OTHER TITLES IN THE SERIES
ON PURE AND APPLIED MATHEMATICS

Vol. 1. WALLACE—*An Introduction to Algebraic Topology*
Vol. 2. PEDOE—*Circles*
Vol. 3. SPAIN—*Analytical Conics*
Vol. 4. MIKHLIN—*Integral Equations*
Vol. 5. EGGLESTON—*Problems in Euclidean Space: Application of Convexity*
Vol. 6. WALLACE—*Homology Theory on Algebraic Varieties*
Vol. 7. NOBLE—*Methods Based on the Wiener-Hopf Technique for the Solution of Partial Differential Equations*
Vol. 8. MIKUSIŃSKI—*Operational Calculus*
Vol. 9. HEINE—*Group Theory in Quantum Mechanics*
Vol. 10. BLAND—*The Theory of Linear Viscoelasticity*
Vol. 11. KURTH—*Axiomatics of Classical Statistical Mechanics*
Vol. 12. FUCHS—*Abelian Groups*
Vol. 13. KURATOWSKI—*Introduction to Set Theory and Topology*
Vol. 14. SPAIN—*Analytical Quadrics*
Vol. 15. HARTMAN and MIKUSIŃSKI—*The Theory of Lebesgue Measure and Integration*
Vol. 16. KULCZYCKI—*Non-Euclidean Geometry*
Vol. 17. KURATOWSKI—*Introduction to Calculus*
Vol. 18. GERONIMUS—*Polynomials Orthogonal on a Circle and Interval*
Vol. 19. ELSGOLC—*Calculus of Variations*
Vol. 20. ALEXITS—*Convergence Problems of Orthogonal Series*
Vol. 21. FUCHS and LEVIN—*Functions of a Complex Variable, Vol. II*
Vol. 22. GOODSTEIN—*Fundamental Concepts of Mathematics*
Vol. 23. KEENE—*Abstract Sets and Finite Ordinals*
Vol. 24. DITKIN and PRUDNIKOV—*Operational Calculus in Two Variables and its Applications*
Vol. 25. VEKUA—*Generalized Analytic Functions*
Vol. 26. AMIR-MOEZ and FASS—*Elements of Linear Spaces*
Vol. 27. GRADSHTEIN—*Direct and Converse Theorems*
Vol. 28. FUCHS—*Partially Ordered Algebraic Systems*
Vol. 29. POSTNIKOV—*Foundations of Galois Theory*
Vol. 30. BERMANT—*Course of Mathematical Analysis, Part II*
Vol. 31. ŁUKASIEWICZ—*Elements of Mathematical Logic*
Vol. 32. VULIKH—*Introduction to Functional Analysis for Scientists and Technologists*
Vol. 33. PEDOE—*Introduction to Projective Geometry*
Vol. 34. TIMAN—*Theory of Approximation of Function of a Real Variable*
Vol. 35. CSÁSZÁR—*Foundations of General Topology*
Vol. 36. BRONSHTEIN and SEMENDYAYEV—*A Guide-Book to Mathematics for Technologists and Engineers*
Vol. 37. MOSTOWSKI and STARK—*Introduction to Higher Algebra*
Vol. 38. GODDARD—*Mathematical Techniques of Operational Research*
Vol. 39. TIKHONOV and SAMARSKII—*Equations of Mathematical Physics*
Vol. 40. MCLEOD—*Introduction to Fluid Dynamics*
Vol. 41. MOISIL—*The Algebraic Theory of Switching Structures*

NOMOGRAPHY

by

EDWARD OTTO

Translated by

JANINA SMÓLSKA

PERGAMON PRESS

OXFORD · LONDON · NEW YORK · PARIS

PWN — POLISH SCIENTIFIC PUBLISHERS

WARSZAWA

1963

PERGAMON PRESS LTD.
Headington Hill Hall, Oxford
4 and 5 Fitzroy Square, London W.1.

PERGAMON PRESS INC.
122 East 55th Street, New York 22, N.Y.

GAUTHIER-VILLARS ED.
55 Quai des Grands-Augustins, Paris 6e

PERGAMON PRESS G.m.b.H.
Kaiserstrasse 75, Frankfurt am Main

Distributed in the Western Hemisphere by
THE MACMILLAN COMPANY · NEW YORK
pursuant to a special arrangement with
Pergamon Press Limited

This book is an enlarged translation of the original Polish *Nomografia*, published by PWN—Polish Scientific Publishers 1956

Library of Congress Card Number 63-10028

Printed in Poland (D.R.P)

FOREWORD

Literature in the field of nomography is nowadays so extensive that in many languages textbooks of nomography and collections of nomograms for various branches of technology are published separately.

This book is not a collection of nomograms but a manual to teach nomography. The examples contained in it are not meant to give ready-made solutions for the use of engineers but serve as illustrations of the methods of constructing nomograms; that is why most of them are given without any comment regarding the technical problems from which they have arisen.

The importance of geometrical transformations, and particularly projective transformations of a plane, has been specially stressed. The traditional method of providing the best form of a nomographic drawing within the given variability limits of the parameters occurring in the equation, a method consisting in a suitable choice of units for various functional scales, has been replaced in this manual by a method of transforming an arbitrary nomogram satisfying the given equation. Thus the finding of the so called modules, which is different for every type of equations dealt with in nomography, has been replaced by one method: a projective transformation of an arbitrary quadrilateral into a rectangle.

Accordingly, Chapter I begins with the necessary information on the projective plane and collineation transformations. They have been approached both from the geometrical and the algebraical point of view: the geometrical approach aims at permitting the use of elementary geometrical methods in drawing collineation nomograms consisting of three rectilinear scales (§§ 10–13) while the algebraical treatment concerns nomograms containing curvilinear scales. The necessary algebraic calculation has been developed as a uniform procedure involving the use of the matrix calculus. The chapter ends with information on duality in the plane.

Chapter II contains the fundamental data concerning functional scales.

In the first part of Chapter III those equations are singled out which can be represented by elementary methods without the use of a system of coordinates. Those equations are most frequent in practice and it has seemed advisable to give the simplest methods for them. The remaining cases (§§ 15–19) require the use of algebraic calculation. The second part of Chapter III deals with nomograms with a binary field (lattice nomograms): it has been stressed that from the algebraical point of view it is only necessary to pass from the coordinates of a point to the coordinates of a straight line.

In Chapter IV the methods discussed in the preceding chapters are used for constructing combined nomograms.

Chapter V is an introduction to mathematical problems which have arisen in the analysis of the methods of constructing nomograms. Besides solutions known in literature, such as the so called Massau method and the criterion of Saint Robert, § 31 contains an algebraic criterion of nomogrammability of functions, which is a realisation of an idea of Duporq (Comptes Rendus 1898). It finally solves a problem which has only partially been solved by other authors, who have been using complicated, practically inapplicable methods.

My manuscript has been revised and corrected in various places by Dr. K. Kominek from Prague, for which I owe him sincere thanks.

