Propagation of electromagnetic waves in a gravitational field

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The Maxwell equations in a weak gravitational field are reduced to one scalar wave equation. A similar result is also obtained for a slowly varying gravitational field of arbitrary strength, up to and including terms of second order in the photon wavelength. The index of refraction, the phase velocity, and the group velocity of the electromagnetic waves have been calculated.

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1. In a gravitational field we write Maxwell's equations in the form

$$g^{\mu\lambda}\partial_\mu F_{\lambda\nu}+L^\lambda F_{\lambda\nu}+L_\nu{}^{\mu\lambda}F_{\mu\lambda}=0, \tag{1}$$

where $g^{\mu\lambda}$ is the metric tensor, and

$$F_{\lambda\nu}=\partial_\lambda A_\nu-\partial_\nu A_\lambda,\quad \partial_\lambda=\partial/\partial x^\lambda,$$

$$L^\lambda=(-g)^{-1/2}\partial_\mu[(-g)^{1/2}g^{\mu\lambda}],\quad L_\nu{}^{\mu\lambda}=g^{\mu\varkappa}g_{\nu\rho}\partial_\varkappa g^{\rho\lambda}.$$

We make use of light-cone coordinates $x^0=t-z$, $x^3=(t+z)/2$. As the x^3 coordinate axis we choose the null geodesic along which the wave propagates in the limit of infinite frequency ω. We additionally define the coordinate system so that along this geodesic the metric tensor should coincide with the metric tensor of a flat space: $g^{\mu\nu}=\eta^{\mu\nu}$, $\eta^{30}=\eta_{30}=1$, $\eta^{mn}=\eta_{mn}=-\delta_{mn}$, $(m,n=1,2)$, all other components of $\eta^{\mu\nu}$ vanish. Furthermore, a choice of the coordinate system allows us to make the Christoffel symbols vanish along the geodesic (Ref. 1) so that the deviation of the metric tensor from the flat one, $f^{\mu\nu}=g^{\mu\nu}-\eta^{\mu\nu}$, is a quantity small of second order in x^α $(\alpha=0,1,2,)$.

We reduce the system of equations (1) in this locally inertial frame to a second-order equation for a single unknown function. We proceed in a manner similar to the one usual in flat-space electrodynamics. We differentiate the equation (1) with $\nu=n=1,2$ with respect to x^0, make use of the second pair of Maxwell equations

$$\partial_\lambda F_{\mu\nu}+\partial_\mu F_{\nu\lambda}+\partial_\nu F_{\lambda\mu}=0 \tag{2}$$

and again of Eq. (1) for $\nu=0$. As a result of all this we obtain

$$(-2\partial_0\partial_3+\partial_m\partial_m)F_{n0}-\partial_n[(f^{\mu\nu}\partial_\mu+L^\nu)F_{\nu0}+L_0{}^{\nu\rho}F_{\nu\rho}]+$$
$$+\partial_0[(f^{\mu\nu}\partial_\mu+L^\nu)F_{\nu n}+L_n{}^{\nu\rho}F_{\nu\rho}]=0. \tag{3}$$

Here m, n take on the values 1, 2, and $\partial_m\partial_m=\partial_1^2+\partial_2^2$.

We shall assume that the wavelength ω^{-1} is small compared to the characteristic distances D over which the gravitational field varies, and compared to the size d of the wave packet; in turn $d\ll D$. Without restricting ourselves to the geometric optics approximation, we retain in the equations terms of the order ω^2d^2, ωd, 1, $\omega^2d^3D^{-1}$, ωd^2D^{-1}, and $\omega^2d^4D^{-2}$ (it is obvious that $|x^\alpha|<d$, $\alpha=0,1,2$). This allows us to calculate the effects of focusing and defocusing of a beam, effects which depend on the polarization, as well as the corrections to the group and phase velocities. We search for the solution in the form

$$F_{n0}\sim\exp\{-i\omega x^0+i\omega\psi+i\varphi+i\omega^{-1}\chi\}, \tag{4}$$

where, as can be seen by substituting into the equations (3), for $x^\alpha\to0$ $(\alpha=0,1,2)$, $\psi\sim(x^\alpha)^2$, and in general φ and χ do not vanish.

