mathbf{l}. \tag{3-4.2}$$

Since the field is everywhere parallel to the *z*-axis, the only contributions to the voltage come from the horizontal sides of the loop. The integration along the horizontal sides amounts to multiplication of $\mathbf{E}_k$ by b (with a reversal of sign for one of the two sides). The result is

$$V_{\text{ind}} = \frac{\mu_0 b}{2\pi}\ln\frac{d+a}{d}\frac{\partial I}{\partial t} \tag{3-4.3}$$

and is the same as obtained in conventional manner by using the coefficient of mutual inductance between the wire and the loop.[11]

Example 3-4.3 A narrow slot is made in the cylinder described in Example 3-2.2, and the rectangular loop described in Example 3-4.2 is inserted in it so that the loop is partly inside the cylinder (Fig. 3.15). The horizontal sides of the loop are parallel to the axis of the cylinder, and the side which is inside the cylinder is at a distance d from the axis. Find the voltage induced in the loop by the current of the cylinder.

As in Example 3-2.2, we find the induced voltage by multiplying $\mathbf{E}_k$ at the horizontal sides of the loop by the width of the loop b, and reversing the sign of the product for one of the sides. Using the

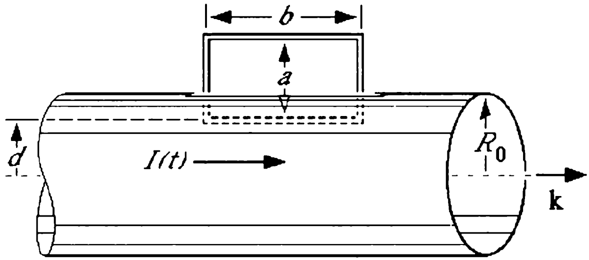

Fig. 3.15 The voltage induced in the rectangular loop partially inserted in a current-carrying cylinder is found by using the electrokinetic field of the cylinder.

expressions found in Example 3-2.2 for $\mathbf{E}_k$ inside and outside the cylinder, we then have

$$V_{\text{ind}} = b\frac{\mu_0}{2\pi}\left(\ln\frac{2L}{R_0} - \ln\frac{2L}{d+a}\right)\frac{\partial I}{\partial t} = \frac{\mu_0 b}{2\pi}\ln\frac{d+a}{R_0}\frac{\partial I}{\partial t}. \qquad (3\text{-}4.4)$$

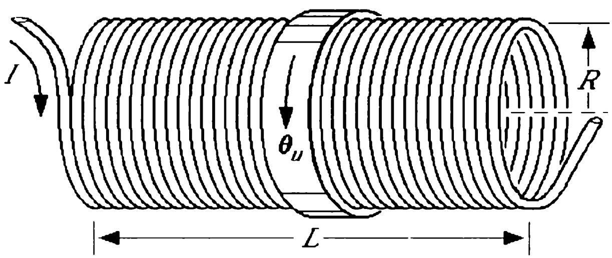

Fig. 3.16 The voltage induced in the ring placed over a solenoid is found by using the electrokinetic field of the solenoid.

Example 3-4.4 A conducting circular ring is placed just outside the long solenoid described in Example 3-2.3 (Fig. 3.16). The radius of the ring and of the solenoid is R. Find the voltage induced in the ring by the current of the solenoid. The electrokinetic field of the solenoid is, by Eq. (3-2.6),

$$\mathbf{E}_k = -\mu_0 \frac{\partial I}{\partial t} \frac{nR}{2L} \theta_u . \tag{3-4.5}$$

The induced voltage is therefore

$$V_{\text{ind}} = \oint \mathbf{E}_k \cdot d\mathbf{l} = \mu_0 \pi \frac{nR^2}{L} \frac{\partial I}{\partial t}, \tag{3-4.6}$$

which also agrees with the result obtained by conventional methods.[12]

▲

3-5. Summary of Chapter 3

In this chapter we have presented examples on calculation and use of electrokinetic fields. In spite of their simplicity, these examples give a remarkable new insight into a variety of electric phenomena. Examples 3-3.1, 3-3.2, 3-3.3, and 3-3.7 unveil some intriguing new electrodynamic effects. Examples 3-3.4, 3-3.5, and 3-3.6, although not new in principle, give a much clearer picture of the phenomenon of induced rotation of charged bodies.[13] Example 3-4.1 shows how useful and effective is the concept of the electrokinetic field for analyzing the induction of current in electric circuits. Examples 3-4.2, 3-4.3, and 3-4.4 show that one can find voltages (emf's) induced in conductors by using the electrokinetic field alone, without invoking any magnetic fields.

The fact that the calculations of induced voltages in Examples 3-4.2, 3-4.3, and 3-4.4 yield the same results as the conventional calculations deserves special attention. As it was mentioned in Section 3-1, the electrokinetic fields used in our illustrative examples are approximate ones, obtained by neglecting the retardation effects. Since the induced voltages obtained in Examples 3-4.2, 3-4.3, and 3-4.4 agreed exactly with the voltages computed by conventional methods, it is clear that the conventional methods do not yield exact solutions. This is an important fact, usually overlooked in textbooks on electricity and magnetism.

Are there some effects of electrokinetic fields and forces that have not yet come to light? There probably are. They would be most prominently associated with very strong and rapidly changing electric currents. Electric spark discharges are good examples of such currents. According to Examples 3-2.1 and 3-3.3, spark discharges should have significant effects on nearby charged particles, causing them to move along the spark, and causing polar particles to move toward or away from the spark and to rotate. As a consequence, the spark itself could spread laterally and could give rise to secondary discharges. It will be interesting to see to what extent such effects do actually accompany bolts of lightning.

The examples presented in this chapter give an unequivocal support to the basic conclusion reached in Chapter 2: Faraday induction is not caused by changing magnetic fields, but by the electrokinetic fields produced by changing electric currents. The essence of the induction phenomenon is that the electrokinetic fields, just like electrostatic fields, are force fields. Acting on charges in conductors, they create conduction currents. Acting on free charges or on charged bodies, they create convection currents or cause the bodies to move. There is no causal linkage between the induced currents or motions and magnetic fields. The illusion of a linkage is a result of the fact that electrokinetic fields and changing magnetic fields, having a common causative source, always appear simultaneously together.

Thus, Faraday's original idea of electric induction was absolutely correct: just as electrostatic fields induce charges on nearby conductors, electric currents, through their electrokinetic fields, induce currents in other conductors. It is therefore correct to call this induction "Faraday induction," but it is incorrect to call it "electromagnetic induction," because no magnetic fields or effects are involved in this induction phenomenon.

References and Remarks for Chapter 3

1. See, for example, Emil J. Konopinski, *Electromagnetic Fields and Relativistic Particles* (McGraw-Hill, New York, 1981) pp. 158-160.

2. See O. D. Jefimenko, *Electricity and Magnetism*, 2nd ed., (Electret Scientific, Star City, 1989) p. 366.

3. Ref. 2, p. 367.

4. Ref. 2, p. 383, Problem 11.2.

5. See Ref. 2, pp. 332-333 and p. 358, Problem 10.8.

6. See Ref. 2, p. 383, Problem 11.1.

7. See Ref. 2, pp. 346-347.

8. See Ref. 2, p. 213.

9. Similar rotation should occur when two coaxial charged disks are placed close to each other and one of them is set in motion.

10. See Ref. 2, p. 381.

11. See Ref. 2, p. 341.

12. For a direct calculation (without using vector potential for the calculation of the electrokinetic field) see O. D. Jefimenko, *Electromagnetic Retardation and Theory of Relativity* (Electret Scientific, Star City, 1997) pp. 38-44

13. Compare T. Bahder and J. Sak, "Elementary solution to Feynman's disk paradox," Am. J. Phys. **53**, 495-497 (1985).

4

ACTION AND REACTION IN ELECTRIC, MAGNETIC, AND GRAVITATIONAL FIELDS

Newton's third law, the law of action and reaction, is one of the very first laws of physics that we learn. It is one of the most unequivocal laws. We readily accept it and do not usually doubt its universal validity. And yet, as we shall see in this chapter, its range of applicability to electric and magnetic systems is severely limited. Does it mean that there is something exclusive about electric and magnetic forces? To answer this question, we shall extend our analysis of the law of action and reaction also to gravitational systems.

4-1. Is the Law of Action and Reaction Always Valid?

Newton's law of action and reaction is usually considered to be one of the most fundamental laws of physics. It is typically stated as follows: "To every action there is always an equal and opposite reaction; that is, whenever a body exerts a certain force on a second body, the second body exerts an equal and opposite force on the first." [1] No exceptions to the law appear possible.

However, there is at least one fairly well known example showing that the law does not always hold: if one calculates the forces between a charged particle moving along the z-axis and another moving along the x-axis, one finds that the forces are not equal.[2,3] Another example is that of a charge moving past a magnetic dipole; here, too, the forces are not equal.[4]

Even more eloquent is the following example. Suppose that a stationary charge is located in the field of another, distant, stationary charge. The two charges exert upon each other equal and opposite forces, as required by the law of action and reaction. Suppose now that the first charge is allowed to move under the action of the field of the second charge and arrives at a new position. But the second charge, being far away from the first, does not yet "know" that the first charge has moved and continues to experience the same force as before. The forces are now unequal in magnitude and direction, and the action and reaction law no longer holds!

And how can the law of action and reaction apply to electrokinetic forces? These forces are caused by electric currents (moving charges), yet they act on all charges, including charges at rest, which cannot possibly generate electrokinetic reaction forces.

But if the law of action and reaction does not hold in one system or another, does it mean that the mechanical momentum in these systems is not conserved? If so, where does the momentum come from, and where does it disappear?

These are perplexing questions. We shall provide answers to them in this chapter.

The law of action and reaction does not differentiate between forces on the basis of their physical nature. Therefore an analysis of the validity of this law cannot be truly conclusive if it is restricted to some particular type of forces. The similarity between electrostatic and gravitational forces is well known. Therefore it is only natural to include gravitational forces in our analysis of the law of action and reaction. As we shall see, this will result in very remarkable conclusions concerning gravitation and its laws.

4-2. Action and Reaction in Electric Systems

What is the range of validity of the law of action and reaction in electric systems?

To answer this question, let us first consider the interaction between two constant, stationary charge distributions ρ_1 and ρ_2. Let the electric field produced by ρ_1 be $\mathbf{E}_1$, and that produced by ρ_2 be $\mathbf{E}_2$. The force exerted by ρ_2 upon ρ_1 is $\int \rho_1 \mathbf{E}_2\, dv$, and the force exerted by ρ_1 upon ρ_2 is $\int \rho_2 \mathbf{E}_1\, dv$. Let us now use vector identity (V-22) (this is an extension of the Gauss's theorem of vector analysis to two vectors, **A** and **B**; the square brackets do not signify retardation),

$$\oint (\mathbf{A}\cdot\mathbf{B})d\mathbf{S} - \oint \mathbf{B}(\mathbf{A}\cdot d\mathbf{S}) - \oint \mathbf{A}(\mathbf{B}\cdot d\mathbf{S}) = \int [\mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) - \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})]dv. \tag{V-22}$$

If we apply this vector identity to the fields $\mathbf{E}_1$ and $\mathbf{E}_2$ and integrate over all space, we obtain

$$\oint (\mathbf{E}_1\cdot\mathbf{E}_2)d\mathbf{S} - \oint \mathbf{E}_2(\mathbf{E}_1\cdot d\mathbf{S}) - \oint \mathbf{E}_1(\mathbf{E}_2\cdot d\mathbf{S}) = \int [\mathbf{E}_1\times(\nabla\times\mathbf{E}_2) + \mathbf{E}_2\times(\nabla\times\mathbf{E}_1) - \mathbf{E}_1(\nabla\cdot\mathbf{E}_2) - \mathbf{E}_2(\nabla\cdot\mathbf{E}_1)]dv. \tag{4-2.1}$$

For constant charges, $\nabla\times\mathbf{E} = 0$, so that the first two terms in the volume integral vanish. Also, according to Eq. (1-4.1), the fields $\mathbf{E}_1$ and $\mathbf{E}_2$ are regular at infinity, so that the surface integrals in Eq. (4-2.1) vanish. Replacing $\nabla \cdot \mathbf{E}_1$ and $\nabla \cdot \mathbf{E}_2$ by ρ_1/ε_0 and ρ_2/ε_0, respectively, and canceling ε_0, we then obtain

$$\int \rho_1 \mathbf{E}_2 dv = - \int \rho_2 \mathbf{E}_1 dv, \qquad (4\text{-}2.2)$$

so that the forces on the two charge distributions are equal in magnitude and opposite in direction, as required by the law of action and reaction. Thus the law of action and reaction holds for interactions between constant stationary charge distributions.

Let us now assume that ρ_2 is moving and/or is time dependent. In this case $\nabla \times \mathbf{E}_1 = 0$, but $\nabla \times \mathbf{E}_2 = - \partial \mathbf{B}_2/\partial t$, where $\mathbf{B}_2$ is the magnetic field produced by ρ_2. Substituting in Eq. (4-2.1) $\mathbf{E}_1$, $\mathbf{E}_2$, $\nabla \cdot \mathbf{E}_1$, $\nabla \cdot \mathbf{E}_2$, $\nabla \times \mathbf{E}_1$, and $\nabla \times \mathbf{E}_2$, we find that the second term in the volume integral vanishes. Because of the finite speed of propagation, $\mathbf{E}_1$ and $\mathbf{E}_2$ are zero at infinity, so that the surface integral vanishes also. Simplifying, we then obtain

$$\int \rho_1 \mathbf{E}_2 dv = - \int \rho_2 \mathbf{E}_1 dv - \varepsilon_0 \int \mathbf{E}_1 \times \frac{\partial \mathbf{B}_2}{\partial t} dv. \qquad (4\text{-}2.3)$$

Hence, when one of the two interacting charge distributions is time variable or is in motion, the law of action and reaction does not hold: the two forces differ by the value of the integral containing $\mathbf{B}_2$.

Let us now assume that both charge distributions are moving and/or are time dependent. In this case $\nabla \times \mathbf{E}_1 = - \partial \mathbf{B}_1/\partial t$ and $\nabla \times \mathbf{E}_2 = - \partial \mathbf{B}_2/\partial t$, where $\mathbf{B}_1$ and $\mathbf{B}_2$ are the magnetic fields produced by the charges ρ_1 and ρ_2, respectively. Substituting in Eq. (4-2.1) $\mathbf{E}_1$, $\mathbf{E}_2$, $\nabla \cdot \mathbf{E}_1$, $\nabla \cdot \mathbf{E}_2$, $\nabla \times \mathbf{E}_1$, and $\nabla \times \mathbf{E}_2$, we obtain

$$\int \rho_1 \mathbf{E}_2 dv + \varepsilon_0 \int \mathbf{E}_2 \times \frac{\partial \mathbf{B}_1}{\partial t} dv = - \int \rho_2 \mathbf{E}_1 dv - \varepsilon_0 \int \mathbf{E}_1 \times \frac{\partial \mathbf{B}_2}{\partial t} dv. \qquad (4\text{-}2.4)$$

Thus, when both charge distributions are time variable or are in motion, the law of action and reaction, in general, does not hold: the two forces differ by the value of the two integrals containing $\mathbf{B}_1$ and $\mathbf{B}_2$. However, if the two integrals happen to be equal in magnitude but have opposite signs, they cancel each other, so that in this case the law of action and reaction does hold even when the two charges vary or move.[5]

It should be pointed out that although the law of action and reaction does not hold for certain types of electric interactions, the law of conservation of momentum is valid for all electric and magnetic interactions, without exceptions. This will be shown later in this chapter.

▼

Example 4-2.1 Consider the two rings described in Example 3-3.6. (a) Ring b is given an angular acceleration α_b. Compare the torque exerted by ring b on ring a with the reaction torque exerted by ring a on ring b. (b) Ring a is given an angular acceleration α_a. Compare the torque exerted by ring a on ring b with the reaction torque exerted by ring b on ring a.

(a) The torque exerted by ring b on ring a is, by Eq. (3-3.14),

$$\mathbf{T}_{ba} = -\, a q_a \mu_0 \alpha_b \frac{q_b a}{8\pi b}\mathbf{k}. \tag{4-2.5}$$

This torque causes ring a to rotate with an angular acceleration given by Eq. (3-3.15)

$$\alpha_a = -\, \mu_0 \frac{q_a q_b}{8\pi m_a b}\alpha_b. \tag{3-3.15}$$

The torque experienced by ring b due to the electrokinetic field $\mathbf{E}_{ka}$ of ring a is then, by Eq. (3-2.12),

$$\mathbf{T}_{ab} = \mathbf{r}\times q_b \mathbf{E}_{ka} = -\, b q_b \mu_0 \alpha_a \frac{q_a a^2}{8\pi b^2}\mathbf{k}. \tag{4-2.6}$$

Substituting Eq. (3-3.15) into Eq. (4-2.6), we obtain for the reaction torque exerted by ring a on ring b

$$\mathbf{T}_{ab\,\text{reaction}} = \mu_0^2 \frac{q_a^2 q_b^2 a^2}{64\pi^2 m_a b^2} \alpha_b \mathbf{k}. \tag{4-2.7}$$

(b) The torque exerted by ring a on ring b is, by Eq. (4-2.6),

$$\mathbf{T}_{ab} = -\mu_0 \frac{q_a q_b a^2}{8\pi b} \alpha_a \mathbf{k}. \tag{4-2.8}$$

This torque causes ring b to rotate with an angular acceleration given by Eq. (3-3.16)

$$\alpha_b = -\mu_0 \frac{q_a q_b a^2}{8\pi m_b b^3} \alpha_a. \tag{3-3.16}$$

The torque experienced by ring a due to the electrokinetic field of ring b is given by Eq. (3-3.14). Substituting Eq. (3-3.16) into Eq. (3-3.14), we obtain for the reaction torque exerted by ring b on ring a

$$\mathbf{T}_{ba\,\text{reaction}} = \mu_0^2 \frac{q_a^2 q_b^2 a^4}{64\pi^2 m_b b^4} \alpha_a \mathbf{k}. \tag{4-2.9}$$

Example 4-2.2 Consider two equally large, charged disks carrying equal uniformly distributed charges Q_1 and Q_2. The disks are placed close to each other on a common axle and are free to rotate about it. When one of the disks is set in motion, the other starts rotating in the opposite direction. Assuming that the angular velocities of the two disks are always equal, does the law of action and reaction hold for the electric forces between the disks?

For simplicity, let us assume that the disks are so close together that the volume of the space between them is negligibly small. Since the electric and magnetic fields in this space are finite, this space does not contribute then to the second and fourth integral of Eq. (4-2.4). Elsewhere in space the "electrostatic" fields [first integral of Eq. (1-4.1)] of the two disks are equal in sign and magnitude, the magnetic fields are equal in magnitude and opposite in sign, and the electrokinetic fields are equal in magnitude and opposite in sign. Hence, the contributions to the second and fourth integral of Eq. (4-2.4) from the "electrostatic" fields of the disks cancel, while the contributions from the electrokinetic fields do not. Consequently, in the system under consideration, the law of action and reaction holds only for "electrostatic" interactions, but not for the electrokinetic interactions.

▲

4-3. Action and Reaction in Magnetic Systems

We shall now examine what happens to the action and reaction law in magnetic systems.

Let there be two constant, stationary current distributions $\mathbf{J}_1$ and $\mathbf{J}_2$. Let the magnetic flux density fields produced by these current distributions be $\mathbf{B}_1$ and $\mathbf{B}_2$, respectively. The force exerted by $\mathbf{J}_2$ on $\mathbf{J}_1$ is then $\int \mathbf{J}_1 \times \mathbf{B}_2 \, dv$ and the force exerted by $\mathbf{J}_1$ on $\mathbf{J}_2$ is $\int \mathbf{J}_2 \times \mathbf{B}_1 \, dv$.

Applying vector identity (V-22) to $\mathbf{B}_1$ and $\mathbf{B}_2$ and integrating over all space, we have

$$\oint (\mathbf{B}_1 \cdot \mathbf{B}_2) d\mathbf{S} - \oint \mathbf{B}_2 (\mathbf{B}_1 \cdot d\mathbf{S}) - \oint \mathbf{B}_1 (\mathbf{B}_2 \cdot d\mathbf{S}) =$$
$$\int [\mathbf{B}_1 \times (\nabla \times \mathbf{B}_2) + \mathbf{B}_2 \times (\nabla \times \mathbf{B}_1) - \mathbf{B}_1 (\nabla \cdot \mathbf{B}_2) - \mathbf{B}_2 (\nabla \cdot \mathbf{B}_1)] dv. \tag{4-3.1}$$

By Eq. (1-4.2), $\mathbf{B}_1$ and $\mathbf{B}_2$ are regular at infinity. Therefore the surface integrals vanish. By Maxwell's Eq. (1-1.2), $\nabla \cdot \mathbf{B} = 0$, so that the last two terms in the volume integral vanish also. Taking into account that $\nabla\times\mathbf{B} = \mu_0\nabla\times\mathbf{H} = \mu_0\mathbf{J}$, we then obtain

$$\int \mathbf{J}_1 \times \mathbf{B}_2 dv = - \int \mathbf{J}_2 \times \mathbf{B}_1 dv. \tag{4-3.2}$$

Thus, for magnetic interactions between two constant stationary currents, the two forces are equal in magnitude and opposite in direction, and the law of action and reaction holds.

Let us now assume that $\mathbf{J}_2$ is variable or is in a state of motion. In this case, by Maxwell's Eq. (1-1.4), $\nabla\times\mathbf{B}_2 = \mu_0\mathbf{J}_2 + \mu_0\partial\mathbf{D}_2/\partial t$, where $\mathbf{D}_2$ is the displacement field produced by $\mathbf{J}_2$ [coming from the last term of Eq. (1-4.1)]. Noting that the surface integrals of Eq. (4-3.1) still vanish, and simplifying the volume integral as before, we obtain

$$\int \mathbf{J}_1 \times \mathbf{B}_2 dv = - \int \mathbf{J}_2 \times \mathbf{B}_1 dv + \int \mathbf{B}_1 \times \frac{\partial \mathbf{D}_2}{\partial t} dv. \tag{4-3.3}$$

Hence, when one of the currents is changing or is in motion, the two forces are not equal and differ by the amount of the integral containing $\mathbf{D}_2$. The law of action and reaction does not hold.

Let us now assume that $\mathbf{J}_1$ is also variable. In this case $\nabla\times\mathbf{B}_1 = \mu_0\mathbf{J}_1 + \mu_0\partial\mathbf{D}_1/\partial t$ and $\nabla\times\mathbf{B}_2 = \mu_0\mathbf{J}_2 + \mu_0\partial\mathbf{D}_2/\partial t$, where $\mathbf{D}_1$ is the displacement field produced by $\mathbf{J}_1$, and $\mathbf{D}_2$ is the displacement field produced by $\mathbf{J}_2$. Using Eq. (4-3.1), we now obtain

$$\int \mathbf{J}_1 \times \mathbf{B}_2 dv - \int \mathbf{B}_2 \times \frac{\partial \mathbf{D}_1}{\partial t} dv = - \int \mathbf{J}_2 \times \mathbf{B}_1 dv + \int \mathbf{B}_1 \times \frac{\partial \mathbf{D}_2}{\partial t} dv. \tag{4-3.4}$$

Thus, when both currents are changing or are moving, the law of action and reaction, in general, does not hold: the forces differ by the values of the integrals containing $\mathbf{D}_1$ and $\mathbf{D}_2$. However, if the two integrals are equal in magnitude but have opposite signs, they cancel, and then the law of action and reaction does hold.

▼

Example 4-3.1 Consider the two disks described in Example 4-2.2. Each disk constitutes a convection current that produces a magnetic field and exerts a magnetic force on the other disk. Does this interaction satisfy the law of action and reaction?

The two disks create magnetic fields opposite in direction but equal in magnitude. The "electrostatic" fields of the disks are equal in sign and in magnitude. The electrokinetic fields are equal in magnitude but opposite in sign. Therefore the contributions from the "electrostatic" component of the electric fields to the second and fourth integral of Eq. (4-3.4) cancel. But the contributions of the electrokinetic components do not. However, if the accelerations of the disks are uniform, $\partial \mathbf{E}_k/\partial t = 0$, and the two integrals cancel. Thus, the law of action and reaction does not hold for magnetic interactions (repulsion) between the two disks, except when their acceleration is constant.

▲

4-4. Action and Reaction in Gravitational Systems

The two equations that formulate Newton's theory of gravitation as a force-field theory are[6]

$$\nabla \times \mathbf{g} = 0, \tag{4-4.1}$$

and

$$\nabla \cdot \mathbf{g} = -4\pi G\rho, \tag{4-4.2}$$

where $\mathbf{g}$ is the gravitational field vector, G is the universal constant of gravitation, and ρ is the mass density.

Suppose now that a mass distribution ρ_1 interacts with a mass distribution ρ_2. The force on the first mass due to the field $\mathbf{g}_2$ of the second mass is $\int \rho_1 \mathbf{g}_2\, dv$, and the force on the second mass due to the field $\mathbf{g}_1$ of the first mass is $\int \rho_2 \mathbf{g}_1\, dv$.

Let us apply vector identity (V-22) to the fields $\mathbf{g}_1$ and $\mathbf{g}_2$, and let us extend the integration over all space. By Newton's gravitational law, the two fields approach zero at infinity as $1/r^2$, so that the surface integrals vanish. By Eq. (4-4.1), the curl terms in the volume integral are zero. Substituting $\nabla \cdot \mathbf{g}_1 = -4\pi\rho_1$ and $\nabla \cdot \mathbf{g}_2 = -4\pi\rho_2$, we then obtain after canceling $4\pi G$

$$\int \rho_1 \mathbf{g}_2\, dv = -\int \rho_2 \mathbf{g}_1\, dv, \qquad (4\text{-}4.3)$$

so that for gravitational forces the action and reaction law appears to hold always. But does it really? We shall provide the answer to this question in Chapter 5.

4-5. The Law of Action and Reaction and the Law of Conservation of Momentum

As we have seen, Newton's third law, the law of action and reaction, has only a limited validity in the domain of electric and magnetic interactions. In general, it holds only for interactions between constant stationary charges and for interactions between constant stationary currents. However, it is not necessary to state Newton's third law as the law of action and reaction. One can state this law more accurately as the law of conservation of momentum. If we examine the time dependent terms appearing in Eqs. (4-2.3), (4-2.4), (4-3.3), and (4-3.4), we recognize that these terms represent rates of change of electromagnetic momentum [7]

$$\mathbf{G} = \varepsilon_0 \mu_0 \int \mathbf{E} \times \mathbf{H}\, dv. \qquad (4\text{-}5.1)$$

Therefore these equations show that, although the forces are different, the total momentum (mechanical plus electromagnetic) of the system is always conserved. An exchange of momentum between a charge or a current and the surrounding field is, of course, necessary since electric and magnetic fields propagate with finite speed, so that no direct interactions between field-producing and field-experiencing charges or currents are possible. Specific examples for momentum exchange have been computed for several electromagnetic systems involving linear[3,5] as well as angular momentum [8,9].

It is important to note that Eqs. (4-2.3), (4-2.4), (4-3.3), and (4-3.4) involve only the interaction, or mutual, momentum rather than the total electromagnetic momentum of the systems under consideration. Specifically, in the case of electric systems, the rate of momentum change is expressed as the cross product of the electric field and the time derivative of the magnetic flux density field. And in the case of magnetic systems, this rate of momentum change is expressed as the cross product of the magnetic field and the time derivative of the electric displacement field. A remarkable feature of these equations is that they only involve partial fields: $\mathbf{E}_1$ and $\mathbf{B}_2$ or $\mathbf{E}_2$ and $\mathbf{B}_1$. This means that even in a region of space where the total field $\mathbf{E} = \mathbf{E}_1+\mathbf{E}_2$ or $\mathbf{B} = \mathbf{B}_1+\mathbf{B}_2$ is zero, there still can be an exchange of electromagnetic and mechanical momentum.