THE AUTHOR

CHAPTER I

INTRODUCTION

§ 1. Nomograms

Nomograms are drawings made on a plane in order to replace cumbersome numerical calculations occurring in technology by simple geometrical constructions. Figure 1 is an example of a nomogram of this kind. It is closely connected with the formula

$$\Delta = 3160 \frac{G^{1\cdot 85}}{d^{4\cdot 97}}. \tag{1.1}$$

The numerical values of the variable G are represented in the figure by points of the segment marked with the letter G; each number contained between 1000 and 10000 has a point of this segment corresponding to it, and *vice versa*; in Fig. 1 only the points corresponding to numbers 1000, 2000, ..., 10000 are marked, but it should be understood of course that intermediate points correspond to intermediate values. The same can be said of the numerical values of the variable d contained between the numbers 40 and 350 and the segment marked by the letter d in the figure, as well as of the numerical values of the variable Δ and the segment Δ in the nomogram. Now the close relation between Fig. 1 and formula (1.1) consists in the fact that the three numbers G_0, d_0 and Δ_0 satisfy equation (1.1) if and only if the three points of the nomogram corresponding to those points lie on the same straight line. By way of example, points $G_0 = 2238$, $d_0 = 82$ and $\Delta_0 = 1\cdot 52$ have been marked on the nomogram. We thus see that the calculation necessary to find Δ_0 with given G_0 and d_0 is equivalent to the determination of a straight line joining points G_0 and d_0 in the nomogram, fixing the point of intersection of that line with segment Δ and reading the corresponding number Δ_0.

Let us disregard for the present the method of executing Fig. 1

(which will be dealt with in Chapter III) and consider the advisability of constructing geometrical figures like the nomogram in Fig. 1. To begin with, it should be observed that the nomogram in question permits us to read number Δ_0 only with a limited accuracy, depending of course on the magnitude of the segment corresponding to the given interval on Δ; if that segment were

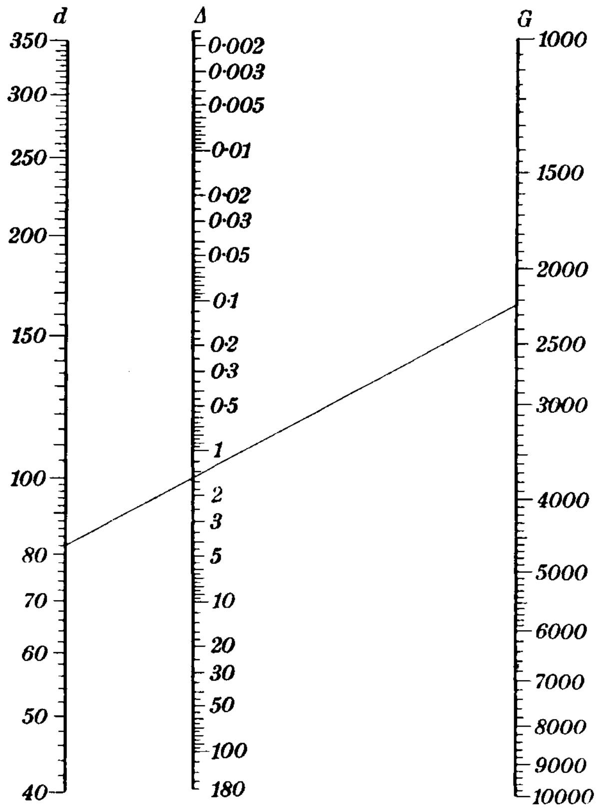

Fig. 1

longer, the accuracy would be greater. The same applies to the remaining two segments, marked in Fig. 1 with letters G and d. In order to increase the reading accuracy we could thus enlarge the drawing (just as we use logarithmic tables with a larger number of digits in order to increase the accuracy of numerical calculations). Obviously, we can restrict ourselves to enlarging only the lengths of the lines for instance, leaving their distance from one another unchanged, i.e. we can make a transformation of the plane of the drawing according to the formulas

$$\xi = x, \quad \eta = ky, \quad \text{where} \quad k > 1 \tag{1.2}$$

(it is assumed here that the segments G, d and Δ are parallel to the axis y); as we know, three points of a plane that lie on a straight line will be changed by this transformation into three new points also lying on a (new) straight line. (The transformation defined by formulas (1.2) produces an elongation of the plane in the direction of the axis y.) Therefore, if three numbers G_0, Δ_0 and d_0 satisfy equation (1.1), then the points corresponding to those numbers after a transformation according to formulas (1.2) will, in the new drawing, also lie on a straight line. The new drawing will also be a nomogram for the given equation. It can thus be seen that a nomogram corresponding to formula (1.1) may yield a new nomogram by being subjected to a suitable transformation. We are of course interested only in those transformations which to each three points that are collinear, i.e. lie on a straight line, assign three new points also lying on a (different) straight line. It can be seen that the transformations defined by formulas (1.2) are not the only transformations of this kind; there are a great many such transformations. The choice of a suitable transformation to obtain the best form of the nomogram is of essential importance in nomography.

Mappings of a plane which transform every triple of collinear points into another triple of collinear points constitute one of the branches of projective geometry. Our exposition of nomography will be preceded by a discussion of the basic notions and theorems of that branch of geometry.

§ 2. Projective plane

2.1. Consider two planes α_1 and α_2 and a point S not lying on either of them (Fig. 2). To a point P_1 of the plane α_1 let us assign such a point P_2 of the plane α_2 as to make the three points P_1, P_2, and S collinear. We can immediately observe that this agreement assigns to every point of the plane α_1 a certain point of the plane α_2 if and only if the plane α_2 is parallel to the plane α_1. If the plane α_2 were not parallel to the plane α_1, the points of the straight line n_1, along which the plane α_1 intersects the plane α_2' parallel to α_2 and passing through the point S,

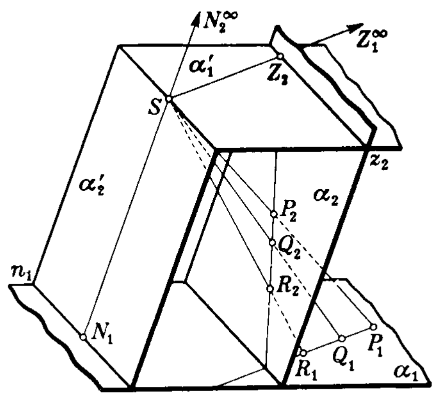

FIG. 2

would have no counterparts on the plane α_2. Indeed, joining the point N_1 lying on the straight line n_1 to the point S in order to find the corresponding point N_2, we should see that the straight line obtained would be parallel to the plane α_2 (as one lying in the plane α_2). Conversely, points corresponding to the points of the plane α_1 do not fill the whole plane α_2, for we see that no point of the straight line z_2 (the intersection edge of the plane α_2 and the plane α_1' parallel to α_1 and passing through the point S) would correspond to any point of the plane α_1.

Observe that if the points P_1, Q_1, and R_1 lying on the plane α_1 have corresponding points P_2, Q_2, and R_2 lying on the plane α_2 and one of these triples of points is collinear, then the other three points are also collinear (the points S, P_1, P_2, R_1, R_2, Q_1, and Q_2 are then lying on the same plane). We thus have here a transformation of the kind discussed in the preceding section.

We find, however, that the use of this kind of transformations involves considerable difficulties, due to the fact that there exist exceptional points and exceptional straight lines (the straight line n_1 on the plane α_1 and the straight line z_2 on the plane α_2), which have no counterparts on the other plane. Thus, for instance, to the straight lines of the plane α_2 which pass through the point Z_2 lying on the straight line z_2 (Fig. 3) there would correspond lines which are parallel on the plane α_1, i.e. such as have no point in common. However, in the set of all straight lines passing through the point Z_2 (and forming a so called *pencil of lines*) there is a straight line z_2 which has no counterpart on the plane α_2; consequently, the parallel lines form a set containing one element less than the set of the straight lines passing through the point Z_2.