Taking account of Eq. (4), Maxwell's equations (2) lead to the following estimates:

$$F_{30}\sim F_{12}\sim(d+\omega^{-1})F_{n0},\quad F_{3n}\sim(d^2D^{-1}+\omega^{-1})F_{n0}.$$

As a result of this we obtain to the desired degree of accuracy two equations that interconnect only the large field components:

$$[-2\partial_0\partial_3+\partial_m\partial_m$$
$$-f^{\mu\nu}\partial_\mu\partial_\nu-(L^0+\partial_0f^{00})\partial_0-(L^m+2\partial_0f^{0m})\partial_m-\partial_0L^0]F_{n0}$$
$$+[(L_n{}^{m0}-L_n{}^{0m}-\partial_nf^{0m})\partial_0-(L_0{}^{m0}-\partial_0f^{0m})\partial_n+(L_n{}^{30}-\partial_nf^{30})\partial_m$$
$$+(L_n{}^{mk}-L_n{}^{km}-\partial_nf^{km})\partial_k-\partial_nL^m-\partial_nL_0{}^{m0}+\partial_0L_n{}^{m0}]F_{m0}=0, \tag{5}$$

where $m,n,k=1,2$.

It is convenient to introduce the new functions E_n by means of the substitution

$$F_{n0}=E_n+{}^1/_2f^{nk}E_k. \tag{6}$$

As a result we obtain to the desired accuracy a system of equations with an antisymmetric off-diagonal part, which can be diagonalized by transforming to a helicity basis:

$$E_\lambda=2^{-1/2}(-\lambda E_1-iE_2),\quad \lambda=\pm1. \tag{7}$$

The final equation has the form

$$\{-2\partial_0\partial_3+\partial_m\partial_m$$
$$-f^{00}\partial_0{}^2-2f_{03}\partial_{03}-f_{km}\partial_k\partial_m+2f_{3k}\partial_0\partial_k+(2\partial_0g_{33}-\partial_mg_{m3}$$
$$+{}^1/_2\partial_3g_{mm})\partial_0-(3\partial_0g_{3m}-\partial_kg_{km}$$
$$-\partial_mg_{03}+{}^1/_2\partial_mg_{kk})\partial_m+\partial_0{}^2g_{33}-\partial_0\partial_mg_{m3}$$
$$+R_{1212}-i\lambda[(\partial_2g_{13}-\partial_1g_{23})\partial_0+(\partial_0g_{23}-\partial_2g_{30}+\partial_1g_{12}$$
$$-\partial_2g_{11})\partial_1-(\partial_0g_{13}-\partial_1g_{30}$$
$$+\partial_2g_{12}-\partial_1g_{22})\partial_2+{}^1/_2(\partial_1\partial_mg_{m2}-\partial_2\partial_mg_{m1})-2R_{1230}]\}E_\lambda=0. \tag{8}$$

Summation is understood over all pairs of repeated indices; $g_{\mu\nu}=\eta_{\mu\nu}-f_{\mu\nu}$; we note that if one takes into account terms of higher order in the gravitational field, the $f_{\mu\nu}$ defined by this equation cannot be obtained from $f^{\mu\nu}=g^{\mu\nu}-\eta^{\mu\nu}$ by simply lowering the indices. Furthermore $f^{00}=f_{33}+2f_{03}f_{33}-f_{3k}f_{3k}$. In our approximation the Riemann curvature tensor equals