The apparent simplicity of Eq. (4-4.3) for the gravitational interactions is misleading. According to this equation, there is no exchange of momentum between a mass and the gravitational field. But what happens if one of two distant interacting masses moves to a new position, and the second mass does not yet "know" that the first mass has moved? The second mass continues to experience the same force as before, in violation of the action and reaction law and in violation of Eq. (4-4.3)! And since, according to Eq. (4-4.3), there is no momentum exchange with the field, the law of conservation of momentum does not hold either. Something is definitely wrong here. We shall discuss this problem in detail in Chapter 5. As we shall see, in order to reconcile Newton's gravitational theory with the law of conservation of momentum and with finite propagation speed of gravitation, certain modifications of Newton's gravitational theory are needed. These modifications will result in some remarkable new equations for gravitational fields and forces.

▼

Example 4-5.1 Consider the system of the two current-carrying parallel plates and the capacitor described in Example 3-3.2. Assuming that the only upward force acting on the capacitor is the electrokinetic force due to the electrokinetic field of the current-carrying plates, and neglecting all edge effects, examine the exchange of momentum in the system.

The magnetic field in the space between the current-carrying plates is

$$\mathbf{H} = \frac{I_0}{w}\mathbf{k}. \tag{4-5.2}$$

The electric field in the capacitor is

$$\mathbf{E} = -\frac{q}{\varepsilon_0 A}\mathbf{i}, \tag{4-5.3}$$

where A is the surface area of the capacitor plates. The initial electromagnetic momentum in the space between the plates of the capacitor is, by Eq. (4-5.1),

$$\mathbf{G} = \varepsilon_0\mu_0\frac{q}{\varepsilon_0 A}\frac{I_0}{w}dA\mathbf{j} = \mu_0\frac{qdI_0}{w}\mathbf{j}. \tag{4-5.4}$$

The final electromagnetic momentum is zero (because, when there is no current, there is no magnetic field). Therefore the electromagnetic momentum lost by the system is the momentum given by Eq. (4-5.4), which, by Example 3-3.2, is the same as the mechanical momentum gained by the capacitor [observe that Eq. (3-3.3) is for only one plate of the capacitor, so that it represents 1/2 of the total momentum gained by the capacitor].

▲

References and Remarks for Chapter 4

1. See, for example, A. P. Arya, *Introduction to Classical Mechanics* (Allyn and Bacon, Boston, 1990) p. 5.
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3. J. J. G. Scanio, "Conservation of momentum in electrodynamics — an example," Am. J. Phys. **43**, 258-260 (1975).
4. A. M. Portis, *Electromagnetic Fields, Sources and Media* (John Wiley & Sons, New York, 1978) pp. 390-392.
5. See O. D. Jefimenko, "A relativistic paradox seemingly violating conservation of momentum in electromagnetic systems," Eur. J. Phys. **20**, 39-44 (1999). In the first part of this article a system of two moving charges satisfying the action-reaction law is discussed. In the second part it is shown that the sum of mechanical and electromagnetic momentum is conserved even if the action-reaction law does not hold.
6. See, for example, Ref. 1, pp. 395-397.
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8. A. S. de Castro, "Electromagnetic momentum for a charged shell," Am. J. Phys. **59**, 180 (1991) and papers cited there.
9. O. D. Jefimenko, "The Trouton-Noble paradox," J. Phys. A: Math. Gen. **32**, 3755-3762 (1999).

5

EXTENDING NEWTON'S THEORY OF GRAVITATION TO TIME-DEPENDENT SYSTEMS

In this chapter we shall generalize Newton's theory of gravitation to time-dependent systems. As a result, we shall obtain time-dependent gravitational equations that are very similar to the basic electromagnetic equations. We shall find that some of the consequences of these equations are very close to certain consequences derived from the general relativity theory.

5-1. Generalization of Newton's Gravitational Theory

Newton's theory of gravitation is based on his gravitational law

$$\mathbf{F}/m = -G\frac{M}{r^2}\mathbf{r}_u, \qquad (5\text{-}1.1)$$

where **F** is the force exerted on the point mass m by the point mass M, G is the constant of gravitation, r is the distance between the two masses, and $\mathbf{r}_u$ is the unit vector directed from M to m. This law is essentially limited to time-independent systems. Two of its major shortcomings are its representation of gravitational interactions as an action-at-a-distance effect (which is in violation of the principle of causality) and, as we saw in Chapter 4, its inability to satisfy the conservation of momentum law for time-dependent interactions. It is clear therefore that Newton's gravitational theory is not entirely correct even within the framework of classical physics. In this chapter we shall generalize Newton's theory to time-dependent systems. We shall start this generalization by reconciling Newton's law of gravitation with the law of conservation of momentum. And we shall complete the generalization by making Newton's gravitational theory compatible with the principle of causality.

We shall base our derivations not on Eq. (5-1.1) directly, but on the two equations that formulate Newton's theory as a force-field theory in terms of the gravitational field vector **g**. These equations are

$$\nabla \times \mathbf{g} = 0, \tag{5-1.2}$$

and

$$\nabla \cdot \mathbf{g} = -4\pi G \rho . \tag{5-1.3}$$

The gravitational field vector **g** is defined as

$$\mathbf{g} = \mathbf{F}/m, \tag{5-1.4}$$

where **F** is the force exerted by the gravitational field on a test mass m, which is at rest relative to an inertial reference frame ("laboratory"). In Eq. (5-1.3), ρ is the mass density defined as

$$\rho = dm/dv, \tag{5-1.5}$$

where dm is a mass element contained in the volume element dv.

Let us consider, as in Section 4-4, two mass distributions ρ_1 and ρ_2 producing, respectively, gravitational fields $\mathbf{g}_1$ and $\mathbf{g}_2$. The force exerted by ρ_2 upon ρ_1 is then $\int \rho_1 \mathbf{g}_2 \, dv$, and the force exerted by ρ_1 upon ρ_2 is $\int \rho_2 \mathbf{g}_1 \, dv$. If we apply to the fields $\mathbf{g}_1$ and $\mathbf{g}_2$ vector identity (V-22),

$$\oint(\mathbf{A}\cdot\mathbf{B})d\mathbf{S} - \oint\mathbf{B}(\mathbf{A}\cdot d\mathbf{S}) - \oint\mathbf{A}(\mathbf{B}\cdot d\mathbf{S}) = \int[\mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) - \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})]dv, \tag{V-22}$$

we obtain, as in Section 4-4,

$$\int\rho_1\mathbf{g}_2 dv = -\int\rho_2\mathbf{g}_1 dv. \tag{5-1.6}$$

Thus, according to Newton's theory of gravitation, the forces of action and reaction are always equal. But, as we know from Section 4-5, the law of action and reaction cannot possibly hold for time-dependent gravitational interactions (unless gravitation propagates instantaneously, which we cannot accept). Therefore our derivation of Eq. (5-1.6) shows that at least one of the two basic field laws of Newton's theory of gravitation, Eqs. (5-1.2) and (5-1.3), is, in general, incompatible with the law of conservation of momentum.

Clearly, there are only three possibilities for modifying the Newtonian theory so that it does not conflict with the law of conservation of momentum: (1) to make $\nabla\times\mathbf{g} \neq 0$, (2) to modify $\nabla\cdot\mathbf{g}$, or (3) to modify both $\nabla\times\mathbf{g}$ and $\nabla\cdot\mathbf{g}$. As we shall see, we can make the theory compatible with the law of conservation of momentum by making $\nabla\times\mathbf{g} \neq 0$.[1]

Taking into account that $\nabla\times\mathbf{g}$ must reduce to $\nabla\times\mathbf{g} = 0$ for time-independent systems, we shall assume that in general

$$\nabla\times\mathbf{g} = -\frac{\partial\mathbf{K}}{\partial t}, \tag{5-1.7}$$

where $\mathbf{K}$ is some function of space and time (its physical significance will be discussed in the next section). If we repeat the derivation used for obtaining Eq. (5-1.6) and use Eq. (5-1.7) instead of Eq. (5-1.2), we find that the derivation yields

$$\begin{aligned}\int \rho_1 \mathbf{g}_2 dv - \frac{1}{4\pi G}\int \mathbf{g}_2 \times \frac{\partial \mathbf{K}_1}{\partial t} dv \\ = - \int \rho_2 \mathbf{g}_1 dv + \frac{1}{4\pi G}\int \mathbf{g}_1 \times \frac{\partial \mathbf{K}_2}{\partial t} dv,\end{aligned} \tag{5-1.8}$$

where $\mathbf{K}_1$ is associated with the field $\mathbf{g}_1$, and $\mathbf{K}_2$ with the field $\mathbf{g}_2$.

We can interpret the two integrals containing the time derivatives as the rates of change of the field momentum, and we can interpret Eq. (5-1.8) as the statement of the conservation of momentum for gravitational interactions. According to this interpretation, the gravitational field is a repository of momentum given by

$$\mathbf{G} = \frac{1}{4\pi G}\int \mathbf{K} \times \mathbf{g} dv, \tag{5-1.9}$$

and the field can exchange momentum with the bodies located in it [although we do not yet have enough information to determine whether the sign in front of the integral of Eq. (5-1.9) should be + or −, we shall presently see that Eq. (5-1.9) is correct as written]. Thus, if we amend Newton's theory by accepting Eq. (5-1.7) as a basic law, the theory becomes fully compatible with the law of conservation of momentum.

5-2. Cogravitational Field K

The function **K** which we introduced in the preceding section constitutes a vector field. What is the physical significance of **K**? As one can see by comparing Eq. (5-1.7) with Eq. (1-1.3), **K** is associated with the gravitational field **g** just like the magnetic field **B** is associated with the electric field **E**. Let us call **K** the *cogravitational*, or *Heaviside's*, field.[2] By analogy with electromagnetism, we could then assume that the cogravitational field represents a force field acting on moving masses. In fact, if Newton's gravitational force, Eq. (5-1.1), obeys the force transformation equations of special relativity, then the existence of the cogravitational field is demanded by these equations.[3,4] The force exerted by **K** on a mass moving with velocity **u** is then

$$\mathbf{F} = m(\mathbf{u} \times \mathbf{K}). \qquad (5\text{-}2.1)$$

This equation can be considered to be the definition of **K** [the order of vectors in the cross product reflects the fact that the gravitational force given by Eq. (5-1.1) is always attractive].[5]

As it is known from Helmholtz's theorem of vector analysis,[6] a vector field requires for its complete specification a definition of its divergence and its curl. We shall assume that

$$\nabla \cdot \mathbf{K} = 0. \qquad (5\text{-}2.2)$$

The curl of **K** will be defined in the next section.

5-3. Gravitational Wave Equation

The similarity of Eqs. (5-1.7) and (5-1.3) with Maxwell's Eqs. (1-1.3) and (1-1.1) suggests that many electromagnetic phenomena should have their gravitational counterparts. In particular, it appears very probable that there should exist gravitational waves similar to the electromagnetic waves. To see if it is really so, let us take the curl of Eq. (5-1.7). We have

$$\nabla \times \nabla \times \mathbf{g} = -\frac{\partial}{\partial t} \nabla \times \mathbf{K}. \qquad (5\text{-}3.1)$$

We can transform this equation into a wave equation by assuming that

$$\nabla \times \mathbf{K} = -\frac{4\pi G}{c^2}\mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{g}}{\partial t}, \tag{5-3.2}$$

where **J** is some function of space and time. We then obtain

$$\nabla \times \nabla \times \mathbf{g} + \frac{1}{c^2}\frac{\partial^2 \mathbf{g}}{\partial t^2} = \frac{4\pi G}{c^2}\frac{\partial \mathbf{J}}{\partial t}, \tag{5-3.3}$$

which is an equation for a **g** wave propagating in space with velocity c.[7]

As one can see from Eqs. (5-3.2) and (5-1.7), the field vector **K** satisfies a similar equation

$$\nabla \times \nabla \times \mathbf{K} + \frac{1}{c^2}\frac{\partial^2 \mathbf{K}}{\partial t^2} = -\frac{4\pi G}{c^2}\nabla \times \mathbf{J}, \tag{5-3.4}$$

which is an equation for a **K** wave propagating in space with velocity c.

5-4. Mass Current J

What is the physical significance of **J**? If we determine the divergence of Eq. (5-3.2), taking into account that the divergence of a curl is zero, we obtain

$$0 = -\frac{4\pi G}{c^2}\nabla \cdot \mathbf{J} + \frac{1}{c^2}\frac{\partial(\nabla \cdot \mathbf{g})}{\partial t}, \tag{5-4.1}$$

which, with Eq. (5-1.3), becomes

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}. \tag{5-4.2}$$

Equation (5-4.2) is a "continuity" equation stating that **J** is a mass current (and therefore also an energy current) coming out of a mass accumulation whenever this accumulation diminishes with time; that is, Eq. (5-4.2) is a statement of the conservation of mass for time-dependent gravitational systems.

By analogy with the electric convection current,[8] the mass current created by a beam of mass particles of density ρ moving with a velocity $\mathbf{v}$ is

$$\mathbf{J} = \rho\mathbf{v}. \tag{5-4.3}$$

5-5. Causal Gravitational Field Equations

We are now ready to complete our basic generalization of Newton's theory of gravitation to time-dependent systems.

The solution of Eq. (5-3.3) is[9,10,11]

$$\mathbf{g} = -\frac{1}{4\pi}\int \frac{\left[\nabla'(\nabla'\cdot\mathbf{g}) - \frac{4\pi G}{c^2}\frac{\partial \mathbf{J}}{\partial t}\right]}{r} dv'. \tag{5-5.1}$$

[As it was explained in Chapter 1, we use square brackets as the retardation symbol to indicate that the quantities between the brackets are to be evaluated for $t' = t - r/c$, where t is the time for which $\mathbf{g}$ is evaluated, r is the distance between the field point (point for which $\mathbf{g}$ is evaluated) and the source point (volume element dv'), c is the propagation velocity of gravitation, and ∇' is the operator del operating on the source-point coordinates.] Substituting $\nabla\cdot\mathbf{g}$ from Eq. (5-1.3), we can write Eq. (5-5.1) as

$$\mathbf{g} = G\int \frac{\left[\nabla'\rho + \frac{1}{c^2}\frac{\partial \mathbf{J}}{\partial t}\right]}{r} dv'. \tag{5-5.2}$$

The solution of Eq. (5-3.4) is, similarly,[9,10,11]

$$\mathbf{K} = -\frac{G}{c^2}\int\frac{[\nabla'\times\mathbf{J}]}{r}dv', \tag{5-5.3}$$

where we have taken into account that, according to Eq. (5-2.2), $\nabla\cdot\mathbf{K} = 0$.

These two equations can be transformed into equations not containing spatial derivatives.[11,12] We then obtain

$$\mathbf{g} = -G\int\left\{\frac{[\rho]}{r^2}+\frac{1}{rc}\frac{\partial[\rho]}{\partial t}\right\}\mathbf{r}_u dv' + \frac{G}{c^2}\int\frac{1}{r}\left[\frac{\partial\mathbf{J}}{\partial t}\right]dv', \tag{5-5.4}$$

and

$$\mathbf{K} = -\frac{G}{c^2}\int\left\{\frac{[\mathbf{J}]}{r^2}+\frac{1}{rc}\frac{\partial[\mathbf{J}]}{\partial t}\right\}\times\mathbf{r}_u dv'. \tag{5-5.5}$$

Equations (5-5.4) and (5-5.5) are the fundamental causal equations of the Newton's gravitational theory generalized to time-dependent systems. Although we have "derived" them with the help of several assumptions and definitions, they should preferably be considered as postulates, and their validity should be judged not by the method by which they have been obtained, but by the agreement (or disagreement) with experimental data and with other laws and theories of proven validity. It is important to emphasize that if Eqs. (5-5.4) and (5-5.5) are regarded as postulates, then Eqs. (5-1.2), (5-1.3), (5-1.7), (5-2.2), (5-3.2)-(5-3.4), and (5-4.2) can be derived directly from them. What is more, the reverse derivations involve nothing but the standard vector analysis (see Chapter 1).

Equations (5-5.4) and (5-5.5) make it possible to calculate gravitational and cogravitational fields produced by continuous mass distributions. They can be transformed, however, into equations for fields of moving point masses. For a point mass m moving with velocity $\mathbf{v}$ and acceleration $\mathbf{a}$, the resulting equations are [13]

$$\mathbf{g} = -G\frac{m}{s^3}\left\{\mathbf{R}\left[1-\frac{v^2}{c^2}\right]+\frac{1}{c^2}[\mathbf{r}]\times(\mathbf{R}\times[\mathbf{a}])\right\}, \qquad (5\text{-}5.6)$$

and

$$\mathbf{K} = -G\frac{m}{c^2 s^3}\left\{[\mathbf{v}]\left[1-\frac{v^2}{c^2}\right]-\frac{1}{c[r]}[\mathbf{r}]\times(\mathbf{R}\times[\mathbf{a}])\right\}\times[\mathbf{r}], \qquad (5\text{-}5.7)$$

with

$$\mathbf{K} = \frac{[\mathbf{r}]\times\mathbf{g}}{c[r]}, \qquad (5\text{-}5.8)$$

where $[\mathbf{r}]$ is the retarded position vector of the moving point mass given by $t' = t - [r]/c$ and directed from the mass to the point of observation (see Fig. 1.1); $\mathbf{R} = [\mathbf{r} - r\mathbf{v}/c]$ is the "projected" present position vector of the point mass (also directed toward the point of observation); $s = [r - \mathbf{r}\cdot\mathbf{v}/c]$; and where the square brackets denote retarded values.

For a point mass moving without acceleration, Eq. (5-5.6) can be expressed in terms of the present position vector $\mathbf{r}_0$ as [14,15]

$$\mathbf{g} = -G\frac{m(1-v^2/c^2)}{r_0^3[1-(v^2/c^2)\sin^2\theta]^{3/2}}\mathbf{r}_0, \qquad (5\text{-}5.9)$$

where θ is the angle between $\mathbf{v}$ and $\mathbf{r}_0$, and Eq. (5-5.8) can be expressed as

$$\mathbf{K} = \frac{\mathbf{v}\times\mathbf{g}}{c^2}. \qquad (5\text{-}5.10)$$

This essentially completes our basic generalization of Newton's theory of gravitation to time-dependent systems. In the next chapter we shall add many more equations to the equations obtained thus far. But first we shall take a look at our results from a historical point of view and shall compare some of the consequences of our time-dependent equations with certain results of the general relativity theory.

5-6. Historical Background

Our generalization of Newton's gravitational theory is based to a large extent on the assumption that there exists a second gravitational field (which we have named the cogravitational, or Heaviside's, field). Let us note that there are several publications in which it is suggested that a second field may be involved in gravitational interactions. The first such publication was by Oliver Heaviside.[2] Since Heaviside's article contains all the basic elements found in most of the later publications on the subject, and since the article appears to have been generally ignored,[16] the entire article is reproduced in Appendix 8. A short summary of the article is presented below.

Heaviside begins his article by considering the energy of the gravitational field and how this energy could propagate in space. He concludes that there should be a gravitational energy flux represented by a vector, similar to Poynting's vector of electromagnetism, equal to the cross product of the gravitational field vector proper and a second field vector analogous to the magnetic field vector. He finds that, in contrast to the electromagnetic energy flux associated with a moving charge, the direction of the gravitational energy flux associated with a moving mass is opposite to the direction in which the mass is moving. He then finds the curl of the second field [except for the sign and notation, it is the same as our Eq. (5-3.2)], and he sets the divergence of this field equal to zero. Next, he shows that if gravitation is propagated with finite speed, then there should exist gravitational waves similar to electromagnetic waves. He then discusses the energy balance in gravitational systems involving gravitational energy flux, modifies the curl equations of the two fields for the case of a moving ether, and considers Maxwellian stresses and tensions in the gravitational field. In the second part of the article he shows how the two fields are modified when the field-producing particles are in the state of uniform motion [except for notation, his equations are the same as our Eqs. (5-5.9) and (5-5.10)] and discusses how planetary motion may be perturbed by the motion of the Sun.

Most of the recent publications that consider the second gravitational field, while treating in greater detail certain aspects of gravitation, do not present substantially new ideas on the second field and do not develop the basic theory much beyond Heaviside's work.[17,18,19,20] An exception is the 786-page book by Popescu,[21] where the author presents a new gravitational theory as an alternative to the general relativity theory.

Why was Heaviside's pioneering work on gravitation largely ignored when it was first published, and why is it practically unknown even today? One possible reason could be the very fact that Heaviside's gravitational theory was based on equations practically identical to Maxwell's curl equations for electric and magnetic fields. These equations were universally believed to represent the phenomenon of electromagnetic induction. But it is almost impossible to imagine that there could be anything similar in the domain of gravitation: there are no conductors of gravitation, there is no gravitational analogue of "electromotive force," there is nothing in gravitation even remotely similar to electromagnetic induction as it is commonly understood by scientists and engineers.[22]

However, as we know from Chapter 1, Maxwell's equations are not causal equations, and they do not really represent any induction phenomena at all. We know from Chapter 1 that the true cause of electromagnetic induction is not the time-variable electric or magnetic fields, but the electrokinetic fields acting on charges in conductors. And we know that the phenomenon of "electromagnetic induction" is just another manifestation of electric forces. Once these simple facts are understood, all initial distrust of Heaviside's idea disappears.

Another reason why Heaviside's work did not attract the attention that it deserved was probably the fact that he did not fully develop his theory of gravitation,[23] and his single article on gravitation was eventually completely eclipsed by Einstein's brilliant and spectacularly successful general relativity theory.

It is noteworthy, however, that Newton's gravitational theory generalized to time-dependent systems (Heaviside's theory was essentially just that) yields several results which heretofore were believed to be the exclusive consequence of the general relativity theory. We shall discuss this very important circumstance in the next section.

5-7. Time-Dependent Gravitation and General Relativity

The most important time-dependent gravitational equations are Eqs. (5-5.4) and (5-5.5). Because of the retarded values involved in these equations, the equations are much more than mere correlations between various physical quantities pertaining to gravitational interactions: they reveal the causative sources of gravitational and cogravitational fields and show the fundamental causal relations governing gravitational interactions. Let us examine some of the consequences of these equations.

First let us see what gravitational effects are disclosed by Eq. (5-5.4). The first term in the first integral of this equation represents the ordinary Newtonian gravitation of the mass distribution ρ corrected for the final propagation speed of gravitational fields. The second term in the first integral takes into account the rate at which a time-dependent mass distribution changes. The second integral represents two different things: (1)

the gravitational effect of a mass current whose density changes with time, and (2) the gravitational effect of a mass current which propagates in space. For a point mass moving with constant velocity, Eq. (5-5.4) reduces to Eq. (5-5.9) so that, in spite of the finite propagation speed, the gravitational field is directed to the field-producing point mass at its present position and therefore in this case there is no gravitational aberration.

Consider now Eq. (5-5.5). The first integral of this equation indicates that a mass current (or a moving mass particle) creates a cogravitational field whose role in gravitational interactions is analogous to that of the magnetic field in electromagnetic interactions. The second integral represents the effect of the rate of change of the mass current on the cogravitational field.

The most important consequence of Eqs. (5-5.4) and (5-5.5) is that these equations give us a new insight into the nature of gravitational interactions between moving bodies. The presence of the five different terms in the integrals of Eqs. (5-5.4) and (5-5.5) means that gravitational interactions between two moving bodies involve, in general, at least *five forces*. The dominant force is the ordinary Newtonian attraction force corrected for retardation. The remaining (usually very much weaker) forces are: a force that basically depends on the time (convective) derivative of the mass of the force-producing body (and hence on the velocity of this body), a force that basically depends on the acceleration of the force-producing body, a force that basically depends on the translational and rotational velocities of the force-experiencing and of the force-producing body, and a force that basically depends on the translational and rotational velocities of the force-experiencing body and on the translational and rotational accelerations of the force-producing body. In general, the five forces are in different directions. Thus, according to the Newton's gravitational theory generalized to time-dependent systems, gravitational interaction between two moving bodies is an intricate juxtaposition of several very different forces.[24] Taking into account that according to the original Newton's theory only one single force is responsible for gravitational interactions, the significance of this result is enormous.

At this time there are no obvious experimental data with which we could compare gravitational effects (except for the ordinary Newtonian attraction) disclosed by Eqs. (5-5.4) and (5-5.5). Therefore, in order to judge the credibility of these equations, we need to compare their consequences with those obtained on the basis of other theories. Although several theories of gravitation are presently in existence, the only generally accepted gravitational theory, other than that of Newton, is now the Einstein's general relativity theory. It is imperative therefore that we compare at least some gravitational effects revealed by Eqs. (5-5.4) and (5-5.5) with similar effects deduced from the general relativity theory, if such similar effects can be identified.

Several such effects are indeed known. Two of them are pictured in Einstein's formulation of "Mach's principle." On the basis of the general relativity theory, Einstein gave a quantitative formulation of Mach's principle in the form of an equation closely resembling the electromagnetic equation representing the force acting on a moving charged particle in the presence of an electric and a magnetic field.[25,26] Einstein's equation is

$$\frac{d}{dt}[(1 + \bar{\sigma})\mathbf{u}] = c^2\nabla\bar{\sigma} + \frac{\partial \mathbf{A}}{\partial t} - [\mathbf{u} \times (\nabla \times \mathbf{A})], \qquad (5\text{-}7.1)$$

where

$$\bar{\sigma} = \frac{G}{c^2}\int \frac{\rho}{r}\,dv', \qquad (5\text{-}7.2)$$

and

$$\mathbf{A} = \frac{4G}{c^2}\int \frac{\rho\mathbf{v}}{r}\,dv', \qquad (5\text{-}7.3)$$

and where **u** and **v** are the velocities of the force-experiencing and force-producing mass, respectively.

The right side of Eq. (5-7.1) is very similar to the well-known "Lorentz's force" equation of electromagnetic theory. In the language of electromagnetic theory, the first term on the right is the gradient of the electric scalar potential, the second term is the rate of change of the magnetic vector potential (these two terms together represent the time-dependent electric field), and the third term is the force exerted by a magnetic field (represented as the curl of the vector potential) on a moving charge of magnitude "one." In the language of gravitational theory, the first term represents the usual Newtonian gravitational attraction, the second term represents the gravitational "drag" due to a changing mass current or due to an accelerating mass, the third term represents the magnetic-like force acting on a moving mass of magnitude "one."

Clearly, all three of these effects are also represented by our Eqs. (5-5.4) and (5-5.5). The gravitational drag due to accelerated bodies is represented by the last term of Eq. (5-5.4), and the magnetic-like force is represented by Eq. (5-5.5) [with Eq. (5-2.1)].[27] Observe that the gravitational drag is the gravitational counterpart of the electrokinetic force. Like the electrokinetic force, it is always parallel to the current that produces it, but in contrast to the electrokinetic force it is in the direction of the current when the current increases, and opposite to the direction of the current when the current decreases. Therefore an accelerating mass drags the neighboring masses with itself, and a decelerating mass pushes the neighboring masses back. Of course, because of c^2 in the denominator of the last term of Eq. (5-5.4), the gravitational drag is a very weak effect except when there is a very rapid change of the velocity of the drag-producing body. It is interesting to note that Einstein derived an expression for the gravitational drag four years before he finally formulated his general relativity theory.[28,29]

Before discussing Eq. (5-5.5), let us point out that this equation represents interactions which are of the second order in v/c. Therefore these interactions are much weaker than the interactions represented by the first integral of Eq. (5-5.4). This explains why Newton's theory based only on Eq. (5-1.1) [the time-independent form of Eq. (5-5.4)] has worked so well.

According to the general relativity theory, magnetic-like gravitational forces definitely exist. Most prominent among them are forces associated with rotating bodies. These forces are sometimes referred to as "forces analogous to magnetic,"[30,31] sometimes as "dragging of the inertial frame,"[30] sometimes as "Coriolis forces,"[32,33,34] and sometimes as "gravitomagnetic, or gravimagnetic, forces."[35,36,37] The best known theoretical effect due to these forces is the Thirring-Lense effect representing an influence of the rotation of a central astronomical body on the motion of planets and satellites.[38,39] All these effects are clearly indicated by our Eq. (5-5.5).