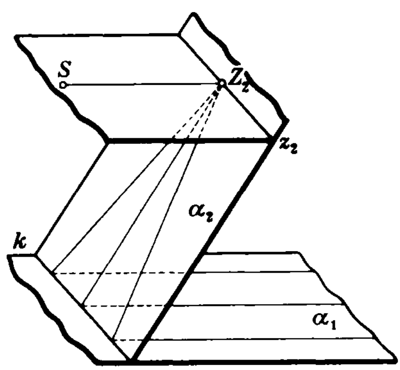

FIG. 3

In order to remove the inconvenience caused by the absence of points which would correspond to the exceptional points (on z_2 or on n_1) of the other plane, we extend the concept of plane in the following manner:

We are accustomed to the use of the notion of direction in geometry. Let us include in the set of all points of a plane the set of all directions. In order to signify that directions will be regarded as elements of the same kind as points, let us call them *points at infinity*. We shall denote them, just as ordinary points (called *ordinary points*), by the letters A, B, ..., P adding the index ∞: thus $A^\infty, B^\infty, \ldots, P^\infty$. Let us make one more agreement: instead of saying that "the straight line p has the direction A^{∞}" we shall say that "the straight line p passes through the point A^{∞}" or that "the point A^{∞} lies on the straight line p", and instead of saying that "two straight lines have a common direction B^{∞}" we shall say that "two straight lines intersect at the point B^{∞}". It will be observed that on a plane extended in this manner every two straight lines have a point in common (i.e. intersect); that point is an ordinary point or a point at infinity. The set of all points at infinity on a plane will be called the *straight line at infinity*. The straight line at infinity has one point in common with every ordinary straight line — it is the point at infinity of that line. A plane extended by points at infinity is called a *projective plane*. An ordinary plane, without points at infinity, is called a *Euclidean plane*.

Now if α_1 and α_2 are projective planes, it can easily be seen that there is a correspondence between the points of the plane α_2 which lie on the straight line z_2 and the points at infinity of the plane α_1 and between the straight line z_2 and the straight line at infinity z_1^{∞} of the plane α_2. Thus the correspondence defined at the beginning of this section and applied to the projective planes α_1 and α_2 is a transformation which changes the whole projective plane α_1 into the whole projective plane α_2; it is termed a *projective transformation* of the planes α_1 and α_2. Henceforth, by a plane we shall mean a projective plane.

2.2. As we know, a Euclidean plane can be represented analytically as a set of ordered pairs of numbers x and y (the so-called *coordinates*): points lie on a straight line if and only if their coordinates satisfy an equation of the first degree, i.e. an equation of the type $ax+by+c = 0$ in which $a^2+b^2 > 0$.

The question arises how to represent analytically a projective plane. If we retain numbers x and y as the coordinates of an ordinary point, what should we assume as the coordinates of a point at infinity? In order to answer this question let us take two straight lines intersecting at a point at infinity:

$$ax+by+c = 0 \quad \text{and} \quad ax+by+c' = 0 \quad \text{where} \quad c \neq c'. \tag{2.1}$$

There are of course no numbers x and y that would satisfy both equations. However, write the ratio x_1/x_3 instead of x and the ratio x_2/x_3 instead of y. We obtain the equations

$$a\frac{x_1}{x_3}+b\frac{x_2}{x_3}+c = 0 \quad \text{and} \quad a\frac{x_1}{x_3}+b\frac{x_2}{x_3}+c' = 0$$

or

$$ax_1+bx_2+cx_3 = 0 \quad \text{and} \quad ax_1+bx_2+c'x_3 = 0.$$

It will be observed that the last two equations are satisfied if and only if we take 0 as x_3. As x_1 and x_2 we can take for instance the numbers b and $-a$. This suggests the idea of regarding the numbers $x_1 = b$, $x_2 = -a$ and $x_3 = 0$ as the coordinates of the point at infnity of the straight lines (2.1) on the projective plane. If $x_3 = 0$, then every three numbers x_1, x_2 and x_3 will be regarded as three coordinates of the new kind of the point $(x_1/x_3, x_2/x_3)$. Such three numbers x_1, x_2, x_3 will be called *homogeneous coordinates* on the projective plane.

The equation of the straight line will then be changed into the homogeneous equation $ax_1+bx_2+cx_3 = 0$. Therefore, if a certain triple x_1, x_2, x_3 satisfies this equation, every proportional triple kx_1, kx_2, kx_3 will also satisfy it. It will be observed that every triple of numbers with the third number equal to 0, which is inadmissible in the substitution $x = x_1/x_3$ and $y = x_2/x_3$, can be regarded, as we have just seen, as three coordinates of a point at infinity, since, if it satisfies the equation of a certain straight line $ax_1+bx_2+cx_3 = 0$, then it satisfies also the equation of every parallel line $ax_1+bx_2+c'x_3 = 0$ for an arbitrary c'.

Thus every triple of numbers x_1, x_2, x_3 with the exception of the triple 0, 0, 0 has a corresponding point on the projective plane, the same point corresponding to proportional threes. Thus the equation of the straight line on a projective plane is the homogeneous equation $u_1x_1+u_2x_2+u_3x_3 = 0$ in which the coefficients u_1, u_2, u_3 are not all equal to zero.

E.g. the equation of a straight line at infinity is of the form

$$0x_1+0x_2+x_3 = 0, \quad \text{i.e.} \quad x_3 = 0,$$

since it is satisfied by every triple of numbers x_1, x_2, 0. Similarly, the axis x has the equation $x_2 = 0$ and the axis y the equation $x_1 = 0$.

§ 3. Projective (collineation) transformations

3.1. A one-to-one correspondence between the points of two projective planes α_1 and α_2 (different or not) in which every triple of collinear points has a triple of collinear points assigned to it on the other plane is called a *collineation transformation.*

The correspondence defined at the beginning of the preceding section (Fig. 2) provides an example of a collineation transformation. In that correspondence the points X_1 and X_2 of the planes α_1 and α_2 correspond to each other only if the straight line X_1X_2 contains a certain fixed point S belonging neither to α_1 nor to α_2; for, as we have seen, in that transformation three collinear points always change into three collinear points.

It will be observed that if we take two collineation transformations (Fig. 4): 1° between the planes α_1 and α_2 and 2° between

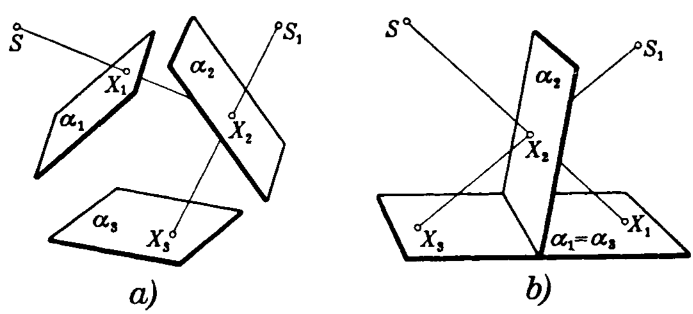

FIG. 4

the planes α_2 and α_3, we can define a new transformation between the planes α_2 and α_3 regarding as corresponding points such two points X_1 and X_2 as correspond on the strength of transformations 1° and 2° to the same point X_2 of the plane α_2.

In Fig. 4 the above is shown in the case where both the transformation of α_1 into α_2 and the transformation of α_2 into α_3 are central projections (from point S and from point S_1) of one plane upon another. In Fig. 4a the planes α_1 and α_3 are different from each other, and in Fig. 4b $\alpha_1 = \alpha_3$.

The transformation of the plane α_1 into the plane α_3 is a collineation transformation because the condition that collinear threes of points should be changed to collinear threes is a transitive property. A transformation of the plane α_1 into the plane α_3 obtained by means of the transormations of α_1 into α_2 and α_2 into α_3 is called a *combined transformation* or a *superposition of two transformations*. Obviously a transformation of the plane α_1 into the plane α_n obtained by a combination of a finite number of projective transformations

$$\alpha_1 \text{ into } \alpha_2, \quad \alpha_2 \text{ into } \alpha_3, \quad \ldots, \quad \alpha_{n-1} \text{ into } \alpha_n,$$

and also called a *projective transformation*, is a collineation transformation.