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$$R_{\mu\nu\varkappa\lambda}={}^1/_2(\partial_\mu\partial_\lambda g_{\nu\varkappa}+\partial_\nu\partial_\varkappa g_{\mu\lambda}-\partial_\mu\partial_\varkappa g_{\nu\lambda}-\partial_\nu\partial_\lambda g_{\mu\varkappa}),$$

even though the gravitational field is not assumed to be weak

2. In the linear approximation with respect to the gravitational field the equation (8) simplifies noticeably, since the derivatives $\partial_3 E_\lambda$ and $\partial_m E_\lambda$ are themselves proportional to the gravitational field. As a result we obtain

$$(-2\partial_0\partial_3+\partial_m\partial_m-\omega^2 g_{33}-i\omega V+W)E_\lambda=0, \tag{9}$$

where

$$V=2\partial_0 g_{33}-\partial_m g_{m3}+{}^1/_2\partial_3 g_{mm}+i\lambda(\partial_1 g_{23}-\partial_2 g_{13}),$$

$$W=R_{1212}+\partial_0{}^2 g_{33}-\partial_0\partial_m g_{m3}$$
$$+{}^1/_2\partial_0\partial_3 g_{mm}+i\lambda(2R_{1230}+{}^1/_2\partial_2\partial_m g_{m1}-{}^1/_2\partial_1\partial_m g_{m2}).$$

Note that this equation is valid without assuming that the photon wavelength is short [this is due to the fact that in the expression for W we have retained the term $(1/2)\partial_0\partial_3 g_{mn}$, which was omitted from Eq. (8)].

We shall search for a solution of Eq. (9) in the form (4), although it is not quite consistent to put $\omega\psi+\varphi+\omega^{-1}\chi$ into the exponent, since the equation (9) is valid only in the weak-field approximation. However, this form is convenient for the calculation of corrections to the phase and group velocities of an electromagnetic wave traveling in a gravitational field. Substituting the expression (4) into the equation (9) we obtain the following system of equations which determine ψ, φ, and χ to lowest order in the gravitational field.

$$\begin{aligned}2\partial_3\psi&=-g_{33},\\ 2\partial_3\varphi&=i(\partial_m\partial_m-2\partial_0\partial_3)\psi-iV,\\ 2\partial_3\chi&=i(\partial_m\partial_m-2\partial_0\partial_3)\varphi+W.\end{aligned} \tag{10}$$

From this it follows, in particular, that

$$\partial_3\chi=\frac{1}{2}(W-\partial_0 V+\partial_0{}^2 g_{33})+\frac{1}{4}\int^{x^3}dx^{3'}(\partial_m\partial_m V-2\partial_0\partial_m\partial_m g_{33})$$
$$+\frac{1}{8}\int^{x^3}dx^{3'}\int^{x^{3'}}dx^{3''}\,\partial_k\partial_k\partial_m\partial_m g_{33}. \tag{11}$$

Making further use of the identities

$$\partial_m\partial_m g_{33}=2R_{33}-\partial_3{}^2 g_{mm}+2\partial_3\partial_m g_{m3},$$

$$\partial_m\partial_m(\partial_1 g_{23}-\partial_2 g_{13})=\partial_3\partial_m(\partial_1 g_{2m}-\partial_2 g_{1m})$$
$$+2(\partial_1 R_{23}-\partial_2 R_{13}+\partial_3 R_{1230}),$$

one can simplify the expression (11) considerably

$$\partial_3\chi={}^1/_2(R_{1212}+i\lambda R_{1230})$$
$$+\frac{i\lambda}{2}\int^{x^3}dx^{3'}(\partial_1 R_{23}-\partial_2 R_{13})+\frac{1}{4}\int^{x^3}dx^{3'}\int^{x^{3'}}dx^{3''}\,\partial_m\partial_m R_{33}. \tag{12}$$

We note that $\partial_3\chi$ does not depend directly either on the metric tensor or on the Christoffel symbols, and is determined only by the Riemann curvature tensor. In the region where there are no sources of the gravitational field, i.e., $R_{\mu\nu}=0$, the integral terms in Eq. (12) vanish, and the result acquires a maximally simple form

$$\partial_3\chi={}^1/_2(R_{1212}+i\lambda R_{1230}). \tag{13}$$

3. We now derive an expression for the phase velocity of a wave in a gravitational field, relative to an observer at rest in the given coordinate frame. The three-velocity of a point experiencing the displacement dx^μ is[2]

$$u^i=(g_{tt})^{1/2}dx^i/(g_{tt}dt+g_{tj}dx^j). \tag{14}$$

Here the indices i,j take on the spatial values x, y, z, while t is the time coordinate. The transition from x^0 and x^3 to z and t is achieved by means of a coordinate-independent rotation of all space, leading to a trivial change of the tensorial quantities and $\Gamma^\mu_{\nu\sigma}$.