It would be desirable to determine to what extent Eqs. (5-5.4) and (5-5.5) are capable of reproducing some representative quantitative results of the general relativity theory, or how these equations may affect celestial mechanics. We shall consider this question in the next chapter. However, a detailed comparison of the time-dependent Newton's gravitational theory with the general relativity theory is outside the purpose of this book or of this chapter. The purpose of this chapter is to extend Newton's gravitational theory to time-dependent systems. It would be irrational not to endeavor to develop Newton's theory to its logical conclusion, regardless what this conclusion and the corresponding quantitative results may be. As it so happens, the theoretical results that we have obtained thus far are intriguing, thought-provoking and are based on a very firm ground. And as it is shown in Chapter 6, our generalization of Newton's theory is fully capable of yielding very useful practical results as well.

Let us now summarize the main results of this chapter: Newton's gravitational law, Eq. (5-1.1), is a special case of the more general gravitational laws for time-dependent systems given by Eqs. (5-5.4) and (5-5.5). Unlike Eq. (5-1.1), these laws are fully compatible with the law of conservation of momentum, satisfy the principle of relativity, satisfy the principle of causality, and reveal the existence of gravitational effects whose presence was previously assumed to be the exclusive results of the general relativity theory. These laws give us an unexpected new insight into the nature of gravitational interactions. The validity of these laws cannot be questioned without questioning the most fundamental principles of physics.

References and Remarks for Chapter 5

1. If one assumes that gravitational energy is itself a source of gravitation, then $\nabla \cdot \mathbf{g}$ becomes $\nabla \cdot \mathbf{g} = -4\pi G\rho + g^2/2c^2$ (see Chapter 8). This would create two additional terms in Eq. (5-1.6). However, the contribution of these terms to the momentum would be extremely small at best, and would not be explicitly time dependent.
2. The existence of a second gravitational field, similar to the magnetic field, was first suggested by Oliver Heaviside in his two-part article "A Gravitational and Electromagnetic Analogy," The Electrician, **31**, 281-282 and 359 (1893). This article is reproduced in Appendix 8.
3. A force field compatible with the special relativity theory must have two subfields, one of which acts on moving bodies only. See O. D. Jefimenko, "Is a magnetic field due to an electric current a

relativistic effect?" Eur. J. Phys. **17**, 180-182 (1996).
4. A short explanation of the nature of the magnetic-like gravitational force is given in P. Lorrain and D. R. Corson, *Electromagnetic Fields and Waves*, 2nd ed., (W. H. Freeman & Co, San Francisco, 1970) p. 251.
5. For the special case of a moving rod this equation is derived in D. Bedford and P. Krumm, "On relativistic gravitation," Am. J. Phys. **53**, 889-890 (1985). See also D. Bedford and P. Krumm, "The gravitational Poynting vector and energy transfer," Am. J. Phys. **55**, 362-363 (1987).
6. See O. D. Jefimenko, *Electricity and Magnetism*, 2nd ed., (Electret Scientific, Star City, 1989) p. 42.
7. Ref. 6, pp. 514-515.
8. Ref. 6, pp. 327-328.
9. Ref. 6, pp. 47 and 515.
10. O. D. Jefimenko, *Electromagnetic Retardation and Theory of Relativity* (Electret Scientific, Star City, 1997) pp. 1-14
11. See Appendix 2. Do not confuse the vector **K** used in Appendix 2 with the cogravitational field vector **K**.
12. Ref. 10, pp. 15-19; Ref. 6, p. 516.
13. Ref. 10, pp. 272-273.
14. This equation was first derived by Oliver Heaviside (Ref. 2). A novel derivation is given in O. D. Jefimenko, "Gravitational field of a point mass moving with uniform linear or circular velocity," Galilean Electrodynamics, **5**, 25-33 (1994). For a graphical representation of the field depicted by Eq. (5-5.9) see Appendix 6.
15. We are using the same letter c for both the velocity of light and the velocity of propagation of gravitation. However, the two velocities are not necessarily the same. See Section 7-3.
16. Léon Brillouin, *Relativity Reexamined* (Academic Press, New York, 1970) pp. 103-104 cites a reprint of the article in Heaviside's *Electromagnetic Theory* (Dover, New York, 1950) pp. 115-118.

17. John Carstoiu, "Les deux champs de gravitation et propagation des ondes gravifiques," Compt. Rend. **268**, series A, 201-204 (1969); "Nouvelles remarques sur les deux champs de gravitation et propagation des ondes gravifiques," Compt. Rend. **268**, series A, 261-264 (1969).

18. W. B. Campbell and T. A. Morgan, "Maxwell form of the linear theory of gravitation," Am. J. Phys. **44**, 356-365 (1976).

19. A. Singh, *Unified Field Theory Based on a New Theory of Gravitation and the Modified Theory of Electromagnetics* (published by Amar Singh, Physics Department, Virginia Polytechnical Institute and State University, Blacksburg, Virginia, 1979).

20. Ref. 5.

21. Ioan N. Popescu, *Gravitation; Pleading for a New Unified Theory of Motion and Fields* (Editrice Nagard, Roma - Italia, 1988).

22. It is interesting to note that Einstein, four years before he published his general relativity theory, published an article on the possibility of a gravitational analogue of electromagnetic induction. Also this article is practically unknown, possibly because it was published in a rather inappropriate journal whose title (translated from the German) was: "Quarterly Journal for Forensic Medicine and Public Sanitation;" see Ref. 28.

23. Heaviside intended to complete his gravitational theory in Volume 4 of *Electromagnetic Theory*. The almost finished manuscript for this Volume was destroyed by thieves and vandals in 1925. See E. Laithwaite, "Oliver Heaviside — establishment shaker," Electrical Review, **211**, 44-45 (1985) and H. J. Josephs, "The Heaviside papers found at Paigton in 1957," Proc. IEE **106C**, Monograph No. 319, pp. 70-76 (1959).

24. Some of these forces are illustrated graphically in Appendix 7.

25. A. Einstein, *The Meaning of Relativity*, 5th ed., (Princeton U. P., Princeton, NJ, 1956) p. 100. Mach's principle played a crucial

role in Einstein's search for a new theory of gravitation and in his formulation of the general relativity theory.

26. A comprehensive discussion of Mach's principle is given in J. David Nightingale, "Special physical consequences of Mach's principle," Am. J. Phys. **45**, 376-379 (1977).

27. The last term in Eq. (5-5.4) can be expressed as the time derivative of the retarded cogravitational vector potential [see p. 108, Eq. (6-2.17)]. However, Einstein's vector potential [Eq. (5-7.3)] is four times as large as the cogravitational vector potential. The reason for this discrepancy is the counter-intuitive expression $\mathbf{J} = 4\rho\mathbf{v}$ for the mass-current density resulting from the general relativity theory (see Ref. 31). As a result, all magnetic-like quantities according to general relativity are four times larger than according to the generalized Newton's theory.

28. A. Einstein, "Gibt es eine Gravitationswirkung, die der elektrodynamischen Induktionswirkung analog ist?" Vierteljahrschrift für gerichtliche Medizin und öffentliches Sanitätswesen, **44**, 1-4 (1912). In this article Einstein computes gravitational drag produced by an accelerating spherical shell on a mass particle at the center of the shell (where there is no gravitational force according to Newton's theory). Later he also points out that a rotation of the shell causes a Coriolis-like force on a pendulum inside the shell (letter to Ernst Mach of June 23, 1913, in Ref. 30, pp. 541-545).

29. We can also compare Eq. (5-5.6) with a similar equation obtained on the basis of the general relativity theory in J. Weber, *General Relativity and Gravitational Waves* (Interscience Publishers, Inc., New York, 1961) p. 160. The relativistic equation has three terms, none of which is retarded: one in the radial direction, one in the direction of the velocity of the field-producing body, and one in the direction of the acceleration of the field-producing body. Each of these terms has a counterpart in Eq. (5-5.6). The relativistic equation is reproduced in Robert L. Forward,

"Guidelines to antigravity," Am. J. Phys. **31**, 166-170 (1963).
30. C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman & Company, San Francisco, 1973) pp. 547-548, 1117-1120.
31. F. G. Harris, "Analogy between general relativity and electromagnetism for slowly moving particles in weak gravitational fields," Am. J. Phys. **59**, 421-425 (1991).
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6

GRAVITATIONAL EQUATIONS

Basic gravitational equations are very similar to basic electromagnetic equations. This similarity makes it possible to convert many electromagnetic equations to gravitational equations. In this chapter we shall use this possibility for obtaining a variety of very useful gravitational equations, many of which were either unknown or were ignored in the past.

6-1. Analogy Between Electromagnetism and Gravitation

Let us summarize basic gravitational equations which we discussed in Chapter 5. We can arrange them in three categories:

(1) *Basic definition equations for gravitational fields*

Gravitational field **g**

$$\mathbf{g} = \mathbf{F}/m, \tag{5-1.4}$$

Cogravitational field **K**

$$\mathbf{F} = m(\mathbf{u} \times \mathbf{K}), \tag{5-2.1}$$

Mass density ρ

$$\rho = dm/dv, \tag{5-1.5}$$

Mass current density **J**

$$\mathbf{J} = \rho\mathbf{v}. \tag{5-4.3}$$

(2) *Basic differential equations for gravitational fields*

$$\nabla \cdot \mathbf{g} = -4\pi G\rho, \tag{5-1.3}$$

$$\nabla \cdot \mathbf{K} = 0, \tag{5-2.2}$$

$$\nabla \times \mathbf{g} = -\frac{\partial \mathbf{K}}{\partial t}, \tag{5-1.7}$$

$$\nabla \times \mathbf{K} = -\frac{4\pi G}{c^2}\mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{g}}{\partial t}. \tag{5-3.2}$$

(3) *Basic causal equations for gravitational fields*

$$\mathbf{g} = -G\int\left\{\frac{[\rho]}{r^2} + \frac{1}{rc}\frac{\partial[\rho]}{\partial t}\right\}\mathbf{r}_u dv' + \frac{G}{c^2}\int\frac{1}{r}\left[\frac{\partial \mathbf{J}}{\partial t}\right]dv', \tag{5-5.4}$$

$$\mathbf{K} = -\frac{G}{c^2}\int\left\{\frac{[\mathbf{J}]}{r^2} + \frac{1}{rc}\frac{\partial[\mathbf{J}]}{\partial t}\right\}\times \mathbf{r}_u dv'. \tag{5-5.5}$$

Let us now list basic electromagnetic equations for fields in a vacuum. Arranging them in categories similar to those used for gravitational equations, we have:

(1) *Basic definitions*

Electric field **E**

$$\mathbf{E} = \mathbf{F}/q, \tag{6-1.1}$$

Magnetic flux density field **B**

$$\mathbf{F} = q(\mathbf{u} \times \mathbf{B}), \qquad (6\text{-}1.2)$$

Electric charge density ρ

$$\rho = dq/dv, \qquad (6\text{-}1.3)$$

Electric convection current density **J**

$$\mathbf{J} = \rho \mathbf{v}. \qquad (6\text{-}1.4)$$

(2) *Maxwell's equations for electromagnetic fields in a vacuum*[1]

$$\nabla \cdot \mathbf{E} = \rho/\varepsilon_0, \qquad (1\text{-}1.1\text{m})$$

$$\nabla \cdot \mathbf{B} = 0, \qquad (1\text{-}1.2)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad (1\text{-}1.3)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}. \qquad (1\text{-}1.4\text{m})$$

(3) *Basic causal equations for electromagnetic fields*

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\int\left\{\frac{[\rho]}{r^2} + \frac{1}{rc}\frac{\partial[\rho]}{\partial t}\right\}\mathbf{r}_u\, dv' - \frac{1}{4\pi\varepsilon_0 c^2}\int\frac{1}{r}\left[\frac{\partial \mathbf{J}}{\partial t}\right]dv', \qquad (1\text{-}4.1)$$

$$\mathbf{B} = \frac{\mu_0}{4\pi}\int\left\{\frac{[\mathbf{J}]}{r^2} + \frac{1}{rc}\frac{\partial[\mathbf{J}]}{\partial t}\right\}\times \mathbf{r}_u\, dv'. \qquad (1\text{-}4.2\text{m})$$

If we compare the gravitational equations listed above with the electromagnetic equations, we find that to each gravitational equation there corresponds an electromagnetic equation. The corresponding equations are identical except for the symbols and constants occurring in them. The relations between the corresponding symbols and constants are shown in Table 6-1.

Table 6-1

Corresponding Electromagnetic and Gravitational Symbols and Constants

Electric	Gravitational
q (charge)	m (mass)
ρ (volume charge density)	ρ (volume mass density)
σ (surface charge density)	σ (surface mass density)
λ (line charge density)	λ (line mass density)
$\mathbf{J}$ (convection current density)	$\mathbf{J}$ (mass current density)
$\mathbf{E}$ (electric field)	$\mathbf{g}$ (gravitational field)
$\mathbf{B}$ (magnetic field)	$\mathbf{K}$ (cogravitational field)
ε_0 (permittivity of space)	$-1/4\pi G$
μ_0 (permeability of space)	$-4\pi G/c^2$
$-1/4\pi\varepsilon_0$ or $-\mu_0 c^2/4\pi$	G (gravitational constant)

It is clear that all equations derivable from the basic electromagnetic equations listed above have their gravitational counterparts, and that gravitational equations can be obtained from the corresponding electromagnetic equations by simply replacing the electromagnetic symbols and constants by the corresponding gravitational symbols and constants in accordance with Table 6-1.

It is important to keep in mind, however, that only electromagnetic equations for fields in a vacuum have their gravitational counterparts, and only the electromagnetic symbols listed in Table 6-1 can be directly replaced by the corresponding gravitational symbols. In all other cases the following conversion procedure should be used:

(1) If an electromagnetic equation is for fields in the presence of material media, reduce the equation to fields in a vacuum.

(2) If the equations contain field vectors **D** or **H**, replace them by **E** or **B**, using the relations $\mathbf{D} = \varepsilon_0\mathbf{E}$ and $\mathbf{B} = \mu_0\mathbf{H}$.

(3) Use Table 6-1 to replace electromagnetic symbols by the corresponding gravitational symbols.

6-2. Gravitational Equations

Listed below are gravitational equations that have been obtained by converting electromagnetic equations in accordance with the procedure explained in Section 6-1. The electromagnetic equations used for conversion were taken from the author's book *Electricity and Magnetism*.[2] Some readers may want to examine these electromagnetic equations and their derivations. For this purpose each gravitational equation appearing below is provided with the number of the page where the corresponding electromagnetic equation appears in *Electricity and Magnetism* (hereafter abbreviated as EM). The equations are arranged in three categories: equations for calculating fields and potentials, equations for calculating energy and forces, and wave equations. To avoid repetition, the list does not include several equations discussed previously. Some of the equations are only applicable to time-independent fields, others are completely general; consult EM if in doubt.

(1) Equations for calculating gravitational fields and potentials:

Basic gravitational laws in integral notation, EM502

$$\oint \mathbf{g} \cdot d\mathbf{S} = -4\pi G \int \rho\, dv. \tag{6-2.1}$$

$$\oint \mathbf{K} \cdot d\mathbf{S} = 0. \tag{6-2.2}$$

$$\oint \mathbf{g} \cdot d\mathbf{l} = -\frac{\partial}{\partial t} \int \mathbf{K} \cdot d\mathbf{S}. \tag{6-2.3}$$

$$\oint \mathbf{K} \cdot d\mathbf{l} = -\frac{1}{c^2} \int \left(4\pi G \mathbf{J} - \frac{\partial \mathbf{g}}{\partial t} \right) \cdot d\mathbf{S}. \tag{6-2.4}$$

Gravitational field of a point mass, EM96

$$\mathbf{g} = -G\frac{m}{r^2}\mathbf{r}_u. \tag{6-2.5}$$

Gravitational field of a mass distribution, EM93

$$\mathbf{g} = -G \int \frac{\rho}{r^2} \mathbf{r}_u\, dv'. \tag{6-2.6}$$

Gravitational field in terms of mass inhomogeneities (constant interior mass), EM103

$$\mathbf{g} = -G\rho \oint \frac{d\mathbf{S}'}{r}. \tag{6-2.7}$$

Gravitational scalar potential (with respect to ∞), EM120

$$\varphi = -G \int \frac{\rho}{r} dv'. \tag{6-2.8}$$

Gravitational potential of a point mass, EM121

$$\varphi = - G\frac{m}{r}. \tag{6-2.9}$$

Gravitational field in terms of scalar potential, EM111

$$\mathbf{g} = - \nabla\varphi. \tag{6-2.10}$$

Gravitational potential in terms of the field, EM112

$$\varphi_a = \int_a^c \mathbf{g}\cdot d\mathbf{l} + \varphi_c. \tag{6-2.11}$$

Poisson's equation for scalar potential, EM142

$$\nabla^2\varphi = 4\pi G\rho. \tag{6-2.12}$$

Gravitational field in terms of vector potential[3]

$$\mathbf{g} = - 4\pi G\nabla \times \mathbf{A}_g. \tag{6-2.13}$$

Cogravitational field of a moving point mass, EM390

$$\mathbf{K} = - G\frac{m(\mathbf{v} \times \mathbf{r}_u)}{c^2r^2}. \tag{6-2.14}$$

Cogravitational field of a current distribution, EM344

$$\mathbf{K} = - \frac{G}{c^2}\int\frac{\mathbf{J} \times \mathbf{r}_u}{r^2}dv'. \tag{6-2.15}$$

Cogravitational field in terms of current inhomogeneities (constant mass-current density), EM352

$$\mathbf{K} = - \frac{G}{c^2}\oint\frac{\mathbf{J} \times d\mathbf{S}'}{r}. \tag{6-2.16}$$

Cogravitational vector potential, EM364

$$\mathbf{A} = -\frac{G}{c^2}\int\frac{\mathbf{J}}{r}dv'. \tag{6-2.17}$$

Cogravitational field in terms of vector potential, EM363

$$\mathbf{K} = \nabla\times\mathbf{A}. \tag{6-2.18}$$

Poisson's equation for cogravitational vector potential, EM364

$$\nabla^2\mathbf{A} = \frac{4\pi G}{c^2}\mathbf{J}. \tag{6-2.19}$$

Cogravitational field in terms of scalar potential, EM373

$$\mathbf{K} = \frac{4\pi G}{c^2}\nabla\varphi_k. \tag{6-2.20}$$

*Cogravitational dipole moment of filamentary mass current I (**S**′ is right-handed relative to I), EM381*

$$\mathbf{m} = -\frac{4\pi G}{c^2}I\mathbf{S}'. \tag{6-2.21}$$

Cogravitational dipole field, EM381

$$\mathbf{K} = \frac{m}{2\pi r^3}\cos\theta\,\mathbf{r}_u + \frac{m}{4\pi r^3}\sin\theta\,\boldsymbol{\theta}_u. \tag{6-2.22}$$

(2) Equations for calculating gravitational energy and forces:

Gravitational force on a mass distribution, EM208

$$\mathbf{F} = \int\rho\mathbf{g}'dv. \tag{6-2.23}$$

Gravitational force in terms of scalar potential[4]
(single mass of constant density), EM211

$$\mathbf{F} = -\rho\oint \varphi' d\mathbf{S}. \tag{6-2.24}$$

Gravitational force in terms of vector potential[4]
(single mass of constant density)

$$\mathbf{F} = 4\pi G\rho\oint \mathbf{A}_g' \times d\mathbf{S}. \tag{6-2.25}$$

Maxwell's stress integral for the gravitational field, EM215

$$\mathbf{F} = \frac{1}{8\pi G}\oint \mathbf{g}^2 d\mathbf{S} - \frac{1}{4\pi G}\oint \mathbf{g}(\mathbf{g}\cdot d\mathbf{S}). \tag{6-2.26}$$

Cogravitational force on a mass current, EM440

$$\mathbf{F} = \int \mathbf{J}\times\mathbf{K}' dv. \tag{6-2.27}$$

Cogravitational force on a mass-current dipole, EM446

$$\mathbf{F} = -\frac{c^2}{4\pi G}(\mathbf{m}\cdot\nabla)\mathbf{K}'. \tag{6-2.28}$$

Cogravitational torque on a mass-current dipole, EM446

$$\mathbf{T} = -\frac{c^2}{4\pi G}\mathbf{m}\times\mathbf{K}'. \tag{6-2.29}$$

Cogravitational force in terms of vector potential[4]
(constant mass-current density), EM453

$$\mathbf{F} = \oint \mathbf{A}'\cdot\mathbf{J}\,d\mathbf{S}. \tag{6-2.30}$$

Cogravitational force in terms of scalar potential[4] *(constant mass-current density)*, EM453

$$\mathbf{F} = \frac{4\pi G}{c^2}\oint \varphi_k'\mathbf{J}\times d\mathbf{S}. \qquad (6\text{-}2.31)$$

Maxwell's stress integral for the cogravitational field, EM447

$$\mathbf{F} = \frac{c^2}{8\pi G}\oint \mathbf{K}^2 d\mathbf{S} - \frac{c^2}{4\pi G}\oint \mathbf{K}(\mathbf{K}\cdot d\mathbf{S}). \qquad (6\text{-}2.32)$$

Gravitational field energy, EM186

$$U = -\frac{1}{8\pi G}\int \mathbf{g}^2 dv. \qquad (6\text{-}2.33)$$

Gravitational energy in terms of potential, EM190

$$U = \frac{1}{2}\int \varphi\rho\, dv. \qquad (6\text{-}2.34)$$

Energy of a system of point masses, EM192

$$U = -\frac{G}{2}\sum_i\sum_k{}' \frac{m_i m_k}{r_{ik}} + U_s. \qquad (6\text{-}2.35)$$

Energy of a mass distribution in an external field, EM195

$$U' = \int \rho\varphi'\, dv. \qquad (6\text{-}2.36)$$

Energy of a point mass in an external field, EM195

$$U' = m\varphi'. \qquad (6\text{-}2.37)$$

Cogravitational field energy, EM427

$$U = -\frac{c^2}{8\pi G}\int \mathbf{K}^2 dv. \qquad (6\text{-}2.38)$$

Cogravitational energy in terms of vector potential, EM430

$$U = \frac{1}{2}\int \mathbf{A}\cdot\mathbf{J}dv. \tag{6-2.39}$$

Cogravitational energy of a mass current in an external field, EM432

$$U' = \int \mathbf{J}\cdot\mathbf{A}'dv. \tag{6-2.40}$$

Gravitational Poynting's vector, EM509

$$\mathbf{P} = \frac{c^2}{4\pi G}\mathbf{K}\times\mathbf{g}. \tag{6-2.41}$$

Gravitational field momentum, EM513

$$\mathbf{G} = \frac{1}{4\pi G}\int \mathbf{K}\times\mathbf{g}dv. \tag{6-2.42}$$

Gravitational field angular momentum

$$\mathbf{L} = \frac{1}{4\pi G}\int \mathbf{r}\times(\mathbf{K}\times\mathbf{g})dv. \tag{6-2.43}$$

(3) Equations for gravitational waves (see also Chapter 5):

Direction of field vectors in a plane wave propagating in the z-direction, EM531

$$\mathbf{K} = \frac{1}{c}\mathbf{k}\times\mathbf{g}. \tag{6-2.44}$$

Energy density in a gravitational wave, EM533

$$U_v = -\frac{1}{4\pi G}\mathbf{g}^2 = -\frac{c^2}{4\pi G}\mathbf{K}^2. \tag{6-2.45}$$

The analogy between electromagnetic and gravitational equations is, of course, not limited to the equations listed above. Not only basic electromagnetic equations, but also any equation representing a solution of an electromagnetic problem for fields or forces not involving conducting, dielectric, or magnetic bodies has its gravitational counterpart. However, if the propagation velocity of gravitation is not equal to the velocity of light (see Section 7-3), then c appearing in the gravitational equations should be, in general, the velocity of the propagation of gravitation rather than the velocity of light.

Until recently it was believed that the analogy between electromagnetic and gravitational equations did not apply to fast moving systems, because the electric charge is not affected by velocity, but the mass of a moving body was thought to vary with velocity. It is now generally accepted that mass does not depend on velocity.

Observe, however, that gravitational equations depicting "nonlinear" gravitational effects (see Chapter 8) do not have their electromagnetic counterparts.

References and Remarks for Chapter 6

1. Letter "m" in the equation number indicates that the equation is a modification of the original equation.

2. O. D. Jefimenko, *Electricity and Magnetism*, 2nd ed., (Electret Scientific, Star City, 1989).

3. The possibility of representing the gravitational field by vector potential is not well known. As in the case of electrostatic field, such a representation is only possible for the gravitational field in a mass-free space, where $\nabla \cdot \mathbf{g} = 0$. See Ref. 4 for details.

4. O. D. Jefimenko, "Direct calculation of electric and magnetic forces from potentials," Am. J. Phys. **58**, 625-631 (1990).

7

GRAVITATIONAL FIELDS AND FORCES

We shall now present illustrative examples demonstrating the use of various gravitational equations introduced in Chapters 5 and 6. We shall mostly use equations that cannot be found in conventional presentations of the theory of gravitation. As is the case with all other illustrative examples in this book, the purpose of the examples is a clear demonstration of the theory rather than a deliberate derivation of new results. Therefore our examples are simple both physically and mathematically.

7-1. Illustrative Examples on Static Gravitational Fields

As we already know, many solutions of electrostatic problems can be converted to solutions of the corresponding gravitational problems. The examples presented in this section make use of such conversion or analogy (consult Table 6-1 as needed).

▼

Example 7-1.1 The electric field on the axis of a thin, uniformly charged disk of radius a and charge q at a distance z from the center of the disk is

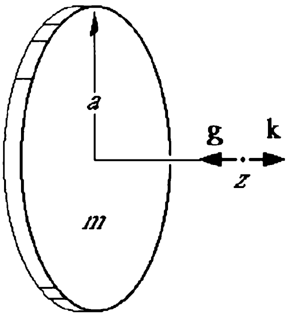

Fig. 7.1 Calculation of the gravitational field on the axis of a disk.

$$\mathbf{E} = \frac{q}{2\pi\varepsilon_0 a^2}\left[1 - \frac{z}{(a^2 + z^2)^{1/2}}\right]\mathbf{k}, \qquad (7\text{-}1.1)$$

where **k** is a unit vector along the axis of the disk pointing away from the disk (EM100). Using the analogy between electric and gravitational equations, find the gravitational field on the axis of a similar disk of mass m (Fig. 7.1).

Replacing **E** by **g** and ε_0 by $-1/4\pi G$, we obtain for the gravitational field of a disk of radius a and mass m

$$\mathbf{g} = -G\frac{2m}{a^2}\left[1 - \frac{z}{(a^2 + z^2)^{1/2}}\right]\mathbf{k}. \qquad (7\text{-}1.2)$$

Example 7-1.2 The electric force between a uniformly charged ring of charge q' and radius a and a thin, uniformly charged rod of charge q and length $2d$ lying along the axis of the ring, is

$$\mathbf{F} = \frac{qq'}{8\pi\varepsilon_0 d}\left\{\frac{1}{[a^2 + (z_0 - d)^2]^{1/2}} - \frac{1}{[a^2 + (z_0 + d)^2]^{1/2}}\right\}\mathbf{k}, \qquad (7\text{-}1.3)$$

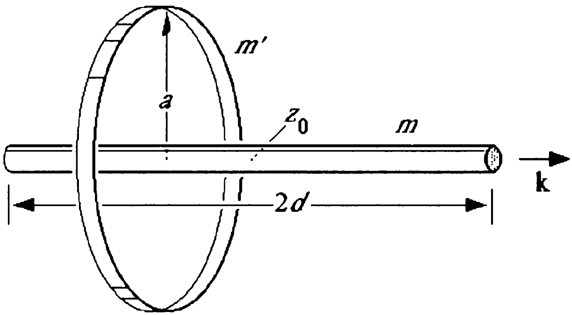

Fig. 7.2 Calculation of the gravitational force on a rod inside a ring.

where z_0 is the distance from the center of the ring to the center of the rod, and **k** is a unit vector along the axis of the ring pointing away from the ring (EM210, EM212). Find the gravitational force between a similar ring of mass m' and a rod of mass m (Fig. 7.2).

Substituting m for q, m' for q', and $-1/4\pi G$ for ε_0, we obtain for the gravitational force between the ring and the rod

$$\mathbf{F} = -G\frac{mm'}{2d}\left\{\frac{1}{[a^2 + (z_0 - d)^2]^{1/2}} - \frac{1}{[a^2 + (z_0 + d)^2]^{1/2}}\right\}\mathbf{k}. \quad (7\text{-}1.4)$$

Example 7-1.3 The electrostatic potential of a spherical charge distribution of uniform density ρ and radius a is, inside and outside the distribution, (EM115)

$$\varphi_{\text{inside}} = \frac{q}{8\pi\varepsilon_0 a^3}(3a^2 - r^2),$$
$$\varphi_{\text{outside}} = \frac{q}{4\pi\varepsilon_0 r}. \quad (7\text{-}1.5)$$

Find the gravitational potential inside and outside a similar spherical mass.