Our further considerations will concern certain properties of projective tranformations and a proof of a theorem that is essential for our purpose:

Let A_1, B_1, C_1 and D_1 be four arbitrary points on the plane α_1 no three of which are collinear, and let A_2, B_2, C_2 and D_2 be four arbitrary points on the plane α_2 no three of which are collinear. Then there exists one and only one projective transformation of the plane α_1 into the plane α_2 such that point A_1 is changed to point A_2, point B_1 to point B_2, point C_1 to point C_2 and point D_1 to point D_2.

In the proof we shall use both geometrical and analytical methods according to which of them give quicker results. We shall also find analytical methods of representing projective transformations.

3.2. On an arbitrary straight line let us take two ordinary points A and B and an arbitrary point C different from point B (Fig. 5). Assume that a unit of measure and the sense on the straight line have been chosen. The fraction

$$\lambda_C = \frac{AC}{BC},$$

in which AC and BC denote the measures of the vectors $\overline{AC}$ and $\overline{BC}$ (i.e. lengths provided with a suitable sign depending on the sense), is called the *division ratio* for the point C with respect to the points A and B.

It will be observed that the number λ_C does not depend either on the unit (by a change of the unit the numerator and the denominator will be increased by the same factor) or on the sense (the change of vectors $\overline{AC}$ and $\overline{BC}$ to the opposite sense will cause a change of the sign both in the numerator and in the denominator). The number λ_C thus depends only on geometrical properties (on the position of the point C with respect to the points A and B). It will be seen that for points lying between

FIG. 5

the points A and B, for instance for the point C, we have $\lambda_C < 0$, while for external points, for instance for the point D, we have $\lambda_D > 0$. It is easy to see that for two different points X and Y we always have $\lambda_X \neq \lambda_Y$. Indeed, if both points were internal, like the points C and C' for instance, then for $AC > AC'$ we should have $\lambda_C > \lambda_{C'}$; if, however, both points were external, like the points D and D' for instance, then for $BD < BD'$ we should have

$$\lambda_{D'} = \frac{AD'}{BD'} = \frac{AB}{BD'} + \frac{BD'}{BD'} < \frac{AB}{BD} + \frac{BD}{BD} = \frac{AD}{BD} = \lambda_D .$$

The definition of the division ratio does not comprise the point at infinity. In view of the fact that for every sequence of points $D_1, D_2, \ldots, D_n, \ldots$ divergent to infinity we have

$$\lim_{n\to\infty} \frac{AD_n}{BD_n} = 1 ,$$

we assume $\lambda_{D\infty} = 1$.

If we are given four points A, B, C, D on a straight line and at least the first two of them are ordinary points, then the number

$$\lambda_C : \lambda_D$$

is called the *cross-ratio* of the four points A, B, C, D and denoted by the symbol $(ABCD)$.

Obviously, if the pair of points A, B separates the pair of points C, D (i.e. if one point of the second pair lies inside the segment AB and the other point lies outside the segment AB), then $(ABCD) < 0$ because the numbers λ_C and λ_D have opposite signs, while if the pairs of points A, B and C, D do not separate each other, i.e. either the points C and D lie inside the segment AB or the points C and D lie outside the segment AB, then $(ABCD) > 0$.

In geometrical constructions fours of points for which the cross-ratio has the value -1 are particularly frequent: we call them *harmonic fours.*

EXAMPLE. Let A and B be two ordinary points, S the mid--point of the segment AB and N^∞ a point at infinity. Since $\lambda_S = AS/BS = -1$ and $\lambda_{N^\infty} = 1$, we have $(ABSN^\infty) = \lambda_S : \lambda_{N^\infty} = -1$ and thus the four points A, B, S, N^∞ are a harmonic four.

Having three arbitrary points A, B, C of a straight line p let us assign to each point of the line p a number $x = (ABCX)$. It can easily be seen that the function defined in this way is *reflexive,* i.e. such that for different points X and X' we have $x \neq x'$. Indeed, $x = \lambda_C : \lambda_X$, and for different points X and X' we have $\lambda_X \neq \lambda_{X'}$; consequently $x \neq x'$.

3.3. Let A, B, C, D (Fig. 6) be ordinary points of the straight line p and W a point that does not lie on the line p. Joining the

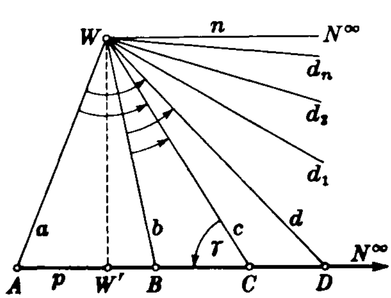

FIG. 6

point W to the points A, B, C, D we obtain the straight lines a, b, c, d. At the vertex W let us choose a sense agreeing with the sense chosen on the line p (i.e. such that the angle (ac)

is positive if the vector $\overline{AC}$ has a positive measure). We shall prove the following equality:

$$(ABCD) = \frac{\sin(ac)}{\sin(bc)} : \frac{\sin(ad)}{\sin(bd)}. \tag{3.1}$$

For this purpose let us draw a straight line WW', orthogonal to the straight line p, and using the well-known formulas for the area of a triangle let us write

$$AC|WW'| = |AW||CW| \sin(ac)$$

and

$$BC|WW'| = |BW||CW| \sin(bc) \text{ (}^1\text{)}.$$

Dividing these expressions we obtain

$$\lambda_C = \frac{AC}{BC} = \left|\frac{AW}{BW}\right| \frac{\sin(ac)}{\sin(bc)}.$$

Analogously we have

$$\lambda_D = \frac{AD}{BD} = \left|\frac{AW}{BW}\right| \frac{\sin(ad)}{\sin(bd)}$$

and thus

$$\lambda_C : \lambda_D = \frac{\sin(ac)}{\sin(bc)} : \frac{\sin(ad)}{\sin(bd)}.$$

It will be observed that formula (3.1) also holds if the point D (or the point C) is a point at infinity (then of course the straight line d or the straight line c is parallel to the straight line p). Indeed, taking the sequence of points $D_1, D_2, \ldots, D_n, \ldots$ and the sequence of corresponding straight lines $d_1, d_2, \ldots, d_n$ tending to the straight line n, we obtain by (3.1)

$$(ABCD_n) = \frac{\sin(ac)}{\sin(bc)} : \frac{\sin(ad_n)}{\sin(bd_n)}.$$

Hence, in view of the continuity of the function $\sin x$, we obtain in the limit

$$\lim_{n\to\infty} (ABCD_n) = (ABCN^{\infty}) = \frac{\sin(ac)}{\sin(bc)} : \frac{\sin(an)}{\sin(bn)}.$$

(1) AC and BC are the measures of the vectors on the axis p; $|WW'|, \ldots$ are the lengths of the corresponding segments.

Since the right side of formula (3.1) depends only on the angles contained between the straight lines a, b, c, d, intersecting these lines by an arbitrary straight line p at the points A_1, B_1, C_1, D_1 (Fig. 7) we obtain the equality

$$(A_1B_1C_1D_1) = (ABCD).$$

This theorem is called the *theorem of Pappus.*

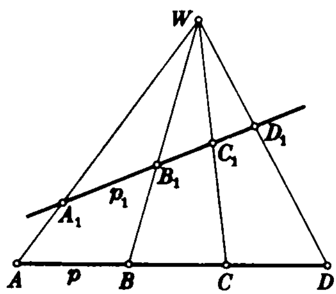

FIG. 7

The cross-ratios of two fours of points of which one is a central projection of the other are equal.