Assume that the surface of constant phase is defined by the equation $\Phi(x,y,z,t)=0$. We represent a displacement of a point of this surface in the form

$$dx^\mu\sim p^\mu+\alpha v^\mu, \tag{15}$$

where $p_\mu=\partial_\mu\Phi$, and $v^\mu=(g_{tt})^{-1/2}(1,0,0,0)$ is the four-velocity of the observer. The condition $d\Phi=0$ that the phase be constant means that $p_\mu dx^\mu=0$, hence

$$dx^\mu\sim p^\mu-v^\mu p_\nu p^\nu/p_\lambda v^\lambda. \tag{16}$$

Substituting Eq. (16) into (14) and calculating the square of the phase three-velocity

$$u^2=(-g_{ij}+g_{ti}g_{jt}/g_{tt})u^iu^j,$$

we obtain[3]

$$u^{-2}=1-g_{tt}g^{\mu\nu}p_\mu p_\nu/p_t{}^2. \tag{17}$$

This quantity may be interpreted as the square of the index of refraction, n^2, for light in an external gravitational field. A simple calculation yields

$$u=n^{-1}=1-\mathrm{Re}\left(\frac{1}{\omega}\partial_3\varphi+\frac{1}{\omega^2}\partial_3\chi\right). \tag{18}$$

As can be seen from Eqs. (10) and (13), along the x^3 axis we have

$$u=n^{-1}=1-\frac{1}{2\omega^2}R_{1212}. \tag{19}$$

However, if one takes into account the explicit expression for u, it follows from Eqs. (10) and (19) that in an arbitrary coordinate system the index of refraction depends only on the curvature tensor. This is quite natural: for example, in a frame in which the x_1 and x_2 axes rotate along the trajectory, n must depend, of course, on the sign of the circular polarization λ.

Utilizing Eq. (18) one can obtain an expression for the group velocity of a wave packet:

$$v=\left[\frac{d}{d\omega}(\omega n)\right]^{-1}=1+\frac{1}{\omega^2}\mathrm{Re}\,\partial_3\chi=1+\frac{1}{2\omega^2}R_{1212}. \tag{20}$$

Thus, the first-order correction to the index of refraction and the group velocity in the coordinate system that was used is determined by the scalar curvature of a two-dimensional surface orthogonal to the trajectory of the wave packet. The sign of the correction is not defined and the group velocity v may be either larger or smaller than unity. This, however, does not violate the causality condition, which implies that the velocity of the wave front is bounded by one. This velocity is[4]

$$\lim_{\omega\to\infty} n^{-1}(\omega)$$

and, as expected, this equals one according to Eq. (19). Here $v>1$ signals only a deformation of wave packet.

The phase Φ has a nonvanishing imaginary part, and accordingly there appears an imaginary part on the refractive index:

$$\operatorname{Im} n=\operatorname{Im}\left(\frac{1}{\omega}\partial_3\varphi+\frac{1}{\omega^2}\partial_3\chi\right)=-\frac{1}{2\omega}\left(\partial_0 g_{33}+\int^{x^3} dx^{3\prime}R_{33}\right)$$
$$+\frac{\lambda}{2\omega^2}\left[R_{1230}+\int^{3} dx^{3\prime}\left(\partial_1 R_{23}-\partial_2 R_{13}\right)\right]. \qquad (21)$$

Along the coordinate axis x^3 we have $\partial_0 g_{33}=0$, and if in addition to that the wave propagates in empty space, the expression for Im n becomes quite simple:

$$\operatorname{Im} n=(\lambda/2\omega^2)R_{1230}. \qquad (22)$$

Thus, if $R_{1230}\neq 0$ then for one of the signs of the circular polarization the wave amplitude increases, and for the other sign it decreases.