Replacing ε_0 by $-1/4\pi G$ and q by m, we obtain for the potentials of a spherical mass of radius a

$$\varphi_{\text{inside}} = -G\frac{m}{2a^3}(3a^2 - r^2), \qquad \varphi_{\text{outside}} = -G\frac{m}{r}. \tag{7-1.6}$$

Example 7-1.4 The electrostatic energy of a uniformly charged spherical shell of charge q and radius a is (EM190)

$$U = \frac{q^2}{8\pi\varepsilon_0 a}. \tag{7-1.7}$$

Find the gravitational energy of a similar shell of mass m.

Replacing q by m and ε_0 by $-1/4\pi G$, we obtain for the gravitational energy of a spherical shell of radius a and mass m

$$U = -G\frac{m^2}{2a}. \tag{7-1.8}$$

Example 7-1.5 Consider a cylinder of uniform mass density ρ, length $2t$, and radius a. The axis of the cylinder is also the z-axis of cylindrical coordinates whose origin is at the center of the cylinder. A spherical cavity is made around an internal axial point of the cylinder at a distance $z = d$ from the center of the cylinder. A particle of mass m is placed at the center of the cavity. Find the gravitational force exerted by the cylinder on the particle (see EM105).

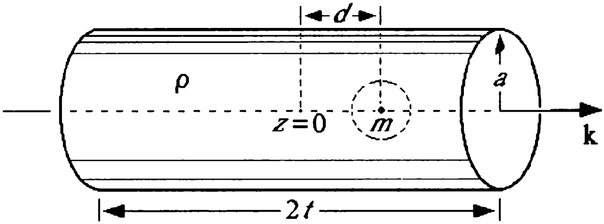

Fig. 7.3 Calculation of the gravitational force on a point mass placed in a spherical cavity inside a cylinder.

This problem is best solved by using Eq. (6-2.7), which makes it possible to find the gravitational field of a mass by integrating over the boundary surfaces of the mass. We see by inspection (Fig. 7.3) that the surface of the cavity makes no contribution to the field at its center, since a spherical surface produces only a radial field, all components of which meet at the center and cancel each other. Likewise, the curved surface of the cylinder makes no contribution to the field. Only the two flat surfaces of the cylinder make a contribution.

The contribution of the closest flat surface ($z > 0$) of the cylinder to the field at the center of the cavity is, by Eq. (6-2.7),

$$\begin{aligned} \mathbf{g}_1 &= -G\rho \int \frac{d\mathbf{S}'}{r} = -G\rho\mathbf{k} \int_0^a \frac{2\pi R\,dR}{[R^2 + (t-d)^2]^{1/2}} \\ &= -2\pi G\rho \{[a^2 + (t-d)^2]^{1/2} - (t-d)\}\mathbf{k}. \end{aligned} \tag{7-1.9}$$

The contribution of the other flat surface ($z < 0$) is, similarly,

$$\begin{aligned} \mathbf{g}_2 &= -G\rho \int \frac{d\mathbf{S}'}{r} = G\rho\mathbf{k} \int_0^a \frac{2\pi R\,dR}{[R^2 + (t+d)^2]^{1/2}} \\ &= 2\pi G\rho \{[a^2 + (t+d)^2]^{1/2} - (t+d)\}\mathbf{k}. \end{aligned} \tag{7-1.10}$$

The field at the center of the cavity is then $\mathbf{g}_1 + \mathbf{g}_2$, or

$$\mathbf{g} = 2\pi G\rho\{[a^2+(t+d)^2]^{1/2} - [a^2+(t-d)^2]^{1/2} - 2d\}\mathbf{k}. \qquad (7\text{-}1.11)$$

The force on the particle of mass m at the center of the cavity is therefore, by Eq. (5-1.4),

$$\mathbf{F} = 2\pi Gm\rho\{[a^2+(t+d)^2]^{1/2} - [a^2+(t-d)^2]^{1/2} - 2d\}\mathbf{k}. \qquad (7\text{-}1.12)$$

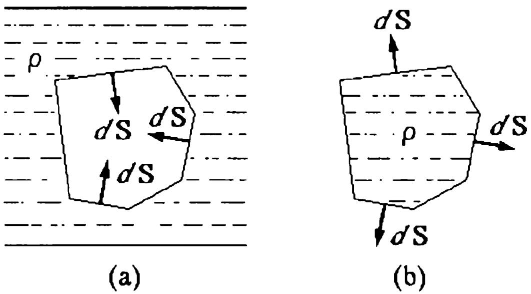

Fig. 7.4 Calculation of the buoyant force on a cavity formed in a liquid. (a) Surface element vectors of the cavity. (b) Surface element vectors of the liquid initially contained in the cavity.

Example 7-1.6 An irregular cavity has formed inside a liquid of density ρ in the region where the gravitational potential is φ'. Find the buoyant force on the cavity (Fig. 7.4).

This problem on Archimedes's principle is usually solved by means of a plausibility argument based on the consideration of the pressure inside the liquid. Here we shall provide a rigorous solution of the problem by means of Eq. (6-2.24). This equation allows one to find the gravitational force on a volume bounded by a given surface. The surface element vector $d\mathbf{S}$ in this equation is directed from the mass under consideration into the empty space, regardless whether the mass is inside or outside the bounding surface.[1] Therefore the force on the cavity and the force on the liquid initially contained in the cavity are exactly the same in magnitude, but opposite in direction. Hence, the buoyant force is equal to the weight of the liquid initially contained in the cavity.

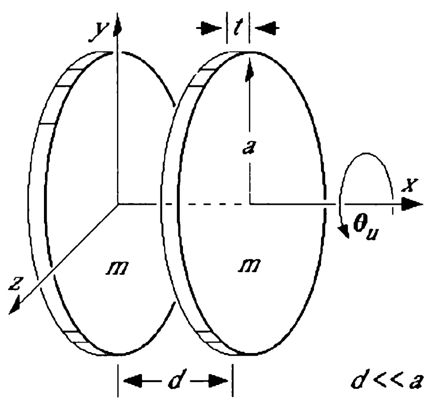

Fig. 7.5 Calculation of the gravitational attraction between two circular plates.

Example 7-1.7 A gravitational "parallel-plate capacitor" consists of two large circular plates of radius a having a uniformly distributed mass m (Fig. 7.5). One of the plates is in the y,z-plane of rectangular coordinates with its center at the origin. The second plate is at a small distance $x=d$ from the first. Using five different methods, and neglecting edge effects, find the gravitational force between the plates.[2]

(a) *Direct calculation.* By Gauss's law, Eq. (6-2.1), and by the symmetry of the system, the gravitational field produced at $x>0$ by the first plate is

$$\mathbf{g}' = -2G\frac{m}{a^2}\mathbf{i}. \tag{7-1.13}$$

By Eq. (6-2.23), the force on the second plate is then

$$\mathbf{F} = m\mathbf{g}' = -2G\frac{m^2}{a^2}\mathbf{i}. \tag{7-1.14}$$

(b) *Force in terms of scalar potential.* The potential produced by the first plate at a distance x from the origin is, by Eqs. (6-2.11) and (7-1.13)

$$\begin{aligned}\varphi' &= \int_x^0 \mathbf{g}'\cdot d\mathbf{l} + \varphi_0 = \int_x^0\left(-2G\frac{m}{a^2}\right)dx + \varphi_0 \\ &= 2G\frac{m}{a^2}x + \varphi_0,\end{aligned} \tag{7-1.15}$$

where φ_0 is a reference potential at $x=0$. Let us assume that the thickness of the second plate is t. The potential at the front surface and at the back surface of the plate is then

$$\varphi'_{\text{front}} = 2G\frac{m}{a^2}d + \varphi_0, \qquad \varphi'_{\text{back}} = 2G\frac{m}{a^2}(d+t) + \varphi_0. \tag{7-1.16}$$

By Eq. (6-2.24), the force on the second plate is therefore [note that, by symmetry, the rim of the plate adds nothing to Eq. (6-2.24)]

$$\begin{aligned}\mathbf{F} = &-\frac{m}{\pi a^2 t}\left(2G\frac{m}{a^2}d + \varphi_0\right)(-\pi a^2\mathbf{i}) \\ &-\frac{m}{\pi a^2 t}\left[2G\frac{m}{a^2}(d+t) + \varphi_0\right]\pi a^2\mathbf{i},\end{aligned} \tag{7-1.17}$$

or

$$\mathbf{F} = -2G\frac{m^2}{a^2}\mathbf{i}. \tag{7-1.18}$$

(c) *Force in terms of vector potential.* For $x>0$, the vector potential of the gravitational field of the first plate is[3]

$$\mathbf{A}_g' = \frac{mr}{4\pi a^2}\boldsymbol{\theta}_u, \tag{7-1.19}$$

where r is a perpendicular distance from the x-axis, and $\boldsymbol{\theta}_u$ is a right-handed circular unit vector around the x-axis. By Eq. (6-2.25), the force on the second plate is then

$$\mathbf{F} = 4\pi G\rho \oint \frac{mr}{4\pi a^2}\boldsymbol{\theta}_u \times d\mathbf{S}. \tag{7-1.20}$$

The surface of integration in Eq. (7-1.20) consists of the two flat surfaces and the circular rim of the second plate. By symmetry, the contributions of the two flat surfaces to the integral of Eq. (7-1.20) cancel. The only nonvanishing contribution to the integral comes from the rim of the plate. If the thickness of the plate is t, the surface element vector of the rim is $d\mathbf{S}=td\mathbf{l}_{out}$ where $d\mathbf{l}_{out}$ is a vector representing a length element of the rim and directed radially outward from the rim. The force on the second plate is therefore

$$\mathbf{F} = 4\pi G\frac{m}{\pi a^2 t}\oint \frac{ma}{4\pi a^2}\boldsymbol{\theta}_u \times t\,d\mathbf{l}_{out}. \tag{7-1.21}$$

Simplifying, we obtain

$$\mathbf{F} = -\ 2G\frac{m^2}{a^2}\mathbf{i}. \tag{7-1.22}$$

(d) *Force in terms of Maxwell's stress integral.* The total gravitational field in the space between the two plates is zero, because there the fields of the plates have opposite directions. The total gravitational field outside the plates, to the right of the second plate, is double the field of each single plate, Eq. (7-1.13), because there the two fields are in the same direction. We thus have

$$\mathbf{g}_{\text{between}} = 0, \qquad \mathbf{g}_{\text{outside}} = -\ 4G\frac{m}{a^2}\mathbf{i}. \tag{7-1.23}$$

Applying Eq. (6-2.26) to a Maxwellian surface enclosing the second plate, we then obtain

$$\mathbf{F} = \frac{1}{8\pi G}\left[0 \cdot \pi a^2(-\mathbf{i}) + \left(-4G\frac{m}{\pi a^2}\right)^2 \cdot \pi a^2 \mathbf{i}\right]$$
$$- \frac{1}{4\pi G}\left[0 \cdot (0 \cdot \pi a^2)(-\mathbf{i}) + \left(-4G\frac{m}{\pi a^2}\right) \cdot \left(-4G\frac{m}{\pi a^2}\right) \cdot \pi a^2 \mathbf{i}\right]$$
$$= -2G\frac{m^2}{a^2}\mathbf{i}. \qquad (7\text{-}1.24)$$

(e) *Force in terms of energy*. The total field in the space between the plates and in the space outside the plates is given by Eq. (7-1.23). According to Eq. (6-2.33), the gravitational energy density is then

$$U_{v \text{ between}} = 0,$$
$$U_{v \text{ outside}} = -\frac{1}{8\pi G}\left(-4G\frac{m}{a^2}\right)^2 = -2G\frac{m^2}{\pi a^4}. \qquad (7\text{-}1.25)$$

Suppose now that the second plate moves through a distance dx. The relation between the force on the plate and the energy change associated with the displacement of the plate is

$$\mathbf{F} = -\frac{\partial U}{\partial x}\mathbf{i} = -\frac{dU}{dx}\mathbf{i}. \qquad (7\text{-}1.26)$$

The energy change associated with the displacement dx is

$$dU = -\left(-2G\frac{m^2}{\pi a^4}\right)\pi a^2 dx. \qquad (7\text{-}1.27)$$

(The minus sign in front of parenthesis reflects the fact that the energy in the space between the plates is zero.) Thus the force on the second plate is

$$\mathbf{F} = -2G\frac{m^2}{a^2}\mathbf{i}. \qquad (7\text{-}1.28)$$

▲

7-2. Illustrative Examples on Dynamic Gravitational Fields

We shall now present illustrative examples involving nonstatic gravitational fields. These examples will depict several remarkable gravitational phenomena not revealed by the conventional Newtonian gravitational theory. As in the preceding section, we shall start with simple conversion of electromagnetic equations to gravitational equations.

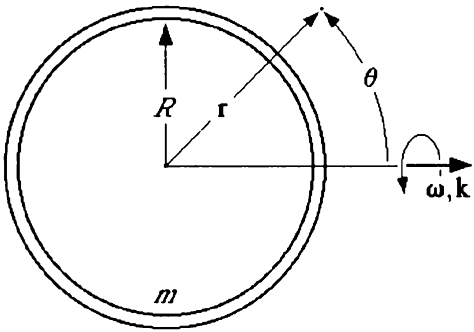

Fig. 7.6 Calculation of the cogravitational field of a rotating spherical shell.

▼

Example 7-2.1 A spherical shell of radius R and uniform surface charge density σ rotates with angular velocity ω about a diameter which is also the polar axis of spherical coordinates whose origin is at the center of the shell. The shell creates a magnetic field in the space inside and outside the shell given by (EM378)

$$\mathbf{H}_{\text{inside}} = \frac{2}{3}\sigma\omega R\mathbf{k},$$
$$\mathbf{H}_{\text{outside}} = \frac{2\sigma\omega R^4}{3r^3}\cos\theta\,\mathbf{r}_u + \frac{\sigma\omega R^4}{3r^3}\sin\theta\,\boldsymbol{\theta}_u. \qquad (7\text{-}2.1)$$

Find the cogravitational field of a similar shell of uniformly distributed mass m (Fig. 7.6).

Since the two equations are for the magnetic field **H** rather than for the flux density field **B**, we must first convert them to **B**, by using $\mathbf{B} = \mu_0\mathbf{H}$. Replacing then **B** by **K**, μ_0 by $-4\pi G/c^2$, and σ by $m/4\pi R^2$, we obtain for the cogravitational field of the shell of radius R and mass m rotating with angular velocity ω

$$\mathbf{K}_{\text{inside}} = -G\frac{2m\omega}{3c^2R}\mathbf{k},$$
$$\mathbf{K}_{\text{outside}} = -G\frac{2m\omega R^2}{3c^2r^3}\cos\theta\,\mathbf{r}_u - G\frac{m\omega R^2}{3c^2r^3}\sin\theta\,\boldsymbol{\theta}_u. \tag{7-2.2}$$

Example 7-2.2 A long beam of charged particles moves with velocity v along its length. The charge density in the beam is ρ, the radius of the beam is a. The beam creates a magnetic field which, inside and outside the beam, is (EM332)

$$\mathbf{H}_{\text{inside}} = \rho\frac{\mathbf{v}\times\mathbf{r}}{2},$$
$$\mathbf{H}_{\text{outside}} = \rho a^2\frac{\mathbf{v}\times\mathbf{r}}{2r^2}, \tag{7-2.3}$$

where **r** is a radius vector directed from the axis of the beam to the point of observation. Find the cogravitational field of a similar beam of mass particles moving with velocity v along its length.

Since the above expressions are for **H** rather than for **B**, we must convert them to **B** by using $\mathbf{B} = \mu_0\mathbf{H}$. Replacing then **B** by **K** and μ_0 by $-4\pi G/c^2$, we obtain for the cogravitational field of a beam of mass particles of density ρ, radius a, and velocity v

$$\mathbf{K}_{\text{inside}} = -G\frac{2\pi\rho}{c^2}\mathbf{v}\times\mathbf{r},$$
$$\mathbf{K}_{\text{outside}} = -G\frac{2\pi\rho a^2}{c^2r^2}\mathbf{v}\times\mathbf{r}. \tag{7-2.4}$$

Example 7-2.3 A long, thin, vertical plate of length L, width w, thickness d, and mass m falls to the ground (Fig. 7.7). The plate remains vertical during the fall. Neglecting end and edge effects, show that the rate at which the kinetic energy of the plate increases is completely accounted for by the influx of the gravitational field energy into the plate.

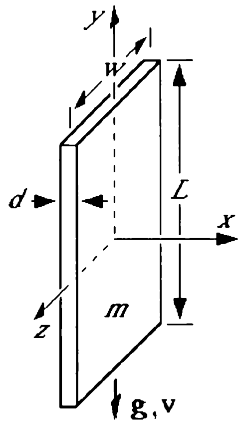

Fig. 7.7 The kinetic energy of the falling plate is completely accounted for by the influx of the gravitational energy into the plate.

Let us use rectangular coordinates for describing the motion of the plate and the corresponding energy relations. Let the x- and the z-axis be in the horizontal plane, and let the y-axis be directed vertically upward. Let us assume that the midplane of the plate is in the y,z-plane.

The gravitational field $\mathbf{g}$ around the plate is equal to the acceleration of gravity g and is directed downward:

$$\mathbf{g} = -g\mathbf{j}. \tag{7-2.5}$$

Let the velocity of the plate be $\mathbf{v}$. The plate constitutes a mass current whose density is, by Eq. (5-4.3),

$$\mathbf{J} = \rho\mathbf{v} = \frac{m}{Lwd}\mathbf{v} = -\frac{mv}{Lwd}\mathbf{j}. \tag{7-2.6}$$

Taking into account that the plate is thin ($d \ll w$) and that $\partial\mathbf{g}/\partial t = 0$ (constant $\mathbf{g}$), the cogravitational field produced by this current is, by Eqs. (6-2.4) and (7-2.6) (compare EM332, EM333 but observe that a cogravitational field, in contrast to a magnetic field, is always left-handed relative to the current by which it is produced),

$$\mathbf{K} = -\frac{4\pi G}{c^2}\frac{Jwd}{2w}\mathbf{k} = -\frac{2\pi G}{c^2}\frac{mv}{Lw}\mathbf{k} \tag{7-2.7}$$

at the front surface of the plate ($x=d/2$) and

$$\mathbf{K} = \frac{2\pi G}{c^2}\frac{mv}{Lw}\mathbf{k} \tag{7-2.8}$$

at the back surface of the plate ($x=-d/2$). The gravitational Poynting's vector at the two surfaces is, by Eqs. (6-2.41), (7-2.8), (7-2.7), and (7-2.5),

$$\mathbf{P} = \frac{c^2}{4\pi G}\mathbf{K}\times\mathbf{g} = \frac{c^2}{4\pi G}\frac{2\pi G}{c^2}\frac{mvg}{Lw}\mathbf{n}_{in} = \frac{mvg}{2Lw}\mathbf{n}_{in}, \tag{7-2.9}$$

where $\mathbf{n}_{in}$ is a unit vector normal to the plate and directed into the plate.

Multiplying $\mathbf{P}$ by the area of the two surfaces, we obtain for the rate of gravitational energy influx into the plate

$$\frac{dU}{dt} = \frac{mvg}{2Lw}2Lw = mvg. \tag{7-2.10}$$

The rate at which the kinetic energy of the plate increases is

$$\frac{dU}{dt} = \frac{d}{dt}\left(\frac{mv^2}{2}\right) = mv\frac{dv}{dt} = mvg. \tag{7-2.11}$$

Thus the rate at which the kinetic energy of the plate increases is completely accounted for by the influx of the gravitational field energy into the plate via the gravitational Poynting's vector.[4] This means that if the plate started to fall from rest, its entire kinetic energy is due to the influx of the gravitational energy into the plate.

Example 7-2.4 Consider two point masses m and m'. The mass m is in free fall toward the ground, the mass m' is at rest below m. At the moment when m passes m', m' is released so that it, too, falls to the ground. (a) What is the acceleration of m'? (b) What is the acceleration of m before and after m' is released? (c) Does the acceleration of a falling body depend on its mass? (Neglect gravitational attraction between the two masses, neglect retardation, and neglect terms of the order v/c or smaller).

(a) Let us designate the acceleration of gravity vector as **a**. Normally, m' would then fall with the acceleration **a**. However, m' is subject not only to the force of gravity, but also to the gravitational force from the accelerating mass m. According to Eq. (5-5.6), this force is (within the limits of accuracy specified in the statement of the problem)

$$\begin{aligned}\mathbf{F} &= -G\frac{mm'\mathbf{r}}{r^3} - G\frac{mm'[\mathbf{r}\times(\mathbf{r}\times\mathbf{a})]}{r^3c^2} \\ &= -G\frac{mm'\mathbf{r}}{r^3} - G\frac{mm'\mathbf{r}(\mathbf{r}\cdot\mathbf{a})}{r^3c^2} + G\frac{mm'\mathbf{a}}{rc^2}.\end{aligned} \tag{7-2.12}$$

The first two terms in the last expression represent gravitational attraction between the two masses. Disregarding these terms, we are left with the "gravikinetic" force (gravitational drag) exerted by m upon m'

$$\mathbf{F}_k = G\frac{mm'}{rc^2}\mathbf{a}, \tag{7-2.13}$$

where r is the distance between the two masses. This force is in the direction of the acceleration of gravity, and it provides an additional acceleration to the mass m'. The total *initial* acceleration of m' is therefore

$$\mathbf{a}_{m'\text{total}} = \left(1 + G\frac{m}{rc^2}\right)\mathbf{a}. \tag{7-2.14}$$

(b) As soon as m' begins to fall, it exerts an additional acceleration on m. Both masses now fall with an acceleration greater than a. The additional acceleration of m enhances even further the initial acceleration of m', and so on. The process converges, however, very rapidly.

(c) According to the results of Parts (a) and (b), falling masses amplify the gravitational acceleration of the neighboring masses. Therefore a large mass should fall with a greater acceleration than a small mass (see Section 7-3).

Example 7-2.5 Consider a single particle of mass m on the surface of the beam of mass particles described in Example 7-2.2. Find the expression for the total force (gravitational and cogravitational) acting on the particle.

Constructing a cylindrical Gaussian surface around the beam and applying Eq. (6-2.1) to this surface, we obtain for the gravitational field at the surface of the beam (compare EM89-90, EM420)

$$\mathbf{g} = -2\pi G\rho a\mathbf{r}_u, \tag{7-2.15}$$

where $\mathbf{r}_u$ is a unit vector pointing away from the axis of the beam at right angles to it. The cogravitational field at the surface of the beam is, by Eq. (7-2.4),

$$\mathbf{K} = -G\frac{2\pi\rho a}{c^2}\mathbf{v}\times\mathbf{r}_u. \tag{7-2.16}$$

The force on the mass particle (mass m) on the surface of the beam is then, by Eqs. (5-1.4) and (5-2.1),

$$\mathbf{F} = m(\mathbf{g} + \mathbf{v} \times \mathbf{K}), \tag{7-2.17}$$

or, after substituting **K**, expanding the cross product, and simplifying,

$$\mathbf{F} = -2\pi G\rho m a\left(1 - \frac{v^2}{c^2}\right)\mathbf{r}_u. \tag{7-2.18}$$

Thus the particle is always attracted to the beam, although the force of attraction is smaller than for a stationary cylinder of the same mass density and radius. The gravitational attraction always dominates over the cogravitational repulsion, so that the beam compresses as it moves. When the speed of the beam approaches c (the speed of the propagation of gravitation), the force on the particle approaches zero.

Example 7-2.6 A rotating sphere of mass m and radius a moves with velocity $v \ll c$ along the x-axis of rectangular coordinates. The angular velocity of the sphere is ω and is directed along the y-axis. A point mass m' is at rest on the z-axis at a distance $r \gg a$ from the origin. Find the force exerted by the sphere on the point mass at the moment when the sphere passes the origin (Fig 7.8).

The force exerted by the sphere on the point mass can be found from Eqs. (5-5.4) and (5-1.4), which yield (omitting the retardation brackets, since $v \ll c$)

$$\mathbf{F} = -m'G\int\left(\frac{\rho}{r^2} + \frac{1}{rc}\frac{\partial\rho}{\partial t}\right)\mathbf{r}_u dv' + m'\frac{G}{c^2}\int\frac{1}{r}\left(\frac{\partial\mathbf{J}}{\partial t}\right)dv'. \tag{7-2.19}$$

Since the radius of the sphere is much smaller than the distance between the sphere and the point mass, the sphere, from the location of the point mass, can be considered to be a cogravitational dipole. As it is shown in Appendix 4, a moving cogravitational dipole appears to acquire an additional mass and acquires a

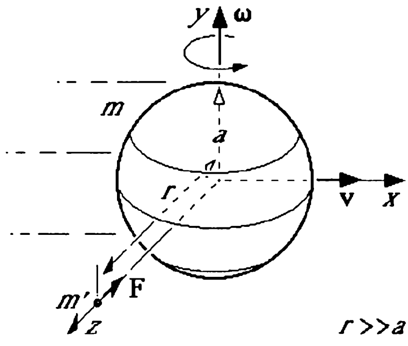

Fig. 7.8 A rotating sphere moving past a stationary point mass exerts on the point mass not only the ordinary Newtonian gravitational force but also a velocity-dependent and a rotation-dependent force.

gravitational dipole field. By Eq. (A-4.11), a cogravitational dipole of mass $4m$ moving with velocity $v \ll c$ appears to have an additional mass $2mv^2/c^2$ [we can ignore the apparent mass given by Eq. (A-4.6), since it is insignificant unless the rotation is extremely fast]. For our sphere, whose mass is m, the additional mass is $mv^2/2c^2$. Designating the apparent gravitational dipole field of the sphere as $\mathbf{g}_{\text{dipole}}$, we can then express the first integral of Eq. (7-2.19) as

$$\mathbf{F}_1 = -G\frac{m'm}{r^2}\mathbf{r}_u - G\frac{m'mv^2}{2c^2r^2}\mathbf{r}_u + m'\mathbf{g}_{\text{dipole}}. \qquad (7\text{-}2.20)$$

The force on the point mass is therefore

$$\mathbf{F} = -G\frac{m'm}{r^2}\mathbf{r}_u - G\frac{m'mv^2}{2c^2r^2}\mathbf{r}_u + m'\mathbf{g}_{\text{dipole}} + G\frac{m'}{c^2}\int\frac{1}{r}\frac{\partial\mathbf{J}}{\partial t}dv'. \qquad (7\text{-}2.21)$$

The last two terms in this equation are the gravitational counterpart of the electric force discussed in Chapter 2, Section 2-6, and in Chapter 3, Example 3-3.7. Therefore, instead of evaluating these terms, we shall determine them by converting the corresponding electric equations. The electric field corresponding to the last two terms of Eq. (7-2.21) is given by Eq. (2-6.9), and the electric force is given by Eq. (2-6.10). Replacing in Eqs. (2-

6.9) and (2-6.10) **B** by **K** and q by m', we obtain for the gravitational force due to the last two terms of Eq. (7-2.21)

$$\mathbf{F}_{\text{dipole}} + \mathbf{F}_k = -m'\mathbf{v} \times \mathbf{K}, \tag{7-2.22}$$

where **K** is the cogravitational field produced by the rotating sphere at the location of the point mass, when the sphere has no translational motion. This field can be found by integrating the expression for the external field of the spherical shell given in Eq. (7-2.2). The result is

$$\mathbf{K} = G\frac{m\omega a^2}{5r^3c^2}\mathbf{j}. \tag{7-2.23}$$

The total force on the point mass is therefore, by Eqs. (7-2.21), (7-2.22), and (7-2.23),

$$\mathbf{F} = -G\frac{m'm}{r^2}\mathbf{k} - G\frac{m'mv^2}{2c^2r^2}\mathbf{k} - G\frac{m'mv\omega a^2}{5r^3c^2}\mathbf{k}. \tag{7-2.24}$$

Thus the force exerted by a slowly moving, rotating sphere on a distant stationary point mass differs from the Newtonian gravitational force by the presence of two additional terms that depend on the linear and angular velocity of the sphere.