3.4. *The value of the cross-ratio of four points lying on a straight line is an invariant of the transformation defined by the so-called homographic function*

$$y = \frac{ax+b}{cx+d} \quad \textit{where} \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0 \ (^1) \qquad (3.2)$$

for any constant a, b, c, d.

This means that any four points X_1, X_2, X_3, X_4 of the axis x have by (3.2) corresponding points Y_1, Y_2, Y_3, Y_4 of the axis y with the same value of the cross-ratio

$$(Y_1Y_2Y_3Y_4) = (X_1X_2X_3X_4).$$

(1) If $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0$, then, as we know, the fraction can be simplified and thus y has a constant value for every value of x. For $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$ the homographic function is reflexive.

P r o o f. It will be observed that if x_i (or y_i) is a coordinate of the point X_i (or Y_i) on the straight line x (or y), then

$$Y_1 Y_3 = y_3 - y_1 = \frac{ax_3+b}{cx_3+d} - \frac{ax_1+b}{cx_1+d} = \frac{(ad-bc)(x_3-x_1)}{(cx_3+d)(cx_1+d)};$$

similarly

$$Y_2 Y_3 = y_3 - y_2 = \frac{ax_3+b}{cx_3+d} - \frac{ax_2+b}{cx_2+d} = \frac{(ad-bc)(x_3-x_2)}{(cx_3+d)(cx_2+d)};$$

consequently

$$\frac{Y_1 Y_4}{Y_2 Y_3} = \frac{x_3-x_1}{x_3-x_2} \cdot \frac{cx_2+d}{cx_1+d}.$$

Replacing the index 3 by 4 we obtain

$$\frac{Y_1 Y_4}{Y_2 Y_4} = \frac{x_4-x_1}{x_4-x_2} \cdot \frac{cx_2+d}{cx_1+d},$$

which immediately gives

$$(Y_1 Y_2 Y_3 Y_4) = \frac{x_3-x_1}{x_3-x_2} : \frac{x_4-x_1}{x_4-x_2} = (X_1 X_2 X_3 X_4). \qquad (3.3)$$

We thus see that the theorem, defined by formula (3.2), on the invariance of the cross-ratio in passing from the straight line x to the straight line y can be reversed; namely the following theorem holds:

Every correspondence between points of two straight lines in which the corresponding fours of points have the same values of the cross-ratio can be written in the form of a homographic function.

P r o o f. If X_1 and Y_1, X_2 and Y_2, and X_3 and Y_3 are pairs of corresponding points, X_1, X_2, X_3 and Y_1, Y_2, Y_3 being threes of different points, then for every pair of corresponding points X, Y we have

$$(Y_1 Y_2 Y_3 Y) = (X_1 X_2 X_3 X),$$

i.e.

$$\frac{y_3-y_1}{y_3-y_2} : \frac{y-y_1}{y-y_2} = \frac{x_3-x_1}{x_3-x_2} : \frac{x-x_1}{x-x_2}.$$

Denoting the fraction $\dfrac{y_3-y_1}{y_3-y_2} : \dfrac{x_3-x_1}{x_3-x_2}$ by $\dfrac{1}{a}$ we have

$$\frac{y-y_2}{y-y_1} = a\,\frac{x-x_2}{x-x_1},$$

whence

$$y = \frac{(y_2-ay_1)x+ax_2y_2-x_1y_2}{(1-a)x+ax_2-x_1}.$$

This is a homographic function. Its determinant is different from zero because

$$\begin{vmatrix} y_2-ay_1 & ax_2y_1-x_1y_2 \\ 1-a & ax_2-x_1 \end{vmatrix} = (x_2-x_1)\,(y_2-y_1)\frac{y_3-y_1}{y_3-y_2}\cdot\frac{x_3-x_2}{x_3-x_1} \neq 0.$$

The homographic function plays an important role in nomography.

The theorem of Pappus and formula (3.3) imply an easy construction of points assigned to one another on the basis of a homographic transformation.

Suppose that we have assigned to three points X_1, X_2, X_3 of one straight line three arbitrary points Y_1, Y_2, Y_3 of another straight line. The theorem which we have proved shows that there exists a homographic transformation (and only one such transformation) assigning to the points X_1, X_2, X_3 and X such points Y_1, Y_2, Y_3 and Y that

$$(Y_1 Y_2 Y_3 Y) = (X_1 X_2 X_3 X). \tag{3.4}$$

We shall now construct a point Y corresponding to an arbitrary point X.

On the straight line p we have the points X_1, X_2, X_3 and on the straight line p_1 the points Y_1, Y_2, Y_3. For simplicity let us assume that $X_1 = Y_1$ is a common point of the straight lines p and p_1 (Fig. 8). Let us join the points X_2 and Y_2, and then X_3 and Y_3 and denote by S the intersection point of the straight lines thus obtained. By the theorem of Pappus the points of intersection X and Y of the straight line passing through the point S with the lines p and p_1 form together with the points X_1, X_2, X_3 and Y_1, Y_2, Y_3 two fours satisfying condition (3.4).

If every pair of corresponding points consisted of different points, as X'_1, X'_2, X'_3 and Y'_1, Y'_2, Y'_3 for instance, it would be sufficient to project the three points X'_1, X'_2, X'_3 onto the straight line passing through the point Y_1 in such a way as to locate the new three points X_1, X_2, X_3 in the same position as that considered before. The passage from the straight line p' to the straight line p_1 requires projecting twice (from the point S_1 and then from the point S).

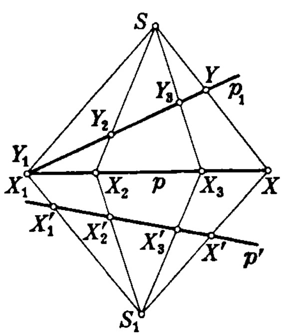

FIG. 8

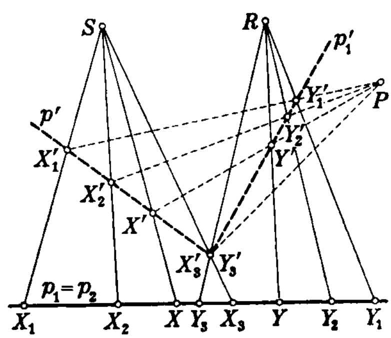

FIG. 9

If the straight lines p and p_1 were not different, it would obviously be necessary to project three times (Fig. 9). We choose intersecting straight lines $\bar{p}$ and $\bar{p}_1$ and project the points X_i onto the line $\bar{p}$ and the points Y_i onto the line $\bar{p}_1$ in such a way as to make the common point of the lines $\bar{p}$ and $\bar{p}_1$ correspond to the point X_1 and to the point Y_1. We then proceed as in the first case.

It will thus be observed that in every case we can construct by a finite number of operations of central projection a homographic correspondence such that given three points X_1, X_2, X_3 have given three points Y_1, Y_2, Y_3 corresponding to them.