4. In connection with what was said so far we would like to dwell on the assertion one can find in the literature that photons in a gravitation field propagate faster than light if radiative corrections are taken into account. In Ref. 5 it was noted that the radiative corrections lead to a change of the characteristics of the wave equation for the photon for frequencies small compared to the electron mass m_e. The addition to the index of refraction turns out to be negative for one of the polarizations and independent of the frequency in the limit $\omega\ll m_e$. Based on the assumption that Im n is positive definite, and making use of the dispersion relation for $n(\omega)$, the authors of Ref. 5 have concluded that $n(\omega=0)>n(\omega\to\infty)$, and consequently, $n(\omega\to\infty)<1$, so that for one of the polarizations causality is explicitly violated. A similar assertion for neutrinos was later made in Ref. 6

From what was said one can infer two inadequacies of the mentioned reasoning. First, the value $n(0)$ has in fact never been computed in a gravitational field. In particular, there is the correction (19) to $n(\omega\to\infty)$ which is much larger than the one which was found in Ref. 5. However, the expression (19) is valid only for $\omega D\gg 1$, which prevents one from realistically calculating $n(0)$. Second, the sign of Im n is undefined when the wave propagates through inhomogeneous media, as can be seen from the expressions (21) and (22). The physical reason for this is clear. In a homogeneous medium without instability (no particle creation) the change of the amplitude of the wave is due only to the elimination of particles from the beam, which is reflected in the condition Im $n>0$. If the medium is not homogeneous focusing (or bunching) processes are possible, leading to a growth of the amplitude; this corresponds to Im $n<0$. One can illustrate this assertion on a simple example from quantum mechanics. The quasiclassical expression for a wave propagating along the x axis in a potential $U(x)$,

$$\psi=\left[\frac{\omega^2}{\omega^2-U(x)}\right]^{1/4}\exp\left\{i\int^{x}dx'\left[\omega^2-U(x')\right]^{1/2}\right\}$$

can be rewritten for $\omega\gg U$ in the form

$$\psi=\exp\left[i\omega x-\frac{i}{2\omega}\int^{x}dx'\,U(x')+\frac{1}{4\omega^2}U(x)\right]. \qquad (23)$$

From this it follows that

$$\operatorname{Im} n(\omega;\,x)=-{}^1/_4\omega^{-2}dU/dx.$$

It is obvious that the sign of this quantity may be arbitrary.

Thus, the conclusion reached in Refs. 5 and 6 that causality is violated, seems unfounded. Unfortunately, our previous attempt[7] to refute this assertion was unsuccessful.

5. We return to an arbitrary gravitational field in which the wave propagation is described by Eq. (8). Substituting the solution in the form (4), it is easy to derive a system of nonlinear equations for the functions ψ, φ, and χ. Since we consider the size of the wave packet to be considerably smaller than the characteristic lengths over which the external field varies, it is convenient to expand these functions in powers of x^α ($\alpha=0,1,2$). In particular, in the expansion of ψ, the approximation we have adopted allows us to retain only terms up to fourth order:

$$\psi={}^1/_2\psi_{\alpha\beta}(x^3)x^\alpha x^\beta+{}^1/_6\psi_{\alpha\beta\gamma}(x^3)x^\alpha x^\beta x^\gamma+{}^1/_{24}\psi_{\alpha\beta\gamma\delta}(x^3)x^\alpha x^\beta x^\gamma x^\delta. \qquad (24)$$