Although we have derived Eq. (7-2.24) for the moving sphere and stationary point mass, it is clear that within the accuracy of our derivations Eq. (7-2.24) also applies for a point mass moving relative to the sphere (this is required by the ordinary Galilean relativity). Thus Eq. (7-2.24) represents a time-dependent generalization of Newton's gravitational law that should be used for computing planetary orbits and for similar problems of celestial mechanics. A remarkable feature of this equation is that it results in elliptical planetary orbits whose major axis experiences a secular advance, or "perihelion precession" (see Section 7-3).

Example 7-2.7 A ring-shaped magnetic dipole antenna of radius a carrying an oscillating current

$$I = I_0 \sin\omega t \qquad (7\text{-}2.25)$$

produces an electric radiation field

$$\mathbf{E} = \mu_0 I_0 \frac{a^2\omega^2 \sin\omega(t - r/c)}{4rc} \sin\theta \boldsymbol{\phi}_u, \qquad (7\text{-}2.26)$$

and a magnetic radiation flux density field

$$\mathbf{B} = -\mu_0 I_0 \frac{a^2\omega^2 \sin\omega(t - r/c)}{4rc^2} \sin\theta \boldsymbol{\theta}_u, \qquad (7\text{-}2.27)$$

where r is the distance from the ring ($r \gg a$), θ is the angle between the axis of the ring and r, $\boldsymbol{\phi}_u$ is an azimuthal unit vector, and $\boldsymbol{\theta}_u$ is a polar unit vector in spherical coordinates whose polar axis is the axis of the ring (EM562-EM565). The time-average power radiated by the antenna is

$$W_{av} = \mu_0 I_0^2 \frac{\pi a^4 \omega^4}{12c^3}. \qquad (7\text{-}2.28)$$

The gravitational counterpart of this antenna is a ring of mass m and radius a oscillating with angular frequency ω about its axis. Find the gravitational radiation field, the cogravitational radiation field, and the time-average power radiated by this ring.

Let the angular amplitude of the oscillating ring be α_0. The angular deflection of the ring from its equilibrium position is then

$$\alpha = \alpha_0 \cos\omega t, \qquad (7\text{-}2.29)$$

and the angular velocity of the ring is

$$\frac{d\alpha}{dt} = -\alpha_0 \omega \sin\omega t. \qquad (7\text{-}2.30)$$

The mass current of the ring is then (note that $I = \lambda v$),

$$I = \frac{m}{2\pi a} a \frac{d\alpha}{dt} = -\frac{m\alpha_0\omega}{2\pi}\sin\omega t = I_0 \sin\omega t, \qquad (7\text{-}2.31)$$

where

$$I_0 = -\frac{m\alpha_0\omega}{2\pi}. \qquad (7\text{-}2.32)$$

Replacing in Eqs. (7-2.26), (7-2.27), and (7-2.28) **E** by **g**, **B** by **K**, μ_0 by $-4\pi G/c^2$, and I_0 by the right side of Eq. (7-2.32), we obtain

$$\mathbf{g} = G\frac{m\alpha_0 a^2\omega^3 \sin\omega(t - r/c)}{2rc^3}\sin\theta\,\boldsymbol{\phi}_u, \qquad (7\text{-}2.33)$$

$$\mathbf{K} = -G\frac{m\alpha_0 a^2\omega^3 \sin\omega(t - r/c)}{2rc^4}\sin\theta\,\boldsymbol{\theta}_u, \qquad (7\text{-}2.34)$$

$$W_{av} = -G\frac{m^2\alpha_0^2 a^4\omega^6}{12c^5}. \qquad (7\text{-}2.35)$$

The minus sign in Eq. (7-2.35) indicates that the ring absorbs energy from space instead of emitting it into space.

▲

7-3. Discussion

In this chapter we have explored the analogy between gravitation and electromagnetism in free space. We have seen that it is possible to convert various equations representing electromagnetic phenomena to corresponding equations representing gravitational phenomena. And we have found that numerous techniques that have been developed for solving problems involving electric and magnetic fields and interactions can be applied to solving gravitational problems.

It is now clear that the classical theory of gravitation is much more complex than commonly believed, and that it has at its disposal many more concepts, tools, and methods than are unveiled in conventional presentations of the theory based exclusively on Newton's gravitational law.

The illustrative examples presented in this chapter have demonstrated several intriguing aspects of classical gravitation that were either ignored or unknown in the past.

The first series of examples (Section 7-1) has demonstrated some effective new techniques for calculating gravitational fields and interactions in time-independent systems. The versatility and usefulness of these techniques has been made particularly clear in Example 7-1.7, where gravitational attraction between two bodies was calculated by five different methods, none of which involved a direct application of Newton's law.

The second series of examples (Section 7-2) has demonstrated a number of time-dependent gravitational effects. Two of these examples are especially important:

Example 7-2.4 shows that the acceleration of a body in a gravitational field depends on the mass of the body. This is in conflict with the original postulate of the general relativity theory: the postulate of the equivalence of the gravitational field and an accelerated reference frame. However, since this postulate is no longer considered an indispensable building block of general relativity,[5] we shall not discuss the significance of this conflict. Of course, Example 7-2.4 also presents the question of the equivalence of gravitational and inertial mass. This particular equivalence appears to hold, because the additional acceleration does not depend on the cause of the primary acceleration.

Example 7-2.6 is even more significant. It shows that in time-dependent systems gravitational interactions involve not only the usual Newtonian attraction but also additional forces associated with the motion of the interacting bodies. This phenomenon is particularly significant because of its relevance for explaining certain discrepancies between observed and calculated properties of planetary motion.

The best known such discrepancy is in the precession of the perihelion of Mercury. According to Newton's gravitational law, this precession is a result of perturbations from the outer planets and should amount to 532 seconds of arc per century.[6] However, the observed value is approximately 575 seconds of arc per century. It was the greatest triumph of Einstein's general relativity theory, when his theory provided an explanation for the residual 43 seconds. To this day Einstein's relativistic correction of Mercury's precession remains the most important observational evidence apparently supporting the theory of general relativity.[7]

According to the general relativity theory, the gravitational force on a mass m_0' moving with velocity $v \ll c$ in an orbit around a mass m is[8]

$$\mathbf{F} = -G\frac{mm_0'}{r^2}\mathbf{r}_u - G\frac{6mm_0'v^2}{c^2r^2}\mathbf{r}_u. \qquad (7\text{-}3.1)$$

The precession is caused by the last term of this equation and is essentially proportional to the numerical coefficient appearing in this term. Let us compare this equation with Eq. (7-2.24). If we omit the last term in Eq. (7-2.24) (because of the slow rotation of the Sun, this term can hardly have a significant effect on planetary motion), we have

$$\mathbf{F} \approx -G\frac{mm'}{r^2}\mathbf{r}_u - G\frac{mm'v^2}{2c^2r^2}\mathbf{r}_u. \qquad (7\text{-}3.2)$$

Except for the numerical coefficient in the last term, Eq. (7-3.2) is the same as Eq. (7-3.1). However, the mass m_0' in Eq. (7-3.1) is the relativistic *rest mass*, while the mass m' in Eq. (7-

3.2) is the ordinary (prerelativistic) mass. When we substitute in Eq. (7-3.2) the so-called "transverse" relativistic mass[9] $m' = m'_0(1 - v^2/c^2)^{-1/2}$, we obtain for the case of $v \ll c$

$$\mathbf{F} \approx - G\frac{mm_0'}{r^2}\mathbf{r}_u - G\frac{mm_0'v^2}{c^2r^2}\mathbf{r}_u, \qquad (7\text{-}3.3)$$

which yields for the precession 7 seconds of arc per century.

If the residual precession of Mercury's perihelion is indeed 43 seconds of arc per century, then Eq. (7-3.3) gives too small a value for the precession. Therefore, if our generalized theory of Newtonian gravitation is correct (and there is no objective reason to doubt its correctness, since it has been built on physical laws and principles of unquestionable validity), then we should be able to explain the precession in spite of the value obtained from Eq. (7-3.3). There are several possibilities to do so.

Let us note that up to now we have assumed that the velocity of the propagation of gravitation is the same as the velocity of light c. But there is no experimental or compelling theoretical evidence that gravitation propagates as fast as light. Therefore, within the generalized time-dependent theory of Newtonian gravitation, it would be at least as natural to use Eq. (7-3.2) for determining the unknown velocity of the propagation of gravitation from the known residual precession as to assume (actually postulate) that gravitation propagates as fast as light.[10] Designating the propagation velocity of gravitation as c_g and using the transverse mass, we can write Eq. (7-3.2) as

$$\mathbf{F} \approx - G\frac{mm_0'}{r^2}\mathbf{r}_u - G\frac{mm_0'v^2}{r^2}\left(\frac{1}{2c_g^2} + \frac{1}{2c^2}\right)\mathbf{r}_u. \qquad (7\text{-}3.4)$$

There is one additional possible correction of Eq. (7-3.2). This correction is associated with the distribution of gravitational energy within and around gravitating bodies. We shall discuss this important effect in the next chapter (Example 8-2.2). At this time we shall only state that with this additional correction Eq. (7-3.2) becomes

$$\mathbf{F} \approx -G\frac{mm_0'}{r^2}\mathbf{r}_u - G\frac{mm_0'v^2}{r^2}\left(\frac{1}{2c_g^2} + \frac{1}{c^2}\right)\mathbf{r}_u. \qquad (7\text{-}3.5)$$

Simple calculations show that to obtain the 43 seconds of perihelion precession for Mercury, c_g should be $0.30c$ according to Eq. (7-3.4) and $0.32c$ according to Eq. (7-3.5). With these values for c_g, Eqs. (7-3.4) and (7-3.5) give for the perihelion precession of all planets exactly the same values as the values provided by the general relativity theory.

There is yet another important consideration concerning our Eqs. (7-3.2)-(7-3.5). The velocity v in these equations is the velocity of the planet relative to the *stationary* Sun. But the Sun is not really at rest — it moves relative to the Milky Way and the Milky Way (our galaxy) moves with respect to the rest of the Universe. Should we not, then, include the velocity of the Sun in our Eqs. (7-3.2)-(7-3.5)? There is no clear answer to this question at this time. One thing is clear, however: the problem with the Mercury's perihelion precession is much more complicated than generally believed.

This brings us once again to the relativistic explanation of the precession. Let us point out that the relativistic explanation is based on the assumption that the *main* Mercury's perihelion precession of 532 seconds of arc per century is correctly explained on the basis of the ordinary Newton's theory of gravitation involving only the ordinary Newtonian gravitational attraction between interacting bodies. But according to the generalized Newton's theory, gravitational interactions involve, in general, at least five different forces (see pp. 91-93 and Appendix 7). Therefore it is highly improbable that the ordinary

Newton's theory can correctly account for any part of Mercury's precession. And, consequently, there are very strong reasons to doubt that a "residual" precession really exists, to say nothing of the residual precession of exactly 43 seconds of arc per century explained by general relativity. The fact is that we really do not know what the correct relativistic value for the Mercury's precession is, since nobody has computed the *entire* Mercury's precession on the basis of the general relativity theory.

Before leaving this chapter, we should mention Example 7-2.7. In this example we found that, in contrast to the electromagnetic radiation, gravitational "radiation" absorbs energy rather than emits it. This result also conflicts with the general relativity theory. According to general relativity, a gravitational "antenna" should release energy in the course of gravitational radiation.[11] It is not likely that the conflict can be resolved experimentally any time soon. The magnitudes of the gravitational radiation fields are much too small to be measured in a laboratory. For example, according to Eqs. (7-2.33)-(7-2.35), a ring of mass 1 kg and radius 1 m oscillating with an amplitude of 1 rad and circular frequency 1 $\sec^{-1}$ would produce at a distance of 10 m from itself a gravitational and cogravitational wave of amplitude $g \approx 1.2 \cdot 10^{-36}$ m/sec^2, $K \approx 4 \cdot 10^{-45}$ sec^{-1}, and average power flow of $W_{av} \approx -2.3 \cdot 10^{-54}$ watts.

References and Remarks for Chapter 7

1. O. D. Jefimenko, *Electricity and Magnetism*, 2nd ed., (Electret Scientific, Star City, 1989) pp. 101-103 and 210-211.

2. This example involves mostly rather novel, unconventional, computations of forces. The reader is advised to consult Ref. 1, Sections 7-9 and 7-11, as well as Ref. 3.

3. O. D. Jefimenko, "Direct calculation of electric and magnetic forces from potentials," Am. J. Phys. **58**, 625-631 (1990).

4. This result is completely general and is not limited to bodies of any special shape. This can be proved by noting that $\oint \mathbf{g}\times\mathbf{K}\cdot d\mathbf{S} = \oint \mathbf{g}\cdot\mathbf{K}\times d\mathbf{S} = \mathbf{g}\cdot\oint \mathbf{K}\times d\mathbf{S} = -\mathbf{g}\cdot\int \nabla\times\mathbf{K}dV = \mathbf{g}\cdot\int (4\pi G/c^2)\mathbf{J}dV = (4\pi G/c^2)\mathbf{g}\cdot\int \rho\mathbf{v}\, dV = (4\pi G/c^2)mvg$ (dV is a volume element). For other examples on gravitational Poynting's vector see Appendix 8 and D. Bedford and P. Krumm, "The gravitational Poynting vector and energy transfer," Am. J. Phys. **55**, 362-363 (1987).

5. See, for example, H. C. Ohanian and R. Ruffini, *Gravitation and Spacetime*, 2nd ed., (W. W. Norton, New York, 1994) pp. 45-54.

6. For an interesting elementary discussion and calculations see M. P. Price and W. F. Rush, "Nonrelativistic contribution to Mercury's perihelion precession," Am. J. Phys. **47**, 531-534 (1979).

7. A critique of the relativistic correction of the precession of Mercury's perihelion can be found in Léon Brillouin, *Relativity Reexamined* (Academic Press, New York, 1970) pp. 54, 99.

8. See, for example, D. R. Stump, "Precession of the perihelion of Mercury," Am. J. Phys. **56**, 1097-1098 (1988) or J. B. Marion, *Classical Dynamics of Particles and Systems* (Academic Press, New York, 1965) pp. 289-294. Strictly speaking, general relativity deals with equations of motion rather than with forces.

9. See, for example, O. D. Jefimenko, *Electromagnetic Retardation and Theory of Relativity* (Electret Scientific, Star City, 1997) pp. 193-196.

10. The idea of determining the propagation velocity of gravitation from the observed perihelion advance is not new. See I. N. Popescu, *Gravitation; Pleading for a New Unified Theory of Motion and Fields* (Editrice Nogard, Roma - Italia, 1988) pp. 685-686.

11. See, for example, Ref. 5, pp. 241-301.

8

GRAVITATION AND ANTIGRAVITATION

According to Einstein's mass-energy relation, any energy has a certain mass. But mass is the source of gravitation. Therefore any energy, including gravitational energy, should be a source of gravitation. In this chapter we shall complete our development and generalization of the Newtonian gravitational theory by investigating how gravitational fields are affected by the gravitational energy contained in them. For simplicity, we shall discuss time-independent fields only.[1]

8-1. Gravitational Energy as a Source of Gravitation

The basic equations of Newton's theory of gravitation are the two field laws

$$\nabla \times \mathbf{g} = 0, \tag{5-1.2}$$

$$\nabla \cdot \mathbf{g} = -4\pi G\rho, \tag{5-1.3}$$

and the energy law

$$U = -\frac{1}{8\pi G}\int \mathbf{g}^2 dv. \tag{6-2.33}$$

The energy law can also be expressed in terms of the energy density $U_v = dU/dv$ as

$$U_v = -\frac{\mathbf{g}^2}{8\pi G}. \tag{8-1.1}$$

However, according to Einstein's mass-energy relation

$$U = mc^2, \tag{8-1.2}$$

any energy has a mass given by $m = U/c^2$, where c is the velocity of light. Hence we must conclude that the gravitational energy density given by Eq. (8-1.1) has a mass density

$$\rho_g = -\frac{\mathbf{g}^2}{8\pi Gc^2}. \tag{8-1.3}$$

But then the source of gravitation should be not just the ordinary mass density ρ but the sum of ρ and ρ_g, in which case Eq. (5-1.3) should be replaced by [2,3]

$$\nabla\cdot\mathbf{g} = -4\pi G\rho + \frac{\mathbf{g}^2}{2c^2}. \tag{8-1.4}$$

From now on, we shall assume that the divergence law of gravitation is given not by Eq. (5-1.3) but by Eq. (8-1.4) instead. Observe that the last term in this equation contains the total gravitational field $\mathbf{g}$. This means that the equation takes into account the effect of the gravitational energy upon itself.

Note that the mass density of the gravitational field, ρ_g, is negative. Thus Eq. (8-1.4) indicates that there may exist not only ordinary attractive gravitational fields but also repulsive, or antigravitational, fields. It also indicates that the field outside a uniform spherical mass distribution depends not only on the magnitude of this distribution but also on its internal field, so that such a mass distribution cannot be replaced by an equal point mass at its center, as it can be done in the conventional Newtonian theory. Finally, Eq. (8-1.4) gives us at least a partial explanation for the behavior of gravitational field lines in the gravitational fields. It is well known that the electric field lines in electrostatic fields always have a beginning (on positive charges) and an end (on negative charges). But, according to Eq. (5-1.3), the gravitational field lines have no beginning, they just end on mass elements. A clue to the mystery of their beginning is now given by Eq. (8-1.4): at least some gravitational field lines begin on ρ_g in the space around and within mass distributions.

The basic field equations of the Newtonian gravitational theory, Eqs. (5-1.2) and (5-1.3), are usually solved by means of the gravitational potential φ, defined by

$$\mathbf{g} = -\nabla\varphi. \tag{6-2.10}$$

By combining Eqs. (6-2.10) and (5-1.3), one obtains

$$\nabla^2\varphi = 4\pi G\rho, \tag{6-2.12}$$

which, subject to appropriate boundary conditions, can be solved for a variety of mass distributions. The field **g** can then be found from φ by means of Eq. (6-2.10). In the Newtonian theory one can also use integral methods for finding **g**. Any of the following expressions can be used for finding **g** given by Eqs. (5-1.2) and (5-1.3) [4]

$$\mathbf{g} = -G\int\frac{\rho}{r^2}\mathbf{r}_u dv', \tag{6-2.6}$$

$$\mathbf{g} = G\int\frac{\nabla'\rho}{r}dv', \tag{8-1.5}$$

or

$$\varphi = -G\int\frac{\rho}{r}dv'. \tag{6-2.8}$$

Unfortunately, none of the above techniques or equations can be employed for finding gravitational fields given by Eqs. (5-1.2) and (8-1.4), since Eq. (8-1.4) is nonlinear in **g**. Thus, when the effect of the gravitational energy on the gravitational field is taken into account, one cannot in general find the gravitational field **g** from a given mass distribution ρ. There is, however, a way out of this difficulty: one can postulate a certain field **g** satisfying Eq. (5-1.2) and then from Eq. (8-1.4) one can find the mass distribution

$$\rho = -\frac{\nabla\cdot\mathbf{g}}{4\pi G} + \frac{\mathbf{g}^2}{8\pi Gc^2} \tag{8-1.6}$$

producing this field. Examples of such calculations are given in the next section. One can also obtain approximate solutions for nonlinear gravitational fields by assuming that gravitational energy is entirely due to the true mass, thus ignoring the effect of the gravitational energy upon itself (the true mass is the mass as such, excluding the associated gravitational energy mass).

8-2. Examples of Nonlinear Gravitational Fields

We shall now present illustrative examples demonstrating basic properties of nonlinear gravitational fields. All fields in these examples are spherically symmetric and are in a radial direction. Hence they automatically have a zero curl and thus satisfy Eq. (5-1.2).[5] Of course, even if a field satisfies Eqs. (5-1.2) and (8-1.4), it still may be physically meaningless. Therefore we shall restrict our choice of fields to those that satisfy the following validity conditions:

(a) the energy of the field must be finite,

(b) the field must be finite at $r=0$,

(c) the true mass density ρ must be either positive or zero,
(d) the field must be everywhere continuous.

▼

Example 8-2.1 Find the mass distribution producing the field

$$\mathbf{g} = -G\frac{m}{a^3}\mathbf{r} \quad \text{for} \quad r \le a, \tag{8-2.1}$$

and

$$\mathbf{g} = -G\frac{m}{r^2}\mathbf{r}_u \quad \text{for} \quad r > a. \tag{8-2.2}$$

Note that in the conventional Newtonian theory this field is produced by a sphere of radius a, mass m, and uniform density $\rho = 3m/4\pi a^3$.

Substituting Eqs. (8-2.1) and (8-2.2) into Eq. (8-1.6) and differentiating, we obtain

$$\rho = \frac{m}{4\pi a^3}\left(3 + G\frac{mr^2}{2c^2a^3}\right) \quad \text{for} \quad r \le a, \tag{8-2.3}$$

and

$$\rho = G\frac{m^2}{8\pi c^2 r^4} \quad \text{for} \quad r > a. \tag{8-2.4}$$

An important consequence of this solution is that a $1/r^2$ field is produced not by a sphere, but by a mass distribution extending all the way to infinity (although the greatest mass density is within the sphere; see Fig. 8.1). Another important consequence is that m in Eqs. (8-2.1) and (8-2.2) is not the true mass of the sphere. The true mass of the sphere, obtained by integrating Eq. (8-2.3), is

$$m_0 = m\left(1 + G\frac{m}{10c^2a}\right), \tag{8-2.5}$$

so that

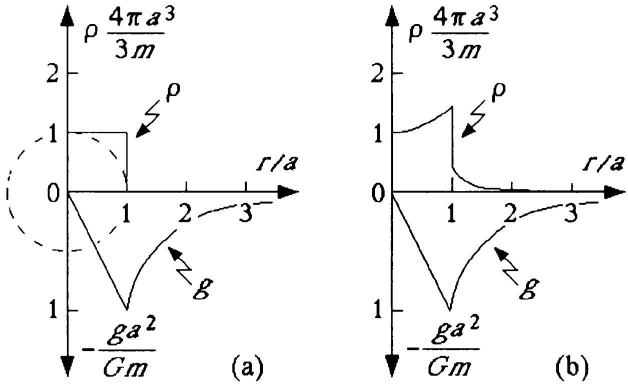

Fig. 8.1 (a) According to Newton's theory, the gravitational field shown in this figure is produced by a mass of uniform density confined to a sphere. (b) According to the nonlinear theory of gravitation, the same field is produced by a mass of variable density occupying all space. (The scale for the field is twice as large as the scale for the mass density.)

$$m = \frac{5c^2a}{G}\left[\left(1 + G\frac{2m_0}{5c^2a}\right)^{1/2} - 1\right]. \tag{8-2.6}$$

If $2Gm_0/5c^2a \ll 1$, which is usually the case, Eq. (8-2.6) can be written as

$$m \approx m_0\left(1 - G\frac{m_0}{10c^2a}\right). \tag{8-2.7}$$

The true mass external to the sphere, obtained by integrating Eq. (8-2.4), is

$$m_e = G\frac{m^2}{2c^2a}. \tag{8-2.8}$$

And the total true mass as seen from $r=\infty$, $m_t=m_0+m_e$, is

$$m_t = m\left(1 + G\frac{3m}{5c^2a}\right). \tag{8-2.9}$$

Example 8-2.2 Find the mass distribution that produces the field given by Eq. (8-2.1) for $r<a$ and produces the field

$$\mathbf{g} = -G\frac{2c^2\mu}{2c^2r^2 - G\mu r}\mathbf{r}_u \quad \text{for} \quad r > a, \tag{8-2.10}$$

where

$$\mu = \frac{2c^2ma}{2c^2a + Gm}. \tag{8-2.11}$$

Note that this expression for μ makes $\mathbf{g}$ continuous at $r=a$.

Since the field for $r\leq a$ is the same as in Example 8-2.1, the mass density ρ for $r\leq a$ is the same as that given by Eq. (8-2.3). Substituting Eq. (8-2.10) into Eq. (8-1.6) and differentiating, we obtain for the mass density in the remaining space

$$\rho = 0, \quad r > a. \tag{8-2.12}$$

Thus the field under consideration is produced by a mass confined to a sphere of radius a.

Let us investigate this field in some detail. If in Eq. (8-2.10) $G\mu/2c^2r \ll 1$, then the equation can be written as

$$\mathbf{g} = -G\frac{\mu}{(1 - G\mu/2c^2r)r^2}\mathbf{r}_u \approx -G\frac{\mu(1 + G\mu/2c^2r)}{r^2}\mathbf{r}_u$$

or

$$\mathbf{g} \approx -G\frac{\mu}{r^2}\mathbf{r}_u - G^2\frac{\mu^2}{2c^2r^3}\mathbf{r}_u. \tag{8-2.13}$$

The first term in this equation is the simple Newtonian gravitational field of a sphere. However, the mass μ in this equation is not the true, or "naked," mass of the sphere. To find the true mass, we need to solve Eq. (8-2.11) for m and then substitute the result into Eq. (8-2.5). This gives for the true mass of the sphere

$$m_0 = \frac{\mu}{1 - G\mu/2c^2a}\left[1 + G\frac{\mu}{10c^2a(1 - G\mu/2c^2a)}\right]. \qquad (8\text{-}2.14)$$

The true mass m_0 is larger than μ, which was to be expected, since μ is the sum of the true mass and the negative mass of the gravitational energy. Of course, the mass responsible for the observable gravitational field outside the sphere is not m_0 but μ.

Let us now assume that the sphere producing the field under consideration is the Sun, and let us change the designation of the mass in Eq. (8-2.13) from μ to the more familiar m. We then have for the gravitational field of the Sun

$$\mathbf{g} \approx -G\frac{m}{r^2}\mathbf{r}_u - G^2\frac{m^2}{2c^2r^3}\mathbf{r}_u. \qquad (8\text{-}2.15)$$

Consider now a planet in an orbit around the Sun. Let us designate the mass of the planet as m'. The gravitational force acting on the planet is then m' multiplied by the right side of Eq. (8-2.15). For a nearly circular orbit, the gravitational force acting on the planet is equal to the centripetal force applied to the planet: $Gmm'/r^2 = m'v^2/r$, where v is the velocity of the planet. Introducing v into Eq. (8-2.15), we therefore can write for the force exerted by the Sun on the planet

$$\mathbf{F} \approx -G\frac{mm'}{r^2}\mathbf{r}_u - G\frac{mm'v^2}{2c^2r^2}\mathbf{r}_u. \qquad (8\text{-}2.16)$$

But this is the same equation as Eq. (7-3.2), except that the last term in it is caused not by the motion of the planet around the Sun, but by the mass of the gravitational energy of the Sun. The effect of this term is the same as in Eq. (7-3.2): it causes a perihelion advance of the planet's orbit. Therefore this term must be added to Eq. (7-3.2), if one wants to determine the total perihelion advance. This is how Eq. (7-3.5) was obtained [Eq. (7-3.5) also takes into account the dependence of planet's mass on the orbital velocity].

Example 8-2.3 Find the mass distribution producing the field

$$\mathbf{g} = -G\frac{m}{a^2}(2e^{1-r/a}-1)\mathbf{r}_u \quad \text{for} \quad a \le r \le 2a. \quad (8\text{-}2.17)$$

Note that this field becomes *antigravitational* for $r > (\ln 2 + 1)a$. Substituting Eq. (8-2.17) into Eq. (8-1.6) and differentiating, we obtain

$$\rho = \frac{m}{4\pi a^3}\left[2(r/a-1)e^{1-r/a} + \frac{2a}{r}(2e^{1-r/a}-1) + G\frac{m}{2c^2 a}(2e^{1-r/a}-1)^2\right]. \quad (8\text{-}2.18)$$

▲

8-3. Properties of Gravitational Fields in Free Space

The most interesting aspect of the effect of the gravitational energy on gravitational fields is the possibility of the existence of mass distributions creating antigravitational fields in free space. Naturally, if such mass distributions are to be stable under gravitational forces alone, the internal gravitational field of the mass distributions must be attractive everywhere within the distributions. The question arises therefore: can there exist a mass distribution producing an attractive field at all points within itself, but a repulsive field outside?

