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The Magnetic Scalar Potential*

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We obtain closed expressions for scalar magnetic potentials due to an arbitrary static current density $\mathbf{J}(\mathbf{x})$. *Simple prescriptions are given for forbidden regions where* $\mathbf{B}\neq-\nabla\phi$; *these forbidden regions make the potential single valued where it can be used. Finally, the complete multipole expansion of the magnetostatic field is derived in a few simple steps.*

I. INTRODUCTION

The static magnetic field in vacuum satisfies the equations

$$\nabla\cdot\mathbf{B}=0, \qquad \nabla\times\mathbf{B}=4\pi\mathbf{J}/c. \tag{1}$$

Because $\nabla\cdot\mathbf{B}=0$, the field is generally derived from a vector potential. On the other hand, wherever $\mathbf{J}=0$, one has the option of deriving $\mathbf{B}$ from a scalar potential, ϕ. Little use is made of this alternative in textbooks on electromagnetism because of two complications. The first is that it is tricky to write a formula relating ϕ to $\mathbf{J}$ because the scalar potential is useless wherever $\mathbf{J}\neq0$. The second complication is that ϕ is in general a multiple-valued function, so a prescription must be supplied specifying where ϕ can and cannot be used. The occurrence of multiple valuedness can be seen from the integral form of Ampere's law, which states that the change in ϕ around a closed path is proportional to the encircled current, even though $\mathbf{J}=0$ along the path:

$$\Delta\phi=-\oint\mathbf{B}\cdot d\mathbf{l}=-(4\pi/c)\iint d\mathbf{S}\cdot\mathbf{J}. \tag{2}$$

These complications are easily overcome, as we shall show in Sec. II, so we are not obliged to give up the scalar potential as an impractical curiosity. This is fortunate for several reasons. First, it is easier to visualize a scalar potential than a vector potential. Surfaces of constant ϕ are normal to $\mathbf{B}$, and their spacing is inversely proportional to $|\mathbf{B}|$. In addition, it may be handier to compute a scalar potential in some cases. Finally, using a scalar potential one can derive the multipole expansion of the magnetostatic field in a general and straightforward way that parallels the treatment of the electrostatic field. It becomes obvious that the magnetic and electric fields of an (l, m) multipole are spatially identical. Such a unified discussion represents a substantial pedagogical advance.

II. TWO MAGNETIC SCALAR POTENTIALS

We can write $\mathbf{B}=-\nabla\phi$ wherever $\mathbf{J}=0$. This equation may be integrated to give an expression for ϕ:

$$\phi(\mathbf{x})=-\int_{\mathbf{a}}^{\mathbf{x}} d\boldsymbol{\xi}\cdot\mathbf{B}(\boldsymbol{\xi}), \tag{3}$$

where $\mathbf{a}$ is a reference point. Two possible integration paths between $\mathbf{a}$ and $\mathbf{x}$ are depicted in Fig. 1.

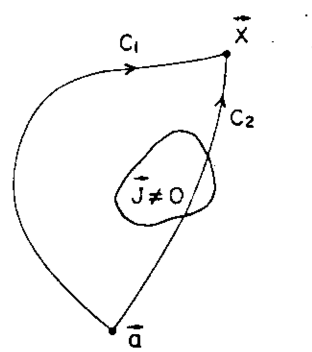

FIG. 1. Possible integration paths for Eq. (3).

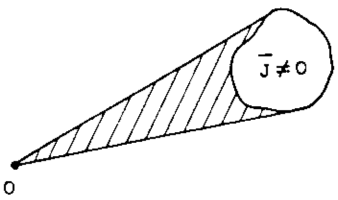

FIG. 2. ϕ_1 cannot be used where $\mathbf{J}\neq 0$, or in the shaded region.

Path C_1 is an acceptable path because $\mathbf{J}=0$ on C_1. Path C_2 can be used only if care is taken to insure that the current flowing between paths C_1 and C_2 does not change as $\mathbf{x}$ is varied. This requirement follows from Eq. (2) and the fact that two potentials yielding $\mathbf{B}$ must differ by a constant.

A potential suitable for deriving the multipole expansion results from the choice $\mathbf{a}\rightarrow\infty$, $\boldsymbol{\xi}=\xi\hat{x}$. We call this potential ϕ_1, and its path of integration in Eq. (3) is along a radius from infinity to $\mathbf{x}$. Clearly, ϕ_1 cannot be used for current distributions which extend to infinity. Since the whole integration contour moves with $\mathbf{x}$, ϕ_1 cannot be used in regions of space lying between currents and the origin, as shown in Fig. 2. This prescription makes ϕ_1 single valued where it can be used.