3.5. Suppose we are given a projective transformation of a plane α_1 onto a plane α_2 obtained by projection from a point

S which does not lie on either of the planes α_1 and α_2 (Fig. 10). On one of these planes, e.g. on the plane α_1, let us take points A_1, B_1, C_1, D_1 lying on a straight line p_1; they have corresponding points A_2, B_2, C_2, D_2 on the straight line p_2 of the plane α_2. By the theorem of Pappus we have

$$(A_1B_1C_1D_1) = (A_2B_2C_2D_2).$$

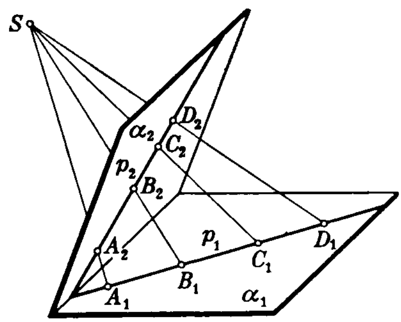

FIG. 10

If the correspondence between the planes α_1 and α_n is a superposition of $n-1$ transformations each of which is a central projection, then we obviously have

$$(A_1B_1C_1D_1) = (A_nB_nC_nD_n).$$

This means that the value of the cross-ratio of four points is an invariant of a projective transformation which is a combination of a finite number of transformations by central projection.

3.6. Four points A, B, C, D of a projective plane no three of which are collinear determine the so-called *complete quadrilateral* (Fig. 11a). It is (unlike the quadrilateral of elementary geometry) a set of six straight lines, i.e. the set of lines AB, AC, AD, BC, BD, and CD each of which contains two of the given points A, B, C, and D. These lines are called the *sides* of the complete quadrilateral and the given points are called its *vertices*. Opposite sides are such pairs of sides as have no common vertex; they are the pairs AB and CD, AC and BD, AD and BC. The intersection point of a pair of opposite sides is called a *diagonal point.* The points P, R, and Q in Fig. 11a are diagonal points.

We shall prove that *every pair of vertices (e.g. B and C), the diagonal point lying on the same side as those vertices (point Q) and the intersection point of that side with the straight line joining the remaining two diagonal points (points M) form a harmonic four.*

For the proof let us take an arbitrary point S which does not lie on the plane of the quadrilateral and denote by β_0 the plane determined by the points A, P, Q; denote by β an arbitrary plane

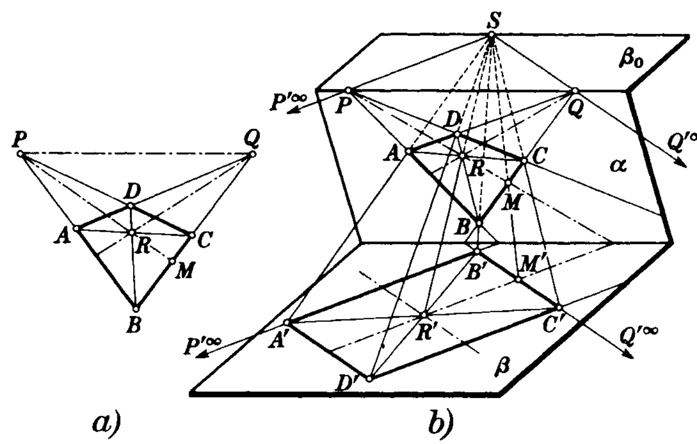

FIG. 11

paralle to the plane β_0 (Fig. 11b). It can easily be seen that the projection of the quadrilateral $ABCD$ from the point S onto the plane β is a complete quadrilateral $A'B'C'D'$ whose diagonal points P' and Q' are points at infinity. Indeed, the fact that $\beta \| \beta_0$ implies $SP \| \beta$ and $SQ \| \beta$. The pair of sides AB and CD intersecting at the point P have a corresponding pair of sides $A'B'$ and $C'D'$ parallel to SP, and similarly the pair of sides AD and BC intersecting at the point Q have a corresponding pair of sides $A'D'$ and $B'C'$ parallel to the straight line SQ; the quadrilateral $A'B'C'D'$ is thus a parallelogram. Consequently the projection M' of the point M is the centre of the segment $B'C'$. As we know

(§ 3.2, example) $(B'C'M'Q'^{\infty}) = -1$; by the theorem of Pappus we have

$$(BCMQ) = (B'C'M'Q'^{\infty}) = -1,$$

and thus $BCMQ$ is a harmonic four.

3.7. Let us take points A, B, C, D on a straight line x and consider whether there exists a pair of points X, Y for which

$$(ABXY) = -1 \quad \text{and} \quad (CDXY) = -1.$$

It will be observed at once that in view of the theorem of Pappus we can restrict ourselves to the case where the points A, B, C, D are all ordinary points (by projection every four of points may be reduced to a four of ordinary points with the same value of the cross-ratio).

In order to solve this problem let us fix the origin of the coordinates at the mid-point of the segment AB and denote the coordinates of the points A, B, C, D successively by x_1, x_2, x_3, x_4; we thus have $x_1 = -x_2$.

Obviously, if we also had $x_3 = -x_4$, the points $X = 0$ and $Y = N^{\infty}$ would be the solution of the problem.

Assume that such points X and Y exist and denote their coordinates by x and y.

Numbers x and y should satisfy the equation

$$(ABXY) = \frac{x-x_1}{x-x_2} : \frac{y-x_1}{y-x_2} = -1,$$

i.e.

$$(x-x_1)(y-x_2)+(y-x_1)(x-x_2) = 0$$

and

$$(CDXY) = \frac{x-x_3}{x-x_4} : \frac{y-x_3}{y-x_4} = -1,$$

i.e.

$$(x-x_3)(y-x_4)+(y-x_3)(x-x_4) = 0.$$

Taking into account the assumption that $x_1 = -x_2$ we obtain the system of equations

$$xy-x_1^2 = 0, \qquad 2xy-(x_3+x_4)(x+y)+2x_3x_4 = 0,$$

whence

$$xy = x_1^2 \quad \text{and} \quad x+y = 2\,\frac{x_3x_4+x_1^2}{x_3+x_4}.$$

It follows that x and y are the roots of the following equation of the second degree:

$$z^2-2\,\frac{x_3x_4+x_1^2}{x_3+x_4}\,z+x_1^2 = 0.$$

This equation has real roots only if

$$\Delta = 4\left(\frac{x_3x_4+x_1^2}{x_3+x_4}\right)^2-4x_1^2 > 0.$$

Therefore we must have $(x_3x_4+x_1^2)^2-x_1^2\,(x_3+x_4)^2 > 0$, i.e. $(x_3^2-x_1^2)\,(x_4^2-x_1^2) > 0$. This means that either $x_3^2 < x_1^2$ and $x_4^2 < x_1^2$ or $x_3^2 > x_1^2$ and $x_4^2 > x_1^2$. In the first case the points C, D lie inside the segment AB, in the second case the points A, B lie inside the segment CD. In such cases we say (§ 3.2) that the pairs A, B and C, D do not separate each other.

Consequently for given pairs of points A, B and C, D a common pair of points X, Y forming harmonic fours with the given points of both pairs exists only if the pairs A, B and C, D do not separate each other.

3.8. *Let us take on a plane α arbitrary points A, B, C, D, no three of which are collinear, and on a plane β arbitrary four points $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$, of which again no three are collinear. Then there exists a projective transformation of the plane α onto the plane β which assigns point $\bar{A}$ to point A, point $\bar{B}$ to point B, point $\bar{C}$ to point C and point $\bar{D}$ to point D.*

In particular we shall prove that this transformation is a superposition of several transformations which are projections of one plane onto the other (from an ordinary point or from a point at infinity).

In the proof we shall speak of a translation of a plane by a vector which is not parallel to it (Fig. 12a) and of a rotation of a plane about a straight line s lying off that plane (Fig. 12b). Assigning to point X, which is a point of the plane in its initial position, point X' after the translation or rotation of the plane, we shall see that the transformation defined in this manner is a projection from the point at infinity S^{∞}. In the case of translation the point S^{∞} is the point at infinity of the vector XX' and in the case of rotation through an angle different from π the centre of projection is the point at infinity of a straight line perpendicular to the plane of symmetry of the rotation angle.