To answer this question, we shall consider the most general expression for a spherically symmetric field,

$$\mathbf{g} = Af(r)\mathbf{r}_u, \quad (8\text{-}3.1)$$

where A is a constant and $f(r)$ is any function of r, and shall determine $f(r)$ for $\rho=0$.

Substituting Eq. (8-3.1) into Eq. (8-1.6) and setting $\rho=0$, we have

$$0 = -\frac{\nabla \cdot \mathbf{g}}{4\pi G} + \frac{\mathbf{g}^2}{8\pi G c^2} = \frac{1}{4\pi G}\left\{-A\nabla \cdot [f(r)\mathbf{r}_u] + \frac{A^2 f^2(r)}{2c^2}\right\}, \qquad (8\text{-}3.2)$$

which upon differentiation and simplification gives

$$\frac{d}{dr}f(r) + \frac{2}{r}f(r) - \frac{A}{2c^2}f^2(r) = 0. \qquad (8\text{-}3.3)$$

The general solution of this equation is

$$f(r) = \frac{2c^2}{Ar + 2Bc^2r^2}, \qquad (8\text{-}3.4)$$

where B is an arbitrary constant.

Thus, by Eqs. (8-3.1) and (8-3.4), the most general expression for a spherically symmetric gravitational field in the region where $\rho = 0$ is

$$\mathbf{g} = \frac{2Ac^2}{Ar + 2Bc^2r^2}\mathbf{r}_u, \qquad (8\text{-}3.5)$$

where A and B are to be determined from the boundary conditions [Condition (d) of Section 8-2].

For this field to be repulsive ($g > 0$) outside some "critical" radius r_c, and attractive ($g < 0$) within r_c, we must have $\mathbf{g} = 0$ at $r = r_c$, or

$$A + 2Bc^2r_c = \infty, \qquad (8\text{-}3.6)$$

which is impossible for a finite r_c. Hence there can be no spherically symmetric antigravitational field outside a mass distribution if the field within the distribution is everywhere attractive. Consequently, a spherical antigravitational body must be held together by some nongravitational forces in addition to the gravitational ones.

Several other important conclusions concerning gravitational fields in mass-free space can be made from Eq. (8-3.5).

First, let us note that for $r \to \infty$, Eq. (8-3.5) reduces to

$$\mathbf{g} = \frac{A}{Br^2}\mathbf{r}_u = \pm G\frac{M}{r^2}\mathbf{r}_u, \tag{8-3.7}$$

where we have set $A/B = \pm GM$. Therefore in the limit $r \to \infty$, the field of a spherical mass m_0 is just a point-mass field of an "effective" positive or negative mass M (M must be determined from boundary conditions at the surface of m_0).

Next, let us consider the possible values of the arbitrary constants in Eq. (8-3.5). To do so, we shall rewrite Eq. (8-3.5) as

$$\mathbf{g} = \frac{2c^2}{r + 2B'c^2r^2}\mathbf{r}_u, \tag{8-3.8}$$

where we have set $B' = B/A$. Let us now assume that the gravitational field represented by Eq. (8-3.8) is created by a spherical mass of radius r_0, and that the field at the surface of the mass is g_0. Substituting r_0 and g_0 into Eq. (8-3.8) and solving it for B', we obtain

$$B' = \frac{2c^2 - g_0r_0}{2g_0c^2r_0^2}. \tag{8-3.9}$$

Assuming that B' in Eq. (8-3.8) is arbitrary, we can have $B' = 0$, $B' < 0$, or $B' > 0$. Let us consider these cases in some detail.

$B' = 0$. Substituting Eq. (8-3.8) with $B' = 0$ into Eq. (6-2.33) (the energy equation) and integrating over all space external to r_0, we obtain $U = -\infty$, in violation of Condition (a) of Section 8-2. Thus $B' = 0$ is impossible, unless the range of validity of Eq. (8-3.8) is limited to a finite region of space, such as a spherical cavity within a spherical mass distribution.

$B' < 0$. This is the condition for the normal (attractive) Newtonian gravitational field. However, there may exist a "critical" distance $r_c = -1/2B'c^2$ for which Eq. (8-3.8) gives

$g=\pm\infty$ ($g>0$ for $r<r_c$ and $g<0$ for $r>r_c$). In this case, the field violates Condition (a) as well as Condition (d). Therefore $B'<0$ with $r_0<r_c$ is also impossible, except, of course, when the region under consideration is a spherical cavity whose radius is smaller than r_c (the field in the cavity is then antigravitational). If $r_0=r_c$, we have the case of a sphere representing a "black hole"[6] of the general relativity theory. However, the resulting field violates Condition (a) and, which is even more important, if $\mathbf{g}_0=-\infty\mathbf{r}_u$ is substituted in Eq. (8-1.6), and Eq. (8-1.6) is integrated over the volume of the sphere (radius r_0), one obtains for the true mass of the sphere $m_0>\infty$, which cannot be. Thus, according to our theory, black holes (and therefore "gravitational collapse"[7]) are impossible (at least for spherically symmetric mass distributions).

$B'>0$. This is the condition for a purely antigravitational field. For this field, Eq. (8-3.9) imposes an important condition on g_0 and r_0:

$$g_0 r_0 < 2c^2. \tag{8-3.10}$$

The significance of this condition will be apparent from the example that follows.

▼

Example 8-3.1 Construct an antigravitational mass distribution by combining mass distributions given by Eqs. (8-2.3) and (8-2.18).

According to Examples 8-2.1 and 8-2.3, the fields associated with the two mass distributions are

$$\mathbf{g} = -G\frac{m}{a^3}\mathbf{r} \qquad \text{for} \qquad r \le a \tag{8-2.1}$$

and

$$\mathbf{g} = -G\frac{m}{a^2}(2e^{1-r/a}-1)\mathbf{r}_u \qquad \text{for} \qquad a \le r \le 2a. \tag{8-2.17}$$

The combined field is continuous at $r=a$ and becomes antigravitational when $r>(\ln 2+1)a$.

For the field to be antigravitational everywhere outside $r=2a$, the condition given by Eq. (8-3.10) must be satisfied. This can be achieved by imposing an appropriate restriction on m. If we substitute g given by Eq. (8-2.17) for g_0 in Eq. (8-3.10) and set $r=r_0=2a$, we find that the restriction is

$$m < \frac{c^2 a}{G(1 - 2/e)}. \qquad (8\text{-}3.11)$$

Let us make

$$m = \frac{3c^2 a}{4G(1 - 2/e)}. \qquad (8\text{-}3.12)$$

The field outside of the mass distribution is given by Eq. (8-3.8). To determine the value for B' appearing in this equation, we use Eqs. (8-3.9) and (8-2.17) with $r=r_0=2a$. After elementary calculation we find that

$$B' = \frac{1}{12c^2 a}. \qquad (8\text{-}3.13)$$

Substituting this B' into Eq. (8-3.8) and eliminating c^2 by means of Eq. (8-3.12), we finally obtain for our antigravitational field

$$\mathbf{g} = G\frac{m}{a^2}\frac{16(1 - 2/e)}{6r/a + (r/a)^2}\mathbf{r}_u \quad \text{for} \quad r \geq 2a. \qquad (8\text{-}3.14)$$

A graphical representation of this field and of the corresponding mass distribution (true mass) is given in Fig. 8.2. Starting at infinity and proceeding toward the origin, we find that from $r=\infty$ to $r=1.69a$ the field is antigravitational (repulsive) with a maximum at $r=2a$. At $r=1.69a$ the field becomes zero. From there on the field is an ordinary gravitational (attractive) field with a minimum at $r=a$ and diminishing to zero at $r=0$.

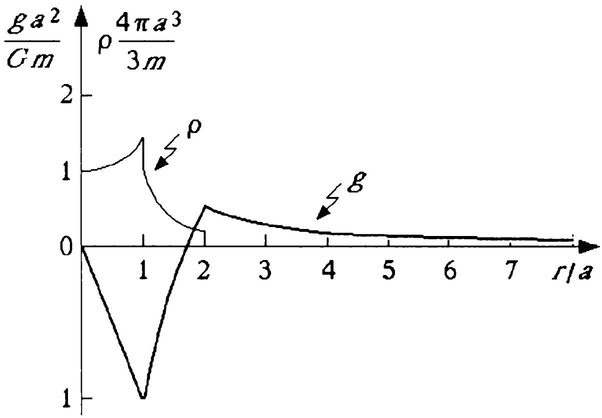

Fig. 8.2 An example of an antigravitational field and of the corresponding mass distribution. (The scale for the field is twice as large as the scale for the mass density.)

Of course, the mass distribution shown in Fig. 8.2 cannot be maintained by gravitational forces alone because, unless the distribution is kept together by some other forces, all the mass located at $r > 1.69a$ would be ejected by antigravitational repulsion, until the radius of the distribution decreases to $r = 1.69a$. The external gravitational (or antigravitational) field of the remaining distribution would then completely disappear, and the distribution would become a "hidden mass" that neither exerts nor experiences any gravitational forces on or from the surrounding bodies.

Observe that m appearing in Eqs. (8-2.1), (8-2.17), and (8-3.14) is not the true mass of the spherical distribution under consideration; it is a quantity associated with the true mass of the central part ($r \leq a$) of the distribution through Eqs. (8-2.5) and (8-2.6). By Eqs. (8-2.5) and (8-3.14), the true mass of the central part is $m_0 \approx 1.3m$.

As can be seen from Fig. 8.2, the maximum density of the true mass occurs at $r=a$. Since, by Eq. (8-3.12), $m=2.48c^2a/G$, the radius of the central part of the mass distribution is $a=Gm/2.48c^2$. By Eq. (8-2.3), we then have for the maximum mass density of the distribution

$$\rho_{\max} \approx \frac{5c^6}{G^3m^2}. \tag{8-3.15}$$

Equation (8-3.15) shows that for a very large mass, the density of an antigravitational mass distribution can be very small. This may be an important factor for the stability of the galaxies in the universe.

▲

8-4. Discussion

There is a widespread belief that the general relativity theory is the definitive theory of gravitation. However, the generalized Newton's theory of gravitation outlined in the three preceding chapters points out a path for an unquestionably viable new inquiry into the nature and properties of gravitational fields and interactions.[8] The generalized Newton's theory is based to a large extent on the idea that the gravitational-cogravitational field is a seat of momentum and energy. One of the consequences of this idea is the supposition, discussed in this chapter, that gravitation is caused not only by a true mass but also by the equivalent mass of the gravitational field energy. Plausible as it is, this supposition is contrary to the general relativity theory. Moreover, even the existence of gravitational field energy is contrary to the general relativity theory. It is important therefore to clarify the reasons why general relativity theory denies the existence of gravitational field energy and it is important to examine the validity of these reasons.

The basic gravitational equation of the general relativity theory is the Einstein's gravitational field equation

$$R_{ik} - \frac{1}{2} R g_{ik} = - G \frac{8\pi}{c^4} T_{ik}. \quad (8\text{-}4.1)$$

The sources of gravitation appear in this equation in the form of the energy-momentum tensor T_{ik}. This tensor includes all types of mass densities and all types of energy densities (electric, magnetic, thermal, etc.) except for the energy density of the gravitational field itself. The determining reason for this is quite simple: in spite of many efforts, no energy-momentum tensor has been found for the gravitational energy (only a "pseudotensor" has been obtained). Various plausibility arguments have therefore been suggested to justify the absence of the gravitational energy as a source of gravitation in Einstein's field equation.[9] Since it would be difficult (if not impossible) to accept the existence of gravitational field energy without accepting this energy as a source of gravitation, these arguments are also the arguments against the presence of gravitational energy in the gravitational field.

The two strongest plausibility arguments for excluding gravitational energy as a source of gravitation are:

(1) Predictions of the general relativity theory obtained with the aid of Einstein's field equation without gravitational energy as a source of gravitation have been found to agree with observations.

(2) Einstein's "equivalence principle" forbids gravitational energy to be a source of gravitation.

However, a careful examination of these arguments shows that neither of them is truly convincing or compelling.

The first argument is easily refuted by the fact that all presently verifiable predictions of the general relativity theory are in the domain of weak fields, where, as it follows from the material presented in this chapter, the effects of the gravitational energy are hardly noticeable.

The second argument appears to be much stronger than the first. What it means is that since, according to Einstein, a gravitational field is equivalent to a certain accelerated frame of reference, and since there apparently is no special energy in the space defined by the accelerated frame of reference, no energy should be present in the space containing the gravitational field (this is known as the "nonlocalizability" of gravitational energy).[9,10] An analysis of this argument shows, however, that it is based on an unprovable premiss and that it can be refuted by reversing it. Indeed, let us suppose that a gravitational field is a seat of gravitational energy. The equivalence principle demands then that a certain energy density would appear in the space defined by the equivalent reference frame. But how will this energy manifest itself? The only presently known way in which it could be detected is by its gravitational effects. However, since the equivalent reference frame is flat and boundless, the "equivalent" energy density, as seen in this frame, must be uniform and must occupy all space. But, as it is well known, a uniformly distributed mass (energy) occupying all space produces no gravitational effects [see Eq. (8-1.5); if $\nabla\rho=0$ or $\nabla\rho_g=0$ everywhere, $\mathbf{g}=0$, too]. Hence the "equivalent" energy is not detectable, or, as an observer in the equivalent reference frame would say, is "absent."

Thus the absence of space energy in an accelerated reference frame does not prove the nonexistence or nonlocalizability of gravitational field energy, and hence the equivalence principle does not forbid its appearance as a source term in Einstein's gravitational field equation. Therefore the exclusion of the gravitational energy as a source of gravitation in the general relativity theory is merely a matter of practical necessity (since no tensor has been found for it). Hence all presently known results of the general relativity theory based on Einstein's field equation cannot be considered as reliable when these results involve gravitational fields whose gravitational-energy mass is comparable with the true mass of the system. And therefore the fact that the results obtained in this chapter are in conflict with the general relativity theory does in no way indicate that these results are wrong. The conflict cannot be resolved by plausibility arguments. Only reliable observational data can truly resolve it.

Let us now summarize what our theory of nonlinear gravitational fields has indicated:

1. The gravitational force acting on a body in a gravitational field is determined not only by the mass of the field-producing body, but also by the gravitational field energy of the field-producing body.

2. Antigravitational bodies can exist in the universe.

3. The mass of the universe, of a galaxy, or of a stellar object can be much larger than the present astrophysical measurements indicate, since there can exist objects of negative or of zero apparent mass. The latter objects would constitute "hidden" masses insofar as they do not produce or experience gravitational effects.

4. "Black holes" cannot exist, and "gravitational collapse" is impossible. Indeed, according to the general relativity theory, a sphere creates an "unescapable" gravitational field and becomes a "black hole" after its radius becomes smaller than the "gravitational radius" [11]

$$r_g = G\frac{2m}{c^2}. \tag{8-4.2}$$

But the radius of the central mass of the mass distribution shown in Fig. 8.2 is *smaller* than the gravitational radius, yet the field at this radius is zero rather than immensely strong, as is required for black holes.

5. Since "gravitational collapse" is impossible, and since antigravitational mass formations are possible, the normal state of the universe appears to be an alternating expansion and contraction.

6. Since a "hidden" mass is an object whose overall rest mass is zero, such a mass could conceivably move with a velocity equal to (or even larger than) the velocity of light.

These are fascinating and intriguing conclusions. Are they true or are they false? Only time will tell.

References and Remarks for Chapter 8

1. This chapter is based on the author's article "Gravitation and Antigravitation" written in 1974 and published in a modified form (after it had been rejected by several journals) in the Proceedings of the West Virginia Academy of Science, **52**, 106-118 (1980). Curiously enough, some time after one of the journals rejected the article (because the article was "unbelievable, of no genuine usefulness, of no aid to learning, of no cultural or historical value, etc.") the same journal accepted for publication an analogous article by a different author. Except for the fact that the latter article did not discuss antigravitation, it was more than just similar to the rejected "Gravitation and Antigravitation" and appeared to had been directly inspired by it.

2. A similar equation including the equivalent mass of the gravitational field energy as well as of the cogravitational field energy [Eq. (6-2.38)] was suggested by F. Hund, "Zugänge zum Verständnis der allgemeinen Relativitätstheorie," Z. Physik, **124**, 742-756 (1947). For simplicity we do not include the cogravitational energy in our discussion, although it may make a significant contribution in the case of rapidly moving or rapidly rotating bodies.

3. A somewhat similar equation (for fields *external* to mass distributions) was suggested by L. Brillouin and R. Lucas, "Le Relation Masse-Énergie en Gravitation," J. Phys. Radium **25**, 229-232 (1966). See also M. Mannheimer, "L'Énergie au Champ de Gravitation," Ann. Phys. (Paris) **1**, 189-194 (1966) and **2**, 57-60 (1967); Léon Brillouin, *Relativity Reexamined* (Academic Press, New York, 1970) pp. 87-95. A related publication is P. C. Peters, "Where is the energy stored in a gravitational field?" Am. J. Phys. **49**, 546-569 (1981).

4. Equation (8-1.5) is the gravitational counterpart of a similar, little-known, electrostatic equation. For the derivation of the electrostatic equation see O. D. Jefimenko, *Electricity and Magnetism,* 2nd ed., (Electret Scientific, Star City, 1989) pp. 101-103.

5. See Ref. 4, p. 60, Problem 2.22.

6. See, for example, H. C. Ohanian and R. Ruffini, *Gravitation and Spacetime*, 2nd ed., (W. W. Norton, New York, 1994) pp. 437-438.

7. See, for example, Ref. 6, pp. 447-448, 489-496.

8. It should be noted that the generalized Newton's theory makes it possible to obtain transformation equations for gravitational and cogravitational fields which make the linear theory of gravitation compatible with the special relativity theory [see O. D. Jefimenko, *Electromagnetic Retardation and Theory of Relativity* (Electret Scientific, Star City, 1997) pp. 158-165, 180, 275-294]. However, the nonlinear gravitational equations discussed in this chapter are not compatible with the special relativity theory. Although there is a widespread opinion that all correct physical theories and equations

must be compatible with the special relativity theory, the incompatibility of the nonlinear gravitational equations with this theory does not mean that the equations are wrong. In this connection it may be noted that there are other examples of perfectly viable equations which are incompatible with the special relativity theory. Maxwell's electromagnetic equations *in their vector form* present the most prominent example of such incompatibility [see O. D. Jefimenko, *Electromagnetic Retardation and Theory of Relativity* (Electret Scientific, Star City, 1997) pp. 158-165]. Furthermore, in the real world, the special relativity theory is itself only approximately correct. This is because this theory is only applicable to inertial systems, but true inertial systems do not really exist. In the end, the only reliable criterion of the correctness (or erroneousness) of our nonlinear gravitational equations is the agreement (or disagreement) of these equations with the experimental data within the range of applicability of the equations.

9. See, for example, C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (W. H. Freeman, San Francisco, 1973) pp. 466-468.

10. Nevertheless, gravitational waves are assumed to carry gravitational energy with them! See, for example, Ref. 6, pp. 241-301.

11. Ref. 6, pp. 438-439.

APPENDIXES

APPENDIX 1
Vector Identities

In the vector identities listed below φ and U are scalar point functions; **A**, **B**, and **C** are vector point functions; X is a scalar or vector point function of primed coordinates and incorporates an appropriate multiplication sign (dot or cross for vectors).

Box product

(V-1) $$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$$

(V-2) $$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}$$

(V-3) $$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = -\mathbf{A}\cdot(\mathbf{C}\times\mathbf{B})$$

"BAC CAB" expansion

(V-4) $$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$

Identities for the calculation of gradient

(V-5) $$\nabla(\varphi U) = \varphi\nabla U + U\nabla\varphi$$

(V-6) $$\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}\times(\nabla\times\mathbf{B}) + (\mathbf{B}\cdot\nabla)\mathbf{A} + \mathbf{B}\times(\nabla\times\mathbf{A})$$

(V-7) $$\nabla\varphi(U_1\cdots U_n) = \sum_{i=1}^{n}\frac{\partial\varphi}{\partial U_i}\nabla U_i$$

Identities for the calculation of divergence

(V-8) $$\nabla\cdot(\varphi\mathbf{A}) = \varphi\nabla\cdot\mathbf{A} + \mathbf{A}\cdot\nabla\varphi$$

(V-9) $$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot\nabla\times\mathbf{A} - \mathbf{A}\cdot\nabla\times\mathbf{B}$$

(V-10) $$\nabla\cdot\mathbf{A}(U_1\cdots U_n) = \sum_{i=1}^{n}\nabla U_i\cdot\frac{\partial\mathbf{A}}{\partial U_i}$$

Identities for the calculation of curl

(V-11) $$\nabla \times (\varphi \mathbf{A}) = \varphi \nabla \times \mathbf{A} + \nabla \varphi \times \mathbf{A}$$

(V-12) $$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A}(\nabla \cdot \mathbf{B}) - (\mathbf{A} \cdot \nabla)\mathbf{B} - \mathbf{B}(\nabla \cdot \mathbf{A})$$

(V-13) $$\nabla \times \mathbf{A}(U_1 \cdots U_n) = \sum_{i=1}^{n} \nabla U_i \times \frac{\partial \mathbf{A}}{\partial U_i}$$

Repeated application of ∇

(V-14) $$\nabla \cdot (\nabla \times \mathbf{A}) = 0$$

(V-15) $$\nabla \times \nabla U = 0$$

(V-16) $$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

Identities for the calculation of line and surface integrals

(V-17) $$\oint \mathbf{A} \cdot d\mathbf{l} = \int \nabla \times \mathbf{A} \cdot d\mathbf{S}$$ (Stokes's theorem)

(V-18) $$\oint U d\mathbf{l} = \int d\mathbf{S} \times \nabla U$$

Identities for the calculation of surface and volume integrals

(V-19) $$\oint \mathbf{A} \cdot d\mathbf{S} = \int \nabla \cdot \mathbf{A}\, dv$$ (Gauss's theorem)

(V-20) $$\oint U d\mathbf{S} = \int \nabla U dv$$

(V-21) $$\oint \mathbf{A} \times d\mathbf{S} = - \int \nabla \times \mathbf{A}\, dv$$

(V-22) $$\oint (\mathbf{A} \cdot \mathbf{B}) d\mathbf{S} - \oint \mathbf{B}(\mathbf{A} \cdot d\mathbf{S}) - \oint \mathbf{A}(\mathbf{B} \cdot d\mathbf{S}) = \int [\mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) - \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})] dv$$

Helmholtz's (Poisson's) theorem

(V-23) $$\mathbf{V} = -\frac{1}{4\pi}\int_{\text{All space}} \frac{\nabla'(\nabla'\cdot\mathbf{V}) - \nabla'\times(\nabla'\times\mathbf{V})}{r}\,dv'$$

Operations with ∇ in Helmholtz's (Poisson's) integrals

(V-24) $$\nabla'\frac{(\mathrm{X})}{r} = \frac{\nabla'(\mathrm{X})}{r} + \mathbf{r}_u\frac{(\mathrm{X})}{r^2}$$

(V-25) $$\nabla\frac{(\mathrm{X})}{r} = -\,\mathbf{r}_u\frac{(\mathrm{X})}{r^2}$$

(V-26) $$\frac{\nabla'(\mathrm{X})}{r} = \nabla\frac{(\mathrm{X})}{r} + \nabla'\frac{(\mathrm{X})}{r}$$

Retarded (causal) integrals

(V-27) $$\mathbf{V} = -\frac{1}{4\pi}\int_{\text{All space}} \frac{\left[\nabla'(\nabla'\cdot\mathbf{V}) - \nabla'\times(\nabla'\times\mathbf{V}) - \dfrac{1}{c^2}\dfrac{\partial^2\mathbf{V}}{\partial t^2}\right]}{r}\,dv$$

(V-28) $$\mathbf{V} = -\frac{1}{4\pi}\int_{\text{All space}} \frac{\left[\nabla'^2\mathbf{V} - \dfrac{1}{c^2}\dfrac{\partial^2\mathbf{V}}{\partial t^2}\right]}{r}\,dv'$$

Operations with ∇ in retarded (causal) integrals

(V-29) $$\nabla'[\mathrm{X}] = [\nabla'\mathrm{X}] + \frac{\mathbf{r}_u}{c}\frac{\partial[\mathrm{X}]}{\partial t}$$

(V-30) $$\nabla[\mathrm{X}] = -\,\frac{\mathbf{r}_u}{c}\frac{\partial[\mathrm{X}]}{\partial t}$$

(V-31) $$[\nabla'\mathrm{X}] = \nabla[\mathrm{X}] + \nabla'[\mathrm{X}]$$

(V-32) $$\frac{[\nabla'\mathrm{X}]}{r} = \nabla\frac{[\mathrm{X}]}{r} + \nabla'\frac{[\mathrm{X}]}{r}$$

APPENDIX 2
Derivations of Some Retarded Integrals

Consider a time-variable vector field **V**. It clearly satisfies the identity

$$\nabla \times (\nabla \times \mathbf{V}) + \frac{1}{c^2}\frac{\partial^2 \mathbf{V}}{\partial t^2} = \nabla \times (\nabla \times \mathbf{V}) + \frac{1}{c^2}\frac{\partial^2 \mathbf{V}}{\partial t^2}. \qquad \text{(A-2.1)}$$

Let us designate the right side of this identity as **K**. We then have

$$\nabla \times (\nabla \times \mathbf{V}) + \frac{1}{c^2}\frac{\partial^2 \mathbf{V}}{\partial t^2} = \mathbf{K}. \qquad \text{(A-2.2)}$$

But the solution of this equation is [see O. D. Jefimenko, *Electricity and Magnetism*, 2nd ed., (Electret Scientific, Star City, 1989) p. 47 and O. D. Jefimenko, *Electromagnetic Retardation and Theory of Relativity* (Electret Scientific, Star City, 1997) pp. 3-14]

$$\mathbf{V} = -\frac{1}{4\pi}\int_{\text{All space}} \frac{[\nabla'(\nabla'\cdot\mathbf{V}) - \mathbf{K}]}{r}dv'. \qquad \text{(A-2.3)}$$

Replacing **K** and omitting the subscript "All space" for simplicity, we obtain

$$\mathbf{V} = -\frac{1}{4\pi}\int \frac{\left[\nabla'(\nabla'\cdot\mathbf{V}) - \nabla'\times(\nabla'\times\mathbf{V}) - \frac{1}{c^2}\frac{\partial^2 \mathbf{V}}{\partial t^2}\right]}{r}dv'. \qquad \text{(A-2.4)}$$

This equation can also be written as

$$\mathbf{V} = -\frac{1}{4\pi}\int \frac{\left[\nabla'^2\mathbf{V} - \frac{1}{c^2}\frac{\partial^2 \mathbf{V}}{\partial t^2}\right]}{r}dv'. \qquad \text{(A-2.5)}$$

Using now vector identity (V-32), we can transform Eq. (A-2.4) into

$$\mathbf{V} = -\frac{1}{4\pi}\int\left\{\nabla\frac{[\nabla'\cdot\mathbf{V}]}{r} + \nabla'\frac{[\nabla'\cdot\mathbf{V}]}{r} - \frac{1}{c^2}\frac{1}{r}\left[\frac{\partial^2\mathbf{V}}{\partial t^2}\right]\right\}dv'$$
$$+ \frac{1}{4\pi}\int\left\{\nabla\times\frac{[\nabla'\times\mathbf{V}]}{r} + \nabla'\times\frac{[\nabla'\times\mathbf{V}]}{r}\right\}dv'. \tag{A-2.6}$$

By vector identities (V-20) and (V-21), the second term in the first integral and the second term in the second integral vanish upon integration, if $\mathbf{V}=0$ at infinity (because of the finite speed of propagation, all meaningful time-dependent fields are zero at infinity). Differentiating the first terms in the two integrals and using vector identity (V-30), we then obtain

$$\mathbf{V} = \frac{1}{4\pi}\int\left\{\left(\frac{[\nabla'\cdot\mathbf{V}]}{r^2} + \frac{1}{rc}\frac{\partial[\nabla'\cdot\mathbf{V}]}{\partial t}\right)\mathbf{r}_u + \frac{1}{c^2}\frac{1}{r}\left[\frac{\partial^2\mathbf{V}}{\partial t^2}\right]\right\}dv'$$
$$+ \frac{1}{4\pi}\int\left\{\frac{[\nabla'\times\mathbf{V}]}{r^2} + \frac{1}{rc}\frac{\partial[\nabla'\times\mathbf{V}]}{\partial t}\right\}\times\mathbf{r}_u dv'. \tag{A-2.7}$$

For electric fields, Eqs. (A-2.3) and (A-2.4) reduce to

$$\mathbf{E} = -\frac{1}{4\pi\varepsilon_0}\int\frac{\left[\nabla'\rho + \dfrac{1}{c^2}\dfrac{\partial\mathbf{J}}{\partial t}\right]}{r}dv'. \tag{A-2.8}$$

For magnetic fields, they reduce to

$$\mathbf{H} = \frac{1}{4\pi}\int\frac{[\nabla'\times\mathbf{J}]}{r}dv'. \tag{A-2.9}$$

Transforming Eqs. (A-2.8) and (A-2.9) in the same way as we transformed Eqs. (A-2.4) and (A-2.6), we obtain Eqs. (1-4.1) and (1-4.2). Similar transformations yield Eqs. (5-5.4) and (5-5.5).