The equation for the function ψ

$$2\partial_3\psi-2\partial_0\psi\partial_3\psi+\partial_n\psi\partial_n\psi+2f^{\mu 0}\partial_\mu\psi-f^{\mu\nu}\partial_\mu\psi\partial_\nu\psi=f^{00} \qquad (25)$$

leads to the following system of equations for $\psi_{\alpha\beta}$:

$$\partial_3\psi_{\alpha\beta}+\psi_{\alpha n}\psi_{\beta n}=R_{3\alpha 3\beta} \qquad (26)$$

(here $\alpha,\beta=0,1,2$, and $n=1,2$). The functions ψ_{11}, ψ_{12}, and ψ_{22} satisfy an independent subsystem of nonlinear equations:

$$\partial_3\psi_{km}+\psi_{kn}\psi_{mn}=R_{3k3m}. \qquad (27)$$

After determining ψ_{km}, the calculation of ψ_{01} and ψ_{02} reduces to solving a linear system

$$\partial_3\psi_{0m}+\psi_{0n}\psi_{mn}=R_{303m}, \qquad (28)$$

and finally, for known ψ_{0n} the function ψ_{00} is found by plain integration

$$\psi_{00}=\int dx^3\left(R_{3030}-\psi_{0n}\psi_{0n}\right) \qquad (29)$$

The equations for the functions $\psi_{\alpha\beta\gamma}$ and $\psi_{\alpha\beta\gamma\delta}$ are also easily obtained from Eq. (25). We do not write them out here, in view of their bulkiness.

The equations for the functions φ and χ which are analogous to Eq. (25) turn out to be nonlinear. In our approximation φ can be written in the form

$$\varphi=\varphi^{(0)}(x^3)+\varphi_\alpha(x^3)x^\alpha+{}^1/_2\varphi_{\alpha\beta}(x^3)x^\alpha x^\beta, \qquad (30)$$

and it suffices to consider χ for $x^\alpha=0$. The equation for $\varphi^{(0)}(x^3)$ has the simple form

$$\partial_3\varphi^{(0)}={}^1/_2 i(\psi_{11}+\psi_{22}), \qquad (31)$$

but the other equations, which are also easily derived, turn out to be clumsy, and we do not write them out here.

In conclusion, we note that making use of Eq. (17) it is easy to find the index of refraction for $x^\alpha=0$:

$$\operatorname{Re} n=1+\frac{1}{2\omega^2}\left[2\operatorname{Re}\partial_3\chi+(\operatorname{Re}\varphi_1)^2+(\operatorname{Re}\varphi_2)^2\right], \qquad (32)$$

where φ_n is defined according to Eq. (30). This yields in particular, that the index of refraction, and with it the phase and group velocities, do not depend on the helicity λ.

By iterations it is not difficult to obtain the nonlinear corrections to the real and imaginary parts of the index of refraction. Further investigations of the solution would require a knowledge of the explicit form of the gravitational field.

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[1]P. K. Rashevskiĭ, Rimanova geometriya i tensornyĭ analiz (Riemannian Geometry and Tensor Analysis). Moscow, Nauka, 1967; p. 428.
[2]L. D. Landau and E. M. Lifshitz, Teoriya polya (The Classical Theory of Fields). Moscow, Nauka, 1973 [Engl. translation: Pergamon Press 1974].
[3]J. L. Synge, Relativity: The General Theory, Amsterdam: North-Holland, 1960 [Russian translation, edited by A. Z. Petrov, Moscow: IIL, 1963, p. 315].
[4]M. A. Leontovich, in the volume: L. I. Mandel'shtam: Lektsii po optike, teorii otnositel'nosti i kvantovoĭ mekhanike (Lectures on Optics, Relativity Theory, and Quantum Mechanics). Moscow: Nauka, 1972, p. 308.
[5]I. T. Drummond and S. J. Hathrell, Phys. Rev. **D22**, 343 (1980).
[6]Y. Ohkuwa, Prog. Theor. Phys. **65**, 1085 (1981).
[7]A. D. Dolgov and I. B. Khriplovich, Nucl. Phys. Inst. Preprint IYaF 81-53.

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