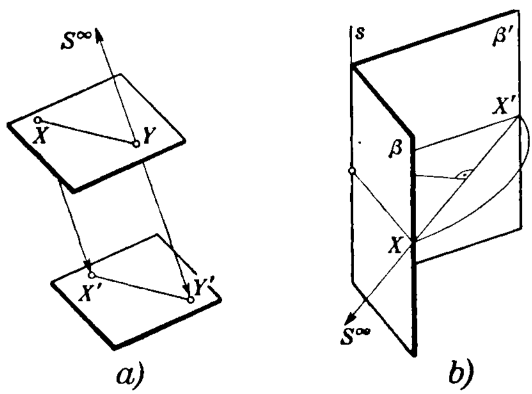

FIG. 12

Proceeding now to the proof let us assume that the planes α and β intersect along an ordinary straight line which does not pass through the point $\bar{A}$ and that both A and $\bar{A}$ are ordinary points (¹). Consider the following transformation, consisting in projections of one plane onto the other:

a. A translation of the plane α (Fig. 13a) by the vector $\overline{A\bar{A}}$ (lying off the plane α) so as to transform the quadrilateral $ABCD$ into the quadrilateral $A'B'C'D'$, with $A' = \bar{A}$.

b. A projection of the plane α' (Fig. 13b) onto the plane α'' passing through the straight line $\bar{A}\bar{B}$ from the point S_1 of intersection S_1 of the straight lines $B'\bar{B}$ and $P'\bar{P}$ (the points P' and $\bar{P}$ are diagonal points of the corresponding complete quadrilaterals). The projection centre S_1 exists because the points

(¹) It follows from the assumption that at least two of the points A, B, C, D are ordinary points.

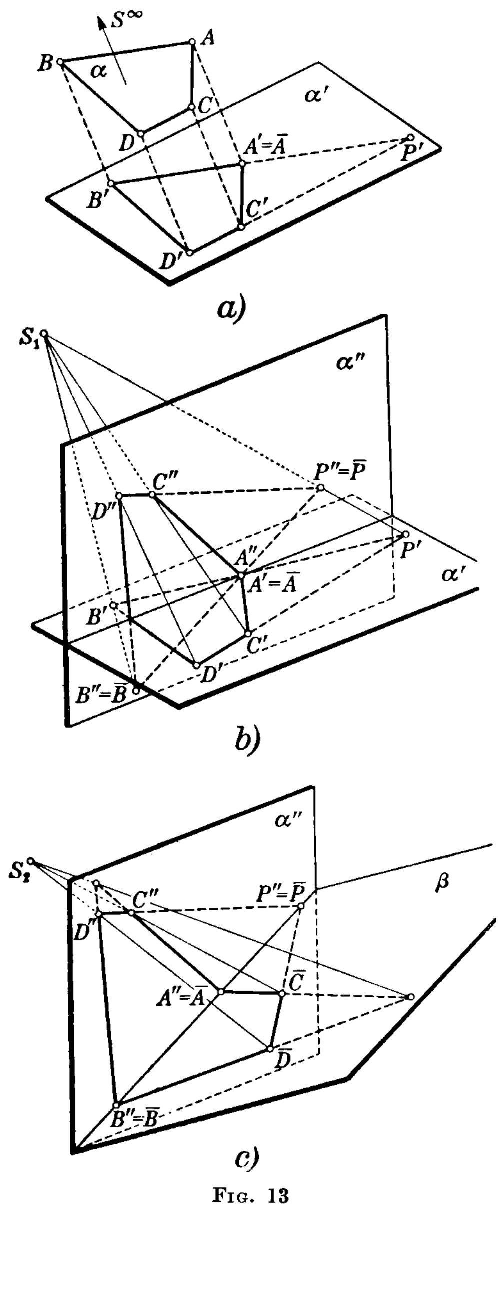

Fig. 13

B', P', A' and the corresponding points $\bar{B}$, $\bar{P}$, $\bar{A}$ lie on the same plane. We thus obtain a quadrilateral $A''B''C''D''$ two vertices of which, A'' and B'', coincide with the points $\bar{A}$ and $\bar{B}$, and moreover the diagonal point P'' coincides with the point $\bar{P}$.

c. A projection from the point of intersection S_2 of the straight lines $C''\bar{C}$ and $D''\bar{D}$, which finally transforms the quadrilateral $A''B''C''D''$ into the quadrilateral $\bar{A}\bar{B}\bar{C}\bar{D}$. The projection centre S_2 exists because the points C'', B'', P'', equal to the points $\bar{C}$, $\bar{D}$, $\bar{P}$ respectively, lie on the same plane.

We have thus proved the theorem.

3.9. A projective transformation is, as we know, a collineation transformation because it transforms every triple of collinear points of one plane into three collinear points of the other plane. We shall prove that the projective transformation, just defined, of the plane α into the plane β, assigning four points $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$ to given four points A, B, C, D, is a collineation transformation satisfying a given condition. This means that if $f(X)$ and $g(X)$ are two collineation transformations which satisfy the conditions $f(A) = \bar{A} = g(A)$, $f(B) = \bar{B} = g(B)$, $f(C) = \bar{C} = g(C)$ and $f(D) = \bar{D} = g(D)$, then we have to prove that for every point X of the plane $g(X) = f(X)$.

We shall denote by $X = f^{-1}(\bar{X})$ a transformation inverse to the transformation $\bar{X} = f(X)$, i.e. such as assigns a point X to every point $\bar{X}$. Thus, according to our agreement, $f^{-1}(f(X)) = X$ and similarly $f(f^{-1}(\bar{X})) = \bar{X}$.

Suppose that we are given two collineation transformations $f(X)$ and $g(X)$ of a plane α into a plane β. We are to show that the transformations $f(X)$ and $g(X)$ are identical, i.e. that for every X we have $f(X) = g(X)$. Accordingly let us consider a mapping $Y = f^{-1}(g(X))$ where $f^{-1}(\bar{X})$ is a transformation inverse to $f(X)$. Obviously this mapping, which will be denoted by $Y = F(X)$, transforms into themselves those points which correspond to the same point $\bar{X}$ by the mappings $f(X)$ and $g(X)$. It follows that $F(X)$ assigns to each of the points A, B, C, D the same points. Since $F(X)$ is of course a collineation mapping, the problem is reduced to the following question: Is the transformation

$F(X)$, which retains collinearity and assigns the same points to the points B, C, D (among which there are no threes of collinear points), necessarily an identity transformation, i.e. does $F(X) = X$ always hold?

To begin with, it will be observed that in every one-to-one correspondence retaining collinearity a harmonic four of points is transformed into a harmonic four. This follows from the fact that for three points A, B, C there exists only one point D such that $(ABCD) = -1$ and that the point D can be obtained by drawing a complete quadrilateral $ABMN$ with vertices A and B and the diagonal point C (Fig. 14); for in a collineation mapping a complete quadrilateral is transformed into a complete quadrilateral. It follows (on the grounds of the considerations of § 3.8) that in a collineation transformation separating fours are transformed into separating fours and non-separating fours are transformed into non-separating fours.