APPENDIX 3
Apparent Electric Charge of Moving Neutral Current-Carrying Conductors

As we know from Ref. 21 of Chapter 2, the "ordinary" time-dependent electric field can be expressed in terms of the retarded scalar potential φ^*. The retarded scalar potential of a point charge moving with velocity v relative to a stationary observer located at a distance r from the charge is [see O. D. Jefimenko, *Electromagnetic Retardation and Theory of Relativity* (Electret Scientific, Star City, 1997) pp. 95-96]

$$\varphi^* = \frac{q}{4\pi\varepsilon_0 r[1-(v^2/c^2)\sin^2\theta]^{1/2}}, \tag{A-3.1}$$

where θ is the angle between the direction of v and the direction of r. For $v \ll c$, the potential can be written as

$$\varphi^* = \frac{q}{4\pi\varepsilon_0 r}\left(1+\frac{v^2}{2c^2}\sin^2\theta\right). \tag{A-3.2}$$

Consider now a short segment of length L of a neutral current-carrying wire initially at rest on the x-axis of rectangular coordinates. Let the midpoint of the segment be at the origin. Let the line density of the positive and the negative charges in the wire be $\lambda = q/L$ and $\lambda' = -q/L$, respectively. Let the current in the wire be due to the motion of positive and negative charges; the positive charges moving with velocity $u \ll c$ in the positive x-direction, the negative charges moving with the same velocity in the negative x-direction. The current in the wire is then

$$I = 2\lambda u = \frac{2qu}{L} \tag{A-3.3}$$

and is in the positive x-direction.

An observer located at a point of the z-axis at a distance $r \gg L$ from the wire measures the electric potential produced by the wire. By Eq. (A-3.2), taking into account that $\sin\theta = 1$, the positive charges of the wire produce at the location of the observer the potential

$$\varphi^{+} = \frac{q}{4\pi\varepsilon_0 r}\left(1 + \frac{u^2}{2c^2}\right), \qquad \text{(A-3.4)}$$

and the negative charges produce the potential

$$\varphi^{-} = -\frac{q}{4\pi\varepsilon_0 r}\left(1 + \frac{u^2}{2c^2}\right). \qquad \text{(A-3.5)}$$

The resulting potential is therefore zero.

Let us now assume that the wire moves with velocity $v \ll c$ along the x-axis. When the wire moves, the velocity of the positive charges in the wire becomes $v+u$, and that of the negative charges becomes $v-u$. Therefore, by Eq. (A-3.2), the potentials produced by the positive and the negative charges of the wire at the point of observation are now, respectively,

$$\varphi^{+} = \frac{q}{4\pi\varepsilon_0 r}\left\{1 + \frac{(v+u)^2}{2c^2}\right\}, \qquad \text{(A-3.6)}$$

and

$$\varphi^{-} = -\frac{q}{4\pi\varepsilon_0 r}\left\{1 + \frac{(v-u)^2}{2c^2}\right\}. \qquad \text{(A-3.7)}$$

The total potential produced by the charges of the moving wire is therefore

$$\varphi = \varphi^{+} + \varphi^{-} = \frac{q}{4\pi\varepsilon_0 r}\,\frac{2uv}{c^2}. \qquad \text{(A-3.8)}$$

Thus a neutral current-carrying wire creates an electric potential (and field) when the wire is moving. The stationary observer attributes this potential to a charge residing on the wire.

By Eqs. (A-3.8) and (A-3.3), the apparent charge on a wire moving in the direction of the current is

$$q^{+}_{\text{apparent}} = \frac{2quv}{c^2} = \frac{ILv}{c^2}. \tag{A-3.9}$$

A similar calculation shows that the apparent charge on a wire moving against the direction of the current is

$$q^{-}_{\text{apparent}} = -\frac{2quv}{c^2} = -\frac{ILv}{c^2}. \tag{A-3.10}$$

Let us now assume that the wire under consideration is the top side of a square-shaped loop of length and width L, whose plane is in the xy-plane of the rectangular coordinates. To the observer at $r \gg L$ the stationary loop constitutes a magnetic dipole of moment

$$\mathbf{m} = -\mu_0 IL^2\mathbf{k}. \tag{A-3.11}$$

But when the loop is moving along the x-axis, the top side of the loop appears to acquire a charge given by Eq. (A-3.9), while the bottom side appears to acquire a charge given by Eq. (A-3.10). The moving loop appears therefore to constitute an electric dipole of moment

$$\mathbf{p}_{\text{apparent}} = \frac{2quvL}{c^2}\mathbf{j} = \frac{IL^2v}{c^2}\mathbf{j} = \frac{mv}{\mu_0 c^2}\mathbf{j} = \frac{\mathbf{v}\times\mathbf{m}}{\mu_0 c^2}. \tag{A-3.12}$$

[The vertical sides of the loop make no contribution to the electric dipole, because $v \perp u$ along the vertical sides.]

Although we have derived Eq. (A-3.12) for a square loop, the result is valid for a flat loop of any shape. This is because any flat loop can be approximated by an array of sufficiently small squares.

See Appendix 4 for a closely related derivation.

APPENDIX 4
Apparent Gravitational Dipole Field of a Moving Cogravitational Dipole

Consider a square frame of zero mass and length L on a side. The frame supports a string of uniformly distributed particles of total mass $4m$ sliding with velocity $u \ll c$ along the sides of the frame (Fig. A4.1). Let the frame be located in the xy-plane of rectangular coordinates with its center at the origin. Let the motion of the particles be as shown in Fig. A4.1.

Just like the "ordinary" time-dependent electric field (see Section 2.5 and Appendix 3), the "ordinary" gravitational field given by the first integral of Eq. (5-5.4) can be expressed in terms of the retarded gravitational scalar potential as

$$\mathbf{g} = -G\int\left\{\frac{[\rho]}{r^2} + \frac{1}{rc}\frac{\partial[\rho]}{\partial\rho}\right\}\mathbf{r}_u dv' = -\nabla\varphi^*. \qquad \text{(A-4.1)}$$

By analogy with Eq. (A-3.2), the retarded gravitational scalar potential for a point mass moving with velocity $v \ll c$ is

$$\varphi^* = -G\frac{m}{r}\left(1 + \frac{v^2}{2c^2}\sin^2\theta\right). \qquad \text{(A-4.2)}$$

Let us find the potential that the sliding particles produce at a point of the y-axis at a distance $r \gg L$ from the frame. Applying Eq. (A-4.2) to the particles on the horizontal sides of the frame ($\theta = \pi/2$), we have

$$\varphi_{\text{horizontal}} = -2G\frac{m}{r}\left(1 + \frac{u^2}{2c^2}\right). \qquad \text{(A-4.3)}$$

Applying Eq. (A-4.2) to the particles on the vertical sides of the frame ($\theta = 0$ or π), we similarly have

$$\varphi_{\text{vertical}} = -2G\frac{m}{r}. \qquad \text{(A-4.4)}$$

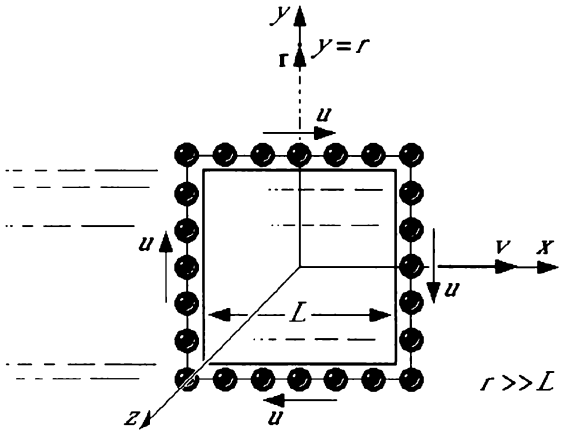

Fig. A4.1 When a closed mass-current drifts as whole, it appears to generate a positive as well as a negative mass. The positive (ordinary) mass appears to be generated by the particles whose resulting velocity is greater than the drift velocity v. The negative mass appears to be generated by the particles whose resulting velocity is smaller than v.

The total potential is therefore

$$\varphi = -4G\frac{m}{r} - G\frac{mu^2}{rc^2}. \qquad \text{(A-4.5)}$$

Thus the particles appear to have acquired a mass

$$m_{\text{apparent}} = \frac{mu^2}{c^2} \qquad \text{(A-4.6)}$$

as a result of their motion along the frame.

Let us now assume that the frame is moving with velocity v in the positive x-direction. The potential due to the particles on the upper horizontal side of the frame ($y>0$) is now

$$\begin{aligned}\varphi_{\text{upper}} &= -G\frac{m}{r}\left\{1 + \frac{(v+u)^2}{2c^2}\right\} \\ &= -G\frac{m}{r}\left\{1 + \frac{v^2}{2c^2} + \frac{uv}{c^2} + \frac{u^2}{2c^2}\right\},\end{aligned} \tag{A-4.7}$$

and the potential due to the particles on the lower horizontal side ($y<0$) is

$$\begin{aligned}\varphi_{\text{lower}} &= -G\frac{m}{r}\left\{1 + \frac{(v-u)^2}{2c^2}\right\} \\ &= -G\frac{m}{r}\left\{1 + \frac{v^2}{2c^2} - \frac{uv}{c^2} + \frac{u^2}{2c^2}\right\}.\end{aligned} \tag{A-4.8}$$

To find the potential due to the particles on the vertical sides of the frame, we must take into account that the velocity of the particles on these sides is $(v^2+u^2)^{1/2}$, and that $\sin\theta$ for these particles is now $v/(v^2+u^2)^{1/2}$. The potential due to these particles is therefore

$$\begin{aligned}\varphi_{\text{vertical}} &= -2G\frac{m}{r}\left(1 + \frac{v^2+u^2}{2c^2}\sin^2\theta\right) \\ &= -2G\frac{m}{r}\left(1 + \frac{v^2}{2c^2}\right).\end{aligned} \tag{A-4.9}$$

Thus the total potential of the particles is now, by Eqs. (A-4.7), (A-4.8), and (A-4.9),

$$\varphi = -4G\frac{m}{r} - 2G\frac{mv^2}{rc^2} - G\frac{mu^2}{rc^2}, \tag{A-4.10}$$

so that, as a result of the motion of the frame, the particles appear to have acquired an additional mass

$$m'_{\text{apparent}} = \frac{2mv^2}{c^2}. \tag{A-4.11}$$

Observe, however, that Eqs. (A-4.7) and (A-4.8) contain inside the parentheses the terms uv/c^2 and $-uv/c^2$. These terms may be interpreted as representing an additional *positive* mass on the upper horizontal side of the frame and an additional *negative* mass on the lower horizontal side of the frame created by the motion of the frame. The two masses give rise to an apparent gravitational *dipole*

$$\mathbf{p}_{\text{apparent}} = \frac{muv}{c^2}L\mathbf{j} = \frac{IL^2v}{c^2}\mathbf{j}, \tag{A-4.12}$$

where I is the mass current of the particles sliding along the frame,

$$I = \lambda u = \frac{m}{L}u. \tag{A-4.13}$$

Equation (A-4.12) is the gravitational counterpart of Eq. (A-3.12) and can be expressed in term of the cogravitational dipole moment $\mathbf{m}$ [see Eq. (6-2.21)] of the particles

$$\mathbf{m} = \frac{4\pi G}{c^2}IL^2\mathbf{k} = \frac{4\pi G}{c^2}\lambda uL^2\mathbf{k} \tag{A-4.14}$$

as

$$\mathbf{p}_{\text{apparent}} = -\frac{\mathbf{v}\times\mathbf{m}}{4\pi G}. \tag{A-4.15}$$

One may think that in our derivation we should have used the relativistic mass of the particles, in which case our equations would have additional u^2/c^2 and v^2/c^2 terms. This is not so. The phenomenon that we have considered here is a consequence of gravitational retardation and has nothing to do with the relativistic equations. What is more, it is now generally accepted that mass does not change with velocity [see O. D. Jefimenko, *Electromagnetic Retardation and Theory of Relativity* (Electret Scientific, Star City, 1997) pp. 193-196 and 205].

APPENDIX 5

Dynamic Electric Field Maps of a Point Charge Moving with Constant Velocity

In 1888, Oliver Heaviside derived the equation for the electric field of a uniformly moving point charge. In modern notation this equation is (see Chapter 1, p. 15)

$$\mathbf{E} = \frac{q(1 - v^2/c^2)}{4\pi\varepsilon_0 r^3[1 - (v^2/c^2)\sin^2\theta]^{3/2}}\mathbf{r}. \qquad \text{(A-5.1)}$$

Heaviside noted that, according to this equation, with increasing velocity of the charge the electric field of the charge concentrates itself more and more about the equatorial plane, $\theta = \pi/2$, and decreases along the line of motion, $\theta = 0$. This effect is shown in Fig. A5.1a by the density of the field lines. It should be noted, however, that field maps such as the map shown in Fig. A5.1a are somewhat misleading. First, it is impossible to represent the intensity of the electric field of a point charge (moving or stationary) on a two-dimensional map by the density of the field lines if *continuous* field lines are used. This is because on a two-dimensional map the radial lines diverge as $1/r$ rather than as $1/r^2$ as they really do in three dimensions. Second, the field shown in Fig. A5.1a is an imaginary "snapshot" that cannot be actually observed (measured) by a single observer, moving or stationary. An observer co-moving with the charge would only see the ordinary electrostatic field of the charge at rest; and a stationary observer would detect a time-dependent electric field rather than the time-independent field shown in Fig. A5.1a. The field shown in Fig. A5.1a could only be observed if many stationary observers (or field-detecting instruments) located around the moving point

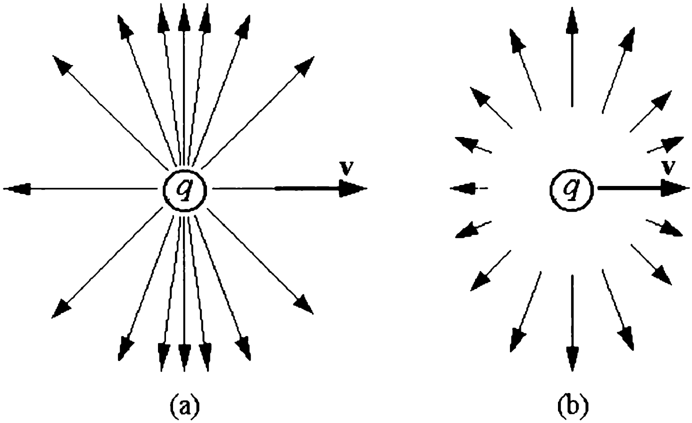

Fig. A5.1 (a) On conventional maps of the electric field of a uniformly moving point charge the magnitude of the electric field is indicated by the density of the field lines. (b) A more accurate way to show the magnitude of the electric field is to use uniformly spaced field vectors of different lengths.

charge would measure simultaneously the field of this charge at the respective points of their location.

An alternative way to represent graphically the electric field of a uniformly moving point charge is to use a map where the intensity of the field at the various points around the charge is represented by the length of the field vectors rather than by the density of the field lines. Such a map is shown in Fig. A5.1b. However, this map, just as the map in Fig. A5.1a, represents the time-independent field that moves with the charge rather than the really important field that a single stationary observer would detect as the charge moves past the observer. To show the latter field, one has to construct a *dynamic electric field map* which depicts the electric field of a moving charge observed at a stationary point as a function of time. Such a map is constructed by using different r's and θ's in Eq. (A-5.1) corresponding to the various positions of the moving charge and by plotting the corresponding electric field vectors, placing their common origin at the point of observation.

A dynamic electric field map is shown in Fig. A5.2. The point of observation is at P. Thirteen different angles θ corresponding to thirteen instantaneous sequential positions

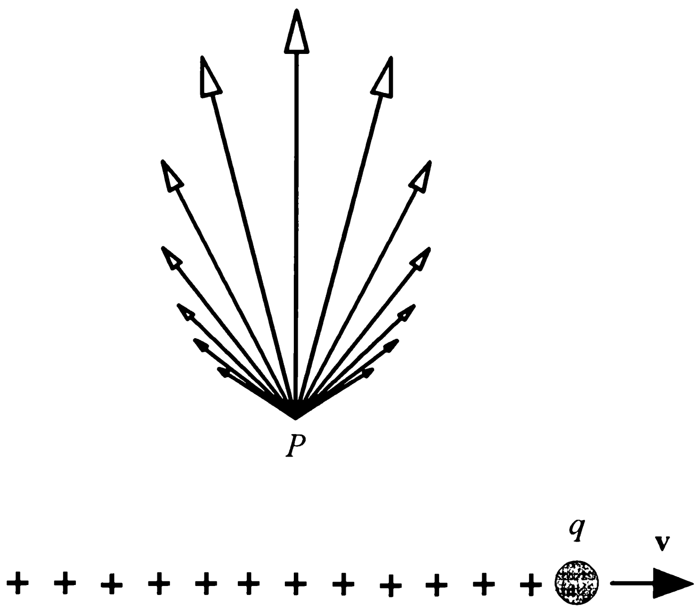

Fig. A5.2 This dynamic map of the electric field of the point charge q moving with velocity v shows electric field vectors at the stationary point of observation P as the charge moves past P. The field vectors correspond to the thirteen sequential positions of the charge indicated on the map. The map is drawn for v = 0.5c.

occupied by the moving charge at the ends of equal time intervals Δt were used for constructing this map. The first twelve sequential positions of the charge are indicated by crosses; the charge is at the last position.

Since the charge moves with constant speed, the instantaneous positions of the charge are separated by equal distances along the trajectory of the charge.

Whereas the two maps shown in Fig. A5.1 are "snapshots" of the electric field co-moving with the charge producing this field, the map shown in Fig. A5.2 is a "multiple exposure" map where the individual field vectors as they would be measured by the stationary observer at time intervals Δt are shown all together. Of course, the entire map represents a very short event. For example, if the point P is located 1 meter above the trajectory of the moving charge, the entire map represents an event that lasts only 10^{-8} seconds.

Closely related to the dynamic electric field map is the "electric field contour curve" representing the locus of the end points of the electric field vectors of a moving point charge as these vectors would be measured by the stationary observer at the point P. The electric field contour curve is strongly affected by the velocity of the charge under consideration. Three electric field contour curves for the same point charge moving with velocities v = 0.01c, v = 0.70c, and v = 0.96c, respectively, are shown in Fig. A5.3.

Dynamic electric field maps and the corresponding contour curves provide a new way of depicting and analyzing the electric field of uniformly moving point charges and reveal several important properties of this field.

It is generally accepted that the field of a point charge moving with a velocity close to the velocity of light becomes a plane wave. However, the dynamic map shown in Fig. A5.2

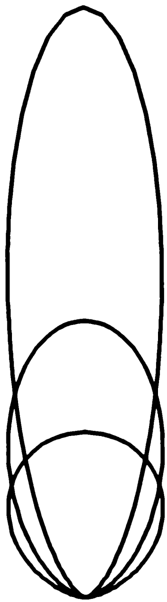

Fig A5.3 The heights of the electric field contour curves are strongly affected by the velocity of the charge, but the widths of the curves do not noticeable depend on v. The three contour curves shown here are for the same point charge moving at velocities v = 0.01c, v = 0.70c, and v = 0.96c, as indicated.

indicates that this is not so. According to Fig. A5.2, the electric field of a fast moving point charge, as seen by a stationary observer, is a momentary pulse, or burst, a sort of electric field explosion, but not a wave in the conventional sense.

It is also generally assumed that for a moving point charge the electric field component in the direction of the motion of the charge rapidly diminishes with increasing velocity of the charge and the component perpendicular to this direction rapidly increases. This assumption is based on Eq. (A-5.1) with $v \to c$ and $\theta = 0$ or $\theta = \pi/2$. However, the contour curves shown in Fig. A5.3 indicate that this assumption is only partially correct. Note that whereas the heights of the curves in Fig. A5.3 are strongly affected by v, the widths of the curves do not noticeable depend on v. Since the half-width of a contour curve represents the maximum value of the field component parallel to the trajectory of the moving charge, it is clear that this value is hardly affected by the speed of the charge. Of course, if P is located on the trajectory of the charge (the x-axis), the only field component observed at P is the x component, and the value of this component diminishes with the distance of the charge from P and with the velocity of the charge, becoming zero for $v \to c$.

Another important effect revealed by the dynamic electric field map shown in Fig. A5.2 and by the contour curves shown in Fig. A5.3 concerns the force exerted by a moving point charge on a stationary charge when the moving charge passes the stationary charge. As is clear from Figs. A5.2 and A5.3, this force lasts only a very short time and is essentially normal to the trajectory of the moving charge. Therefore its main effect on the stationary charge is to give a sudden thrust to the stationary charge in the direction normal to the trajectory of the moving charge. This effect has not yet been discussed in the literature. An interesting possible consequence of this effect is that a rapidly moving electric charge passing through a charged ring can cause a violent, explosion-like, destruction of the ring.

Clearly, of the three graphical representations of the electric field of a uniformly moving point charge discussed above, the dynamic field map is by far the most important and the most informative representation.

APPENDIX 6

Dynamic Gravitational Field Maps of a Point Mass Moving with Constant Velocity

In 1893, Oliver Heaviside obtained the equation for the gravitational field of a uniformly moving point mass (see Chapter 5, p. 88 and Appendix 8, p. 199). In modern notation this equation is

$$\mathbf{g} = -G\frac{m(1 - v^2/c^2)}{r^3[1 - (v^2/c^2)\sin^2\theta]^{3/2}}\mathbf{r}. \qquad \text{(A-6.1)}$$

Heaviside noted that, according to this equation, with increasing velocity of the mass the gravitational field of the mass, just like the electric field of a uniformly moving point charge (see Appendix 5, p. 175), becomes stronger in the direction normal to the line of motion and becomes weaker along the line of motion. This effect is shown in Fig. A6.1a by the density of the gravitational field lines.

The same effect is shown in Fig. A6.1b by the length of the gravitational field vectors rather than by the density of the field lines. However, the two gravitational field maps shown in Fig. A6.1, just like the electric field maps shown in Fig. A5.1, represent the time-independent gravitational field that moves with the mass rather than the really important field that a single stationary observer would detect as the mass moves past the observer. To show the latter field, one has to construct a dynamic gravitational field map (the gravitational analogue of the dynamic electric field map described in Appendix 5). Such a map depicts the gravitational field of the moving mass at a stationary point as a function of time, or, which is the same, as the function of the distance r and the angle θ in Eq. (A-6.1) corresponding to the various positions of the moving mass.

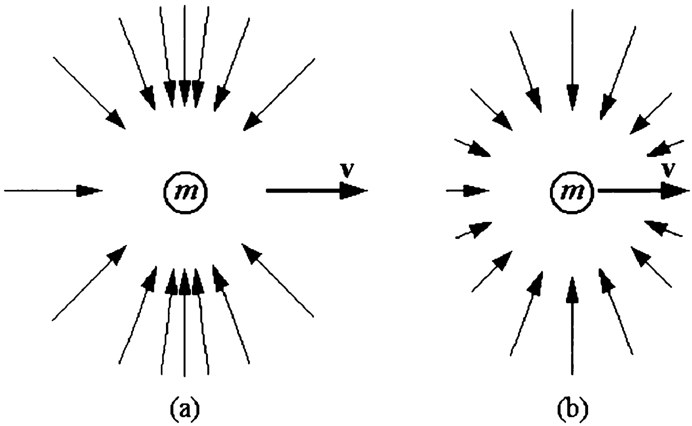

Fig. A6.1 (a) On this map the magnitude of the gravitational field of a uniformly moving point mass m is indicated by the density of the field lines; the points of observation are at the ends of the arrowheuds. (b) A more accurate way to show the gravitational field is to use uniformly spaced field vectors of different lengths.

A dynamic gravitational field map is shown in Fig. A6.2. The point of observation is at P. Thirteen different values for r and θ corresponding to thirteen instantaneous sequential position occupied by the moving mass at the ends of equal time intervals Δt were used for constructing this map. The first position of the mass is indicated by the hollow circle, the mass is at the last (thirteenth) position.

The map shown in Fig. A6.2 is a "multiple exposure" map where the individual field vectors as they would be measured by the stationary observer at equal time intervals Δt are shown all together. Of course, the entire map represents a very short event.

Closely related to the dynamic gravitational field map is the "gravitational field contour curve" representing the locus of the

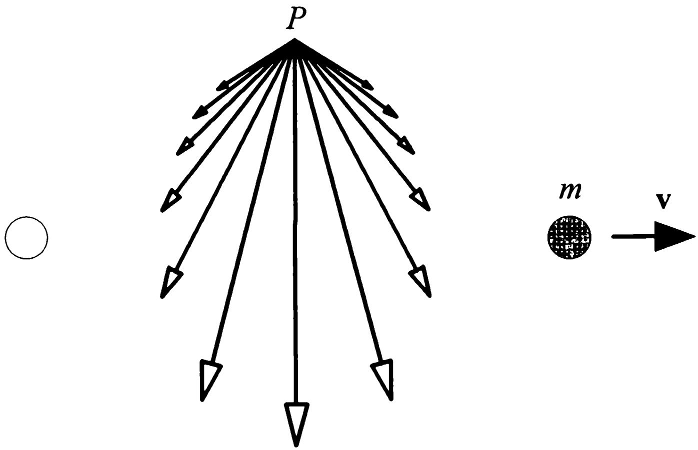

Fig. A6.2 This dynamic map of the gravitational field of the point mass m moving with velocity **v** *shows gravitational field vectors at the stationary point of observation P as the mass moves past P. The field vectors correspond to thirteen sequential positions of the mass. The first position is indicated by the light circle. The mass (dark circle) is at the last position. The map is drawn for* $v = 0.5c$.

end points of the gravitational field vectors of a moving point mass as these vectors would be measured by the stationary observer at the point P. The gravitational field contour curve is strongly affected by the velocity of the mass under consideration. Three gravitational field contour curves for the same point mass moving with velocities $v = 0.01c$, $v = 0.70c$, and $v = 0.96c$, respectively, are shown in Fig. A6.3.

Dynamic gravitational field maps and the corresponding contour curves provide a new way of depicting and analyzing the gravitational field of uniformly moving point masses and reveal several important properties of this field. In particular, it is generally accepted that a moving point mass exerts a gradually

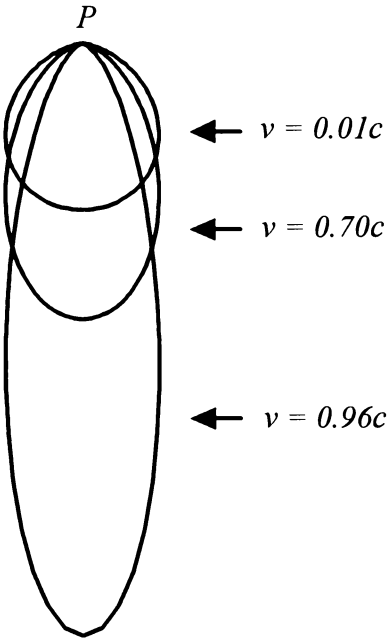

Fig. A6.3 The lengths of the gravitational field contour curves are strongly affected by the velocity of the mass, but the widths of the curves do not noticeable depend on v. The three contour curves shown here are for the same point mass moving at velocities v = 0.01c, v = 0.70c, and v = 0.96c, as indicated.

changing force on a stationary mass. However, according to Figs. A6.2 and A6.3, the gravitational field of a fast moving point mass, as seen by a stationary observer, is a momentary pulse, or burst, a sort of gravitational field explosion. Hence the force exerted by a fast moving mass passing close to a stationary mass may have a distractive effect on the stationary mass, breaking up the stationary mass by the very strong tidal forces. This effect has not yet been discussed in the literature.