We use the standard representation for $\mathbf{B}$ in free space, and find:

$$\phi_1(\mathbf{x}) = -c^{-1}\int_\infty^x d\xi \times\int \frac{d^3x'\hat{x}\cdot\mathbf{J}(\mathbf{x}')\times(\xi\hat{x}-\mathbf{x}')}{|\xi\hat{x}-\mathbf{x}'|^3},$$

$$= -c^{-1}\int d^3x'\mathbf{J}(\mathbf{x}')\cdot(\hat{x}\times\mathbf{x}')\times\int_\infty^x \frac{d\xi}{|\xi\hat{x}-\mathbf{x}'|^3}. \quad (4)$$

The integral over ξ is elementary, and after a little algebra we obtain the formula

$$\phi_1(\mathbf{x}) = c^{-1}\times\int\frac{d^3x'\mathbf{J}(\mathbf{x}')\cdot(\mathbf{x}\times\mathbf{x}')}{|\mathbf{x}'-\mathbf{x}|[x|\mathbf{x}-\mathbf{x}'|+x^2-\mathbf{x}\cdot\mathbf{x}']}. \quad (5)$$

A scalar potential useful in the neighborhood of the origin, and therefore of any finite point, is obtained by choosing $\mathbf{a}=0$, $\boldsymbol{\xi}=\xi\hat{x}$ in Eq. (3). This potential, which we call ϕ_2, cannot be used in regions of space which can be sighted from the origin only by looking through current. The forbidden region is shown in Fig. 3. ϕ_2 is easily found to be

$$\phi_2(\mathbf{x}) = -c^{-1}\int\frac{d^3x'\mathbf{J}(\mathbf{x}')\cdot(\mathbf{x}\times\mathbf{x}')[2\mathbf{x}\cdot\mathbf{x}'-x^2]}{x'|\mathbf{x}'-\mathbf{x}|[(\mathbf{x}\cdot\mathbf{x}')|\mathbf{x}'-\mathbf{x}|+x'(\mathbf{x}\cdot\mathbf{x}')-x^2x']}. \quad (6)$$

III. THE MULTIPOLE EXPANSION OF THE MAGNETOSTATIC FIELD

We review the derivation of the multipole expansion of the electrostatic field, because it establishes the standard of conciseness and generality we wish to emulate. We begin with the electrostatic potential,

$$\phi_E(\mathbf{x}) = \int[d^3x'\rho(\mathbf{x}')/|\mathbf{x}'-\mathbf{x}|]. \quad (7)$$

Into this we substitute the expansion

$$|\mathbf{x}'-\mathbf{x}|^{-1} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{x'^l}{x^{l+1}}\times Y_{lm}(\theta,\phi)Y_{lm}^*(\theta',\phi'). \quad (8)$$

The result is the multipole expansion of the electrostatic field:

$$\phi_E(\mathbf{x}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{E_{lm}}{x^{l+1}}Y_{lm}(\theta,\phi),$$

$$E_{lm} = \int d^3x'x'^lY_{lm}^*(\theta,\phi)\rho(\mathbf{x}'). \quad (9)$$

To treat the magnetostatic case we use ϕ_1, which is valid outside a localized current distribution. The first formula in Eq. (4) can be put in the form

$$\phi_1(\mathbf{x}) = c^{-1}\int_\infty^x\frac{d\xi}{\xi}\int d^3x'\times[\mathbf{J}(\mathbf{x}')\times\mathbf{x}']\cdot\nabla'|\xi\hat{x}-\mathbf{x}'|^{-1}. \quad (10)$$

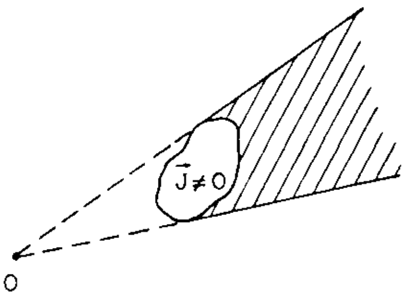

FIG. 3. ϕ_2 cannot be used where $\mathbf{J}\neq 0$, or in the shaded region.

We integrate over the primed variables by parts and interchange orders of integration:

$$\phi_1(\mathbf{x}) = c^{-1}\int d^3x'\,\nabla'\cdot[\mathbf{x}'\times\mathbf{J}(\mathbf{x}')]$$

$$\times\int_\infty^x \frac{d\xi}{\xi\,|\xi\hat{x}-\mathbf{x}'|}. \quad (11)$$

The expansion of Eq. (8) is now inserted, and the integral over ξ is evaluated term by term. The result is the multipole expansion of ϕ_1:

$$\phi_1(\mathbf{x}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{M_{lm}}{x^{l+1}}Y_{lm}(\theta,\phi),$$

$$M_{lm} = -\int d^3x'x'^lY_{lm}^*(\theta',\phi')$$

$$\times\frac{\nabla'\cdot[\mathbf{x}'\times\mathbf{J}(\mathbf{x}')]}{(l+1)c}. \quad (12)$$

To appreciate the simplicity of this derivation, it should be compared with the relatively awkward manipulations of the vector potential given in textbooks. Those manipulations yield only the first few terms of the expansion, and further work is required to show the identity of the spatial distributions of the electric and magnetic fields. Here one has a general treatment, and the parallel between the electric and magnetic cases is immediately obvious. The expression for M_{lm} given in Eq. (12) is derived in the textbooks only after developing the full machinery of vector spherical harmonics.[1] This formalism is not required for the static problem, as we have seen.

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[1] J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1962), Eq. (16.96). See also the comments on p. 145.