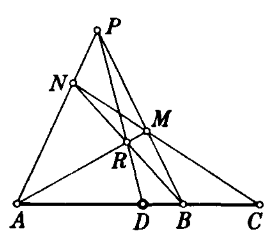

FIG. 14

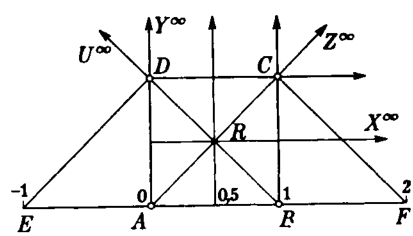

FIG. 15

We can assume without loss of generality that the points A, B, C, D have the coordinates 0, 0; 1, 0; 1, 1; 0, 1 respectively (Fig. 15). It will be seen that the sides of the quadrilateral obtained are transformed into themselves, whence also the diagonal points, i.e. the points X^∞, Y^∞ and R, are transformed into themselves. Now this implies that the straight lines RX^∞, RY^∞ and $X^\infty Y^\infty$ are transformed into themselves and that the points of intersection Z^∞ and U^∞ of the lines AC and BD with the straight line at infinity are transformed into themselves. It is now obvious that the points of intersection E and F of the axis x with the straight lines DZ and CU are transformed into themselves;

their abscissas are -1 and 2. It can be shown by induction that every point of the axis x with a positive abscissa is transformed into itself. But the fact that the straight lines RX^{∞} and RY^{∞} are transformed into themselves implies that the point of the axis x with the abscissa $1/2$ is transformed into itself, and, generally, all points of the axis x with abscissas $n/2$ where n is an arbitrary integer are transformed into themselves. Repeating this procedure we can see that all points of the axis x with abscissas $n/4$ are transformed into themselves, and then also all points with abscissas $n/2^k$ where n and k are arbitrary integers; they are the so-called *dyadic rational numbers*. As we know, they constitute a dense set on a straight line (1).

We shall also prove that, if x is not a dyadic rational number, then the point $X(x, 0)$ is also transformed into itself. It would be sufficient of course to prove that if numbers d_1 and d_2 are dually rational, with $d_1 < x < d_2$ and x' denoting the abscissa of a point corresponding to the point X, then also $d_1 < x' < d_2$. This, however, results directly from the proposition that the pairs of points $D_1(d_1, 0)$, $D_2(d_2, 0)$ and X, X^{∞} separate one another and from the fact that in every collineation transformation of a plane harmonic fours are transformed into harmonic fours. Indeed, in view of the theorem of 3.7, § 3, a pair that would be harmonic with the pair D_1, D_2 on one hand and with the pair X, X on the other hand does not exist.

It follows that there exists no pair harmonic with D_1', D_2' on one hand and with X', X' on the other hand. Since $D_1' = D_1$, $D_2' = D_2$ and $X' = X$, there exists no pair harmonic both with D_1, D_2 and with X', X. Thus D_1D_2 separates the pair X', X, i.e. x' lies between D_1 and D_2 or $d_1 < x' < d_2$. This, however, holds for every pair of dyadic rational numbers d_1 and d_2, i.e. the inequalities $d_1 < x < d_2$ always imply the inequalities $d_1 < x' < d_2$. Hence $X' = X$. Thus every point of the axis x is transformed into itself; similarly every point of the

(1) A set Z of numbers is called *dense* if in every interval (x_1, x_2) there exists a number of that set, i.e. a number x belonging to Z such that $x_1 < x < x_2$.

axis y is transformed into itself. This finally implies that every point of a plane is transformed into itself, q.e.d.([1]).

We have proved at the same time that *every transformation of a plane which retains the collinearity of points is a projective transformation, i.e. can be obtained by a finite number of projections of a plane onto a plane.*

§ 4. Analytical representation of a projective transformation

4.1. Consider two planes, α and β. Denote the homogeneous coordinates on the plane α by x_1, x_2, x_3, and the homogeneous coordinates on the plane β by y_1, y_2, y_3.

Suppose we are given a tranformation of the plane α into the plane β by means of homogeneous linear equations

$$\begin{aligned} y_1 &= a_{11}x_1+a_{12}x_2+a_{13}x_3, \\ y_2 &= a_{21}x_1+a_{22}x_2+a_{23}x_3, \\ y_3 &= a_{31}x_1+a_{32}x_2+a_{33}x_3 \end{aligned} \tag{4.1}$$

with the determinant

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0 \tag{4.2}$$

where a_{ik} are numerical coefficients. From the assumption that $A \neq 0$ it follows that every non-zero triple y_1, y_2, y_3 (i.e. a triple that is different from the triple (0, 0, 0)) has a corresponding non-zero triple x_1, x_2, x_3, i.e. every point Y of the plane β has a corresponding point X of the plane α. The plane α is thus transformed into the whole plane β, the transformation being one-to-one. The coordinates x_1, x_2, x_3 of the point X corresponding to the point Y are defined by the formulas

$$\begin{aligned} Ax_1 &= A_{11}y_1+A_{21}y_2+A_{31}y_3, \\ Ax_2 &= A_{12}y_1+A_{22}y_2+A_{32}y_3, \\ Ax_3 &= A_{13}y_1+A_{23}y_2+A_{33}y_3, \end{aligned}$$

where A_{ik} is a minor corresponding to the term a_{ik} in the determinant A. (If $A = 0$, then, as we know, equations (4.1) would have a solution x_1, x_2, x_3 only for numbers y_1, y_2, y_3 satisfying

([1]) The proof given here is due to Professor S. Straszewicz.

a certain linear equation $my_1+ny_2+py_3 = 0$; the whole plane α would then be transformed into a certain straight line of the plane β. Since that case does not interest us here, we have assumed that $A \neq 0$.)

We shall show that in transformation (4.1) straight lines are transformed into straight lines.

Indeed, if we are given a straight line

$$b_1y_1+b_2y_2+b_3y_3 = 0,$$

where the coefficients b_1, b_2, b_3 are not simultaneously equal to zero, then substituting in this equation the right sides of equations (4.1) we obtain

$$b_1(a_{11}x_1+a_{12}x_2+a_{13}x_3)+b_2(a_{21}x_1+a_{22}x_2+a_{23}x_3) \\ +b_3(a_{31}x_1+a_{32}x_2+a_{33}x_3) = 0$$

i.e., an equation of a straight line since the coefficients of x_1, x_2, x_3, i.e. the numbers

$$a_{11}b_1+a_{21}b_2+a_{31}b_3 = B_1,$$
$$a_{12}b_1+a_{22}b_2+a_{32}b_3 = B_2,$$
$$a_{13}b_1+a_{23}b_2+a_{33}b_3 = B_3,$$

are not simultaneously equal to zero if b_1, b_2, b_3 are not equal to zero.

We thus have a transformation retaining collinearity. We have shown that every transformation defined by formulas (4.1) is a projective transformation. We shall now prove that, conversely, every projective transformation can be expressed by formulas (4.1). For this purpose it is sufficient to prove that there exists a transformation of the form (4.1) which, for instance, transforms a quadrilateral with vertices $X_1(1, 0, 0)$, $X_2(0, 1, 0)$, $X_3(0, 0, 1)$, $X_4(1, 1, 1)$ into a quadrilateral with arbitrarily given vertices $Y_1(a_1, a_2, a_3)$, $Y_2(b_1, b_2, b_3)$, $Y_3(c_1, c_2, c_3)$, $Y_4(d_1, d_2, d_3)$ on the plane β.

Before we define the coefficients a_{ik} of this transformation let us observe that there exist numbers u, v, w, none of them equal to zero, which satisfy the equations

$$d_1 = a_1u+b_1v+c_1w,$$
$$d_2 = a_2u+b_2v+c_2w,$$
$$d_3 = a_3u+b_3v+c_3w.$$

This follows from the fact that each of the determinants

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, \quad \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, \quad \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

is different from zero because none of the threes of points Y_1, Y_2, Y_3; Y_4, Y_2, Y_3; Y_1, Y_4