APPENDIX 7

Gravitational Forces According to the Generalized Newton's Theory

As was explained in Chapter 5 (pp. 87 and 92), when Newton's gravitational theory is generalized to time-dependent system, gravitational interaction between two bodies is described not by one single force, as in the original Newton's theory, but by an intricate juxtaposition of several different forces. Mathematically, these forces result from Eqs. (5-5.4), (5-5.5), (6-2.23) and (6-2.27). When Eqs. (5-5.4) and (5-5.5) are written as five separate integrals, they become

$$\mathbf{g} = -G\int \frac{[\rho]}{r^2}\mathbf{r}_u dv' - G\int \frac{1}{rc}\frac{\partial[\rho]}{\partial t}\mathbf{r}_u dv' + \frac{G}{c^2}\int \frac{1}{r}\left[\frac{\partial \mathbf{J}}{\partial t}\right] dv', \quad \text{(A-7.1)}$$

and

$$\mathbf{K} = -\frac{G}{c^2}\int \frac{[\mathbf{J}]}{r^2}\times \mathbf{r}_u dv' - \frac{G}{c^2}\int \frac{1}{rc}\frac{\partial[\mathbf{J}]}{\partial t}\Big\}\times \mathbf{r}_u dv'. \quad \text{(A-7.2)}$$

Each of these integrals represents a force field. Therefore, according to the generalized Newton's theory, gravitational interactions between two bodies involve at least five different forces. Let us consider the physical nature of these forces.

First let us consider Eq. (A-7.1). The field represented by the first integral of this equation is the ordinary Newton's gravitational field created by the mass distribution ρ corrected for the finite speed of the propagation of the field (this is indicated by the square brackets — the retardation symbol — in the numerator). The field represented by the second integral is created by a mass whose density varies with time. Like the ordinary Newton's gravitational field, these two fields are directed toward the masses which create them. The field represented by the last integral in Eq. (A-7.1) is created by a mass current whose magnitude and/or direction varies with time. The direction of this field is parallel to the direction along which the mass current increases. All three fields in Eq. (A-7.1) act on stationary as well as on moving masses.

Consider now Eq. (A-7.2). The first integral in this equation represents the cogravitational field created by the mass current. The direction of this field is normal to the mass current vector. The second integral represents the field created by a time-variable mass current. The direction of this field is normal to the direction along which the mass current increases. Both fields in Eq. (A-7.2) act on moving masses only.

If the mass under consideration does not move and does not change with time, then there is no retardation and no mass current. In this case both integrals in Eq. (A-7.2) vanish and only the first integral remains in Eq. (A-7.1). As a result, one simply obtains the integral representing the ordinary Newton's gravitational field. Thus, the ordinary Newton's gravitational theory is a special case of the generalized theory, as it should be.

As far as the gravitational interaction between two masses is concerned, the meaning of the five integrals discussed above can be explained with the help of Fig. A7.1. The upper part of Fig. A7.1 shows the force which the mass m_1 experiences under the action of the mass m_2 according to the ordinary Newton's theory. The lower part of Fig. A7.1 shows five forces which the same mass m_1 experiences under the action of the mass m_2 according to the generalized Newton's theory. The time for which the positions of the two masses and the force experienced by m_1 are observed is indicated by the letter t. Let us note first of all that, according to the ordinary Newton's theory, the mass m_1 is subjected to one single force directed to the mass m_2 at its present location, that is, to its location at the time t. However,

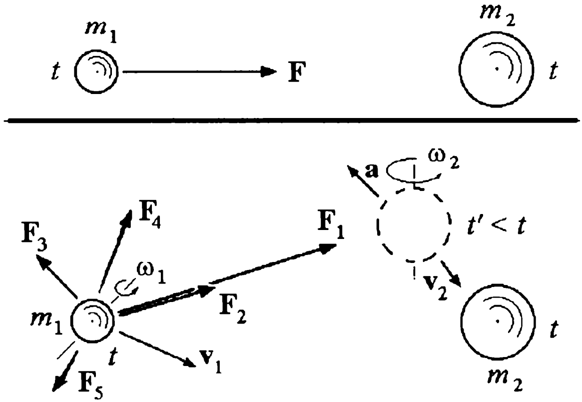

Fig. A7.1 The upper part of this figure shows the force that the mass m_1 experiences under the action of the mass m_2 according to the ordinary Newton's theory. The lower part shows five forces which the same mass m_1 experiences under the action of the mass m_2 according to the generalized Newton's theory. See text for complete explanation.

according to the generalized Newton's theory, all forces acting on the mass m_1 are associated not with the position of the mass m_2 at the time of observation, but with the position of m_2 at an earlier time $t' < t$. Therefore, the magnitude of the mass m_2, its position and its state of motion at the present time t have no effect at all on the mass m_1.

The subscripts identifying the five forces shown in the lower part of Fig. A7.1 correspond to the five integrals in the Eqs. (A-7.1) and (A-7.2). The force $\mathbf{F}_1$ is associated simply with the mass m_2 and differs from the ordinary Newton's gravitational force only insofar as it is directed not to the mass m_2 at its present position, but to the place where m_2 was located at the past time t'. The force $\mathbf{F}_2$ is associated with the variation of the density of the mass m_2 with time; the direction of this force is the same as that of $\mathbf{F}_1$. The force $\mathbf{F}_3$ is associated with the time variation of the mass current produced by m_2; this force is directed along the acceleration vector $\mathbf{a}$ (or along the velocity vector $\mathbf{v}_2$) which the mass m_2 had at the time t'. The three forces are produced by the gravitational field $\mathbf{g}$ (if m_2 is a point mass moving at constant velocity, $\mathbf{g}$ and the resultant of the three forces are directed toward the *present position* of m_2; see p. 92).

The forces $\mathbf{F}_4$ and $\mathbf{F}_5$ are due to the cogravitational field $\mathbf{K}$. The force $\mathbf{F}_4$ is associated with the mass current created by the mass m_2 and with the velocity of the mass m_1. Its direction is normal to the velocity vector $\mathbf{v}_2$ which the mass m_2 had at the time t' and normal to the velocity vector $\mathbf{v}_1$ which the mass m_1 has at the present time t. The force $\mathbf{F}_5$ is associated with the velocity of the mass m_1 and with the variation of the mass current of the mass m_2 with time; the direction of this force is normal to the acceleration vector (or to the velocity vector) that the mass m_2 had at the time t' and normal to the velocity vector that the mass m_1 has at the present time t. Although not shown in Fig. A7.1, additional forces associated with the rotation of m_2 and m_1 (angular velocities ω_2 and ω_1) are generally involved in the interaction between the two masses due to the field $\mathbf{K}$.

The forces $\mathbf{F}_2$, $\mathbf{F}_3$, $\mathbf{F}_4$, and $\mathbf{F}_5$ are usually much weaker than the force $\mathbf{F}_1$ because of the presence of the speed of gravitation c (generally assumed to be the same as the speed of light) in the denominators of the integrals representing the fields responsible for these four forces. This means that only when the translational or rotational velocity of m_2 or m_1 is close to c, the forces $\mathbf{F}_2$, $\mathbf{F}_3$, $\mathbf{F}_4$, and $\mathbf{F}_5$ are significant. Of course, the cumulative effect of these forces in long-lasting gravitational systems (such as the Solar system, for example) may be significant regardless of the velocities of the interacting masses.

APPENDIX 8

This reproduction of Heaviside's article is an unedited copy of the original, except that some formulas and all vector equations have been converted to modern notation.

A GRAVITATIONAL AND ELECTROMAGNETIC ANALOGY.

BY OLIVER HEAVISIDE.

[Part I, The Electrician, **31**, 281-282 (1893)]

To form any notion at all of the flux of gravitational energy, we must first localise the energy. In this respect it resembles the legendary hare in the cookery book. Whether the notion will turn out to be a useful one is a matter for subsequent discovery. For this, also, there is a well-known gastronomical analogy.

Now, bearing in mind the successful manner in which Maxwell's localisation of electric and magnetic energy in his ether lends itself to theoretical reasoning, the suggestion is very natural that we should attempt to localise gravitational energy in a similar manner, its density to depend upon the square of the intensity of the force, especially because the law of the inverse squares is involved throughout.

Certain portions of space are supposed to be occupied by matter, and its amount is supposed to be invariable. Furthermore, it is assumed to have personal identity, so that the position and motion of a definite particle of matter are definite, at any rate relative to an assumed fixed space. Matter is recognised by the property of inertia, whereby it tends to persist in the state of motion it possesses; and any change in the motion is ascribed to the action of force, of which the proper measure is, therefore, the rate of change of quantity of motion, or momentum.

Let ρ be the density of matter, and **e** the intensity of force, or the force per unit matter, then

$$\mathbf{F} = \mathbf{e}\rho \tag{1}$$

expresses the moving force on ρ, which has its equivalent in increase of the momentum. There are so many forces nowadays of a generalised nature, that perhaps the expression "moving force" may be permitted for distinctness, although it may have been formerly abused and afterwards tabooed.

Now the force **F**, or the intensity **e**, may have many origins, but the only one we are concerned with here is the gravitational force. This appears to depend solely upon the distribution of the matter, independently of other circumstances, and its operation is concisely expressed by Newton's law, that there is a mutual attraction between any two particles of matter, which varies as the product of their masses and inversely as the square of their distance. Let **e** now be the intensity of gravitational force, and **F** the resultant moving force, due to all the matter. Then **e** is the space-variation of a potential, say,

$$\mathbf{e} = \nabla P \tag{2}$$

and the potential is found from the distribution of matter by

$$P = \mathrm{Pot}\,\frac{\rho}{c} = \sum \frac{\rho}{4\pi c r}, \tag{3}$$

where c is a constant. This implies that the speed of propagation of the gravitative influence is infinitely great.

Now when matter is allowed to fall together from any configuration to a closer one, the work done by the gravitational forcive is expressed by the increase made in the quantity $\Sigma(\rho P/2)$. This is identically the same as the quantity $\Sigma(ce^2/2)$ summed through all space. If, for example, the matter be given initially in a state of infinitely fine division, infinitely widely separated, then the work done by the gravitational forcive in passing to any other configuration is $\Sigma(\rho P/2)$ or $\Sigma(ce^2/2)$, which therefore expresses the "exhaustion of potential energy." We may therefore assume that $ce^2/2$ expresses the exhaustion of potential energy per unit volume of the medium. The equivalent of the exhaustion of potential energy is, of course, the gain of kinetic energy, if no other forces have been in action.

We can now express the flux of energy. We may compare the present problem with that of the motion of electrification. If moved about slowly in a dielectric, the electric force is appreciably the static distribution. Nevertheless, the flux of energy depends upon the magnetic force as well. It may, indeed, be represented in another way, without introducing the magnetic force, but then the formula would not be sufficiently comprehensive to suit other cases. Now what is there analogous to magnetic force in the gravitational case? And if it have its analogue, what is there to correspond with electric current? At first glance it might seem that the whole of the magnetic side of electromagnetism was absent in the gravitational analogy. But this is not true.

Thus, if $\mathbf{u}$ is the velocity of ρ, then $\rho\mathbf{u}$ is the density of a current (or flux) of matter. It is analogous to a convective current of electrification. Also, when the matter ρ enters any region through its boundary, there is a simultaneous convergence of gravitational force into that region proportional to ρ. This is expressed by saying that if

$$\mathbf{C} = \rho\mathbf{u} - c\frac{\partial \mathbf{e}}{\partial t}, \tag{4}$$

then **C** is a circuital flux. It is the analogue of Maxwell's true current; for although Maxwell did not include the convective term $\rho\mathbf{u}$, yet it would be against his principles to ignore it. Being a circuital flux, it is the curl of a vector, say

$$\nabla \times \mathbf{h} = \rho\mathbf{u} - c\frac{\partial \mathbf{e}}{\partial t}. \tag{5}$$

This defines **h** except as regards its divergence, which is arbitrary, and may be made zero. Then **h** is the analogue of magnetic force, for it bears the same relation to flux of matter as magnetic force does to convective current. We have

$$\mathbf{h} = \nabla \times (\text{Pot}\,\mathbf{C}) = \nabla \times \mathbf{A}, \tag{6}$$

if $\mathbf{A} = \text{Pot}\,\mathbf{C}$. But, since instantaneous action is here involved, we may equally well take

$$\mathbf{A} = \text{Pot}(\rho\mathbf{u}), \tag{7}$$

and its curl will be **h**. Thus, whilst the ordinary potential P is the potential of the matter, the new potential **A** is that of its flux.

Now if we multiply (5) by **e**, we obtain

$$\mathbf{e}\cdot(\nabla \times \mathbf{h}) = \mathbf{e}\cdot(\rho\mathbf{u}) - \mathbf{e}\cdot c\frac{\partial \mathbf{e}}{\partial t}, \tag{8}$$

or, which is the same,

$$\nabla\cdot(\mathbf{e}\times\mathbf{h}) = \mathbf{F}\cdot\mathbf{u} - \frac{\partial U}{\partial t}, \tag{9}$$

if $U = c\mathbf{e}^2/2$. But $\partial U/\partial t$ represents the rate of exhaustion of potential energy, so $-\,\partial U/\partial t$ represents its rate of increase, whilst $\mathbf{F}\cdot\mathbf{u}$ represents the activity of the force on ρ, increasing its kinetic energy. Consequently, the vector $\mathbf{e} \times \mathbf{h}$ expresses the flux of gravitational energy. More strictly, any circuital flux whatever may be added. This $\mathbf{e} \times \mathbf{h}$ is analogous to the electromagnetic $\mathbf{E} \times \mathbf{H}$ found by Poynting and myself. But there is a reversal of direction. Thus, comparing a single moving particle of matter with a similarly-moving electric charge, describe a sphere round each. Let the direction of motion be the axis, the positive pole being at the forward end. Then in the electrical case the magnetic force follows the lines of latitude with positive rotation about the axis, and the flux of energy coincides with the lines of longitude from the negative pole to the positive. But in the gravitational case, although $\mathbf{h}$ still follows the lines of latitude positively, yet since the radial $\mathbf{e}$ is directed to instead of from the centre, the flux of energy is along the lines of longitude from the positive pole to the negative. This reversal arises from all matter being alike and attractive, whereas like electrifications repel one another.

The electromagnetic analogy may be pushed further. It is as incredible now as it was in Newton's time that gravitative influence can be exerted without a medium; and, granting a medium, we may as well consider that it propagates in time, although immensely fast. Suppose, then, instead of instantaneous action, which involves

$$\nabla \times \mathbf{e} = 0, \tag{10}$$

we assert that the gravitational force $\mathbf{e}$ in ether is propagated at a single finite speed v. This requires that

$$v^2 \nabla^2 \mathbf{e} = \frac{\partial^2 \mathbf{e}}{\partial t^2}, \tag{11}$$

for this is the general characteristic of undissipated propagation at finite speed. Now,

$$\nabla^2 = \nabla(\nabla \cdot) - \nabla \times (\nabla \times),$$

so in space free from matter we have

$$-\ v^2\nabla\times(\nabla\times\mathbf{e}) = \frac{\partial^2\mathbf{e}}{\partial t^2}. \tag{12}$$

But we also have, by (5),

$$-\ \nabla\times\mathbf{h} = c\frac{\partial\mathbf{e}}{\partial t}, \tag{13}$$

away from matter. This gives a second value to $\partial^2\mathbf{e}/\partial t^2$, when we differentiate (13) to the time, say

$$\frac{\partial^2\mathbf{e}}{\partial t^2} = -\ \frac{1}{c}\nabla\times\frac{\partial\mathbf{h}}{\partial t}. \tag{14}$$

So, by (12) and (14), and remembering that we have already chosen **h** circuital, we derive

$$cv^2\nabla\times\mathbf{e} = \frac{\partial\mathbf{h}}{\partial t}. \tag{15}$$

Or, if μ is a new constant, such that

$$\mu c v^2 = 1, \tag{16}$$

then (15) may be written in the form

$$\nabla\times\mathbf{e} = \mu\frac{\partial\mathbf{h}}{\partial t}. \tag{17}$$

To sum up, the first circuital law (5), or

$$\nabla\times\mathbf{h} = \rho\mathbf{u} - c\frac{\partial\mathbf{e}}{\partial t}. \tag{18}$$

leads to a second one, namely (17), if we introduce the hypothesis of propagation at finite speed. This, of course, might be inferred from the electromagnetic case.

In order that the speed v should be not less than any value that may be settled upon as the least possible, we have merely to make μ be of the necessary smallness. The equation of activity becomes, instead of (9),

$$\nabla \cdot (\mathbf{e} \times \mathbf{h}) = \mathbf{F} \cdot \mathbf{u} - \frac{\partial U}{\partial t} - \frac{\partial T}{\partial t}, \qquad (3)$$

if $T = \mu \mathbf{h}^2/2$. The negative sign before the time-increase of this quantity points to exhaustion of energy, as before. If so, we should still represent the flux of energy by $\mathbf{e} \times \mathbf{h}$. But, of course, T is an almost vanishing quantity when μ is small enough, or v big enough. Note that $\mathbf{h}$ is not a negligible quantity, though the product $\mu\mathbf{h}$ is. Thus results will be sensibly as in the common theory of instantaneous action, although expressed in terms of wave-propagation. Results showing signs of wave-propagation would require an inordinately large velocity of matter through the ether. It may be worth while to point out that the lines of gravitational force connected with a particle of matter will no longer converge to it uniformly from all directions when the velocity v is finite, but will show a tendency to lateral concentration, though only to a sensible extent when the velocity of the matter is not an insensible fraction of v.

The gravitational-electromagnetic analogy may be further extended if we allow that the ether which supports and propagates the gravitational influence can have a translational motion of its own, thus carrying about and distorting the lines of force. Making allowance for this convection of $\mathbf{e}$ by the medium, with the concomitant convection of $\mathbf{h}$, requires us to turn the circuital laws (17), (18) to

$$\nabla \times (\mathbf{e} + \mu \mathbf{q} \times \mathbf{h}) = \mu \frac{\partial \mathbf{h}}{\partial t}, \qquad (19)$$

$$\nabla\times(\mathbf{h} + c\mathbf{e}\times\mathbf{q}) = \rho\mathbf{u} - c\frac{\partial\mathbf{e}}{\partial t}, \tag{20}$$

where **q** is the velocity of the medium itself.

It is needless to go into detail, because the matter may be regarded as a special and simplified case of my investigation of the forces in the electromagnetic field, with changed meanings of the symbols. It is sufficient to point out that the stress in the field now becomes prominent as a working agent. It is of two sorts, one depending upon **e** and the other upon **h**, analogous to the electric and magnetic stresses. The one depending upon **h** is, of course, insignificant. The other consists of a pressure parallel to **e** combined with a lateral tension all round it, both of magnitude $c\mathbf{e}^2/2$. This was equivalently suggested by Maxwell. Thus two bodies which appear to attract are pushed together. The case of two large parallel material planes exhibits this in a marked manner, for **e** is very small between them, and relatively large on their further sides.

But the above analogy, though interesting in its way, and serving to emphasise the non-necessity of the assumption of instantaneous or direct action of matter upon matter, does not enlighten us in the least about the ultimate nature of gravitational energy. It serves, in fact, to further illustrate the mystery. For it must be confessed that the exhaustion of potential energy from a universal medium is a very unintelligible and mysterious matter. When matter is infinitely widely separated, and the forces are least, the potential energy is at its greatest, and when the potential energy is most exhausted, the forces are most energetic!

Now there is a magnetic problem in which we have a kind of similarity of behaviour, viz., when currents in material circuits are allowed to attract one another. Let, for completeness, the initial state be one of infinitely wide separation of infinitely small filamentary currents in closed circuits. Then, on concentration to any other state, the work done by the attractive forces is represented by $\Sigma\mu\mathbf{H}^2/2$, where μ is the inductivity and **H** the magnetic force. This has its equivalent in the energy of motion of the circuits, or may be imagined to be so converted, or else wasted by friction, if we like. But, over and above this energy, the same amount, $\Sigma\mu\mathbf{H}^2/2$, represents the energy of the magnetic field, which can be got out of it in work. It was zero at the beginning. Now, as Lord Kelvin showed, this double work is accounted for by extra work in the batteries or other sources required to maintain the currents constant. (I have omitted reference to the waste of energy due to electrical resistance, to avoid complications.) In the gravitational case there is a partial analogy, but the matter is all along assumed to be incapable of variation, and not to require any supply of energy to keep it constant. If we asserted that $c\mathbf{e}^2/2$ was stored energy, then its double would be the work done per unit volume by letting bodies attract from infinity, without any apparent source. But it is merely the exhaustion of potential energy of unknown amount and distribution.

Potential energy, when regarded merely as expressive of the work that can be done by forces depending upon configuration, does not admit of much argument. It is little more than a mathematical idea, for there is scarcely any physics in it. It explains nothing. But in the consideration of physics in general, it is scarcely possible to avoid the idea that potential energy should be capable of localisation equally as well as kinetic. That the potential energy may be itself ultimately kinetic is a separate question. Perhaps the best definition of the former is contained in these words :—Potential energy is energy that is not known to be kinetic. But, however this be, there is a practical distinction between them which it is found useful to carry out. Now, when energy can be distinctly localised, its flux can also be traced (subject to circuital indeterminateness, however). Also, this flux of energy forms a useful working idea when action at a distance is denied (even though the speed of transmission be infinitely great, or be assumed to be so). Any distinct and practical localisation of energy is therefore a useful step, wholly apart from the debatable question of the identity of energy advocated by Prof. Lodge.

From this point of view, then, we ought to localise gravitational energy as a preliminary to a better understanding of that mysterious agency. It cannot be said that the theory of the potential energy of gravitation exhausts the subject. The flux of gravitational energy in the form above given is, perhaps, somewhat more distinct, since it considers the flux only and the changes in the amount localised, without any statement of the gross amount. Perhaps the above analogy may be useful, and suggest something better.

[Part II, The Electrician, **31**, 359 (1893)]

In my first article on this subject (*The Electrician,* July 14, 1893, p.281), I partly assumed a knowledge on the part of the reader of my theory of convective currents of electrification ("Electrical Papers," Vol. II., p. 495 and after), and only very briefly mentioned the modified law of the inverse squares which is involved, viz., with a lateral concentration of the lines of force. The remarks of the Editor[1] and of Prof. Lodge[2] on gravitational aberration, lead me to point out now some of the consequences of the modified law which arises when we assume that the ether is the working agent in gravitational effects, and that it propagates disturbances at speed v in the manner supposed in my former article. There is, so far as I can see at present, no aberrational

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[1] *The Electrician*, July 14, p. 277, and July 23, p. 340.

[2] *The Electrician*, July 28, p. 347.

effect, but only a slight alteration in the intensity of force in different directions round a moving body considered as an attractor.

Thus, take the case of a big Sun and small Earth, of masses S and E, at distance r apart. Let f be the unmodified force of S on E, thus

$$f = \frac{SE}{4\pi r^2 c}, \tag{1}$$

using rational units in order to harmonise with the electromagnetic laws when rationally expressed. Also, let F be the modified force when the Sun is in motion at speed u through the ether. Then[3]

$$F = f \times \frac{1 - s}{(1 - s\sin^2\theta)^{3/2}}, \tag{2}$$

where s is the small quantity u^2/v^2, and θ is the angle between r and the line of motion. ("Electrical Papers," Vol. II., pp. 495, 499).

Therefore, if the Sun is at rest, there is no disturbance of the Newtonian law, because its " field of force" is stationary. But if it has a motion through space, there is a slight weakening of the force in the line of motion, and a slight strengthening equatorially. The direction is still radial.

To show the size of the effect, let

$$\begin{aligned} u &= 3 \times 10^7 \text{ centim. per sec.} \\ v &= 3 \times 10^{10} \text{ centim. per sec.} \end{aligned} \tag{3}$$

This value of u is not very different from the speed attributed to fast stars, and the value of v is the speed of light itself.

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[3] This is the case of steady motion. There is no simple formula when the motion is unsteady.

So we have

$$s = \frac{u^2}{v^2} = \frac{1}{10^6}, \tag{4}$$

i.e., one millionth. All perturbing forces of the first order are, therefore, of the order of magnitude of only one-millionth of the full force, even when the speed of propagation is as small as that of light.

The simplest case is when the common motion of the Sun and Earth is perpendicular to the plane of the orbit. Then $\theta = \pi/2$, all round the orbit, and

$$F = f(1 + s/2), \tag{5}$$

showing increase in the force of attraction of S on E of one two-millionth part, without alteration of direction or variation in the orbit.[4]

But when the common motion of the Sun and Earth is in their plane, θ varies from 0 to 2π in a revolution, so that the attraction on E, whilst towards the Sun's centre, always undergoes a periodic variation from

$$F = f(1 - s) \tag{6}$$

when $\theta = 0$, to

$$F = f(1 + s/2), \tag{7}$$

---

[4] But Prof. Lodge tells me that our own particular Sun is considered to move only 10·9 miles per second. This is stupendously slow. The size of s is reduced to about 1/360 part of that in the text, and the same applies to the corrections depending upon it.

when $\theta = \pi/2$. The extreme variation is, therefore, $3sf/2$, according to the data used. The result is a slight change in the shape of the orbit.

But, to be consistent, having made v finite by certain suppositions, we should carry out the consequences more fully, and allow not merely for the change in the Newtonian law, as above, but for the force brought in by the finiteness of v which is analogous to the "electromagnetic force." This is very small truly, but so is the above change in the Newtonian law, and since they are of the same order of magnitude, we should also count the auxiliary force. Call it $\mathbf{G}$. Then

$$\mathbf{G} = F\frac{xqu}{v^2}\mathbf{q}_1 \times (\mathbf{r}_1 \times \mathbf{u}_1), \tag{8}$$

where F is as before, in (2) above, q is the actual speed of the Earth (not the same as u), and in the third vectorial factor $\mathbf{q}_1$, $\mathbf{u}_1$, and $\mathbf{r}_1$ are unit vectors drawn parallel to the direction of the Earth's motion, of the Sun's motion, and from the Sun to the Earth. We see at once that the order of magnitude cannot be greater than that of the departure of F from f before considered, because u and q will be of the same order, at least when u is big. As for x, it is simply a numerical factor, which cannot exceed 1, and is probably 2/3

The simplest case is when the motion of the Sun is perpendicular to the orbit of the Earth. Then

$$G = Fxs \tag{9}$$

gives the tensor[5] or size of the auxiliary force. It is radial, but

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[5] Heaviside uses the word "tensor" for the magnitude of the force vector (O. D. J.).

outwards, so that the result is merely to reduce the size of the previous correction, viz., the difference of F from f in the same motional circumstances.

But when the line of motion of Sun is in the plane of the orbit, the case is much more complicated. The force **G** is neither constant (for the same distance) nor radial, except in four positions, viz., two in the line of motion of the Sun, when the auxiliary force vanishes, and two when $\theta = \pm \pi/2$, when it is greatest. But this force is still in the plane of the orbit, which is an important thing, and is, moreover, periodic, so that the tangential component is as much one way as the other in a period.

All we need expect, then, so far as I can see from the above considerations, are small perturbations due to the variation of the force of gravity in different directions, and to the auxiliary force. Of course, there will be numerous minor perturbations

If variations of the force of the size considered above are too small to lead to observable perturbations of motion, then the striking conclusion is that the speed of gravity may even be the same as that of light. If they are observable, then, if existent, they should turn up, but if non-existent then the speed of gravity should be greater. Furthermore, it is to be observed that there may be other ways of expressing the propagation of gravity.

But I am mindful of the good old adage about the shoemaker and his last, and am, therefore, reluctant to make any more remarks about perturbations. The question of the ether in its gravitational aspect must be faced, however, and solved sooner or later, if it be possible. Perhaps, therefore, my suggestions may not be wholly useless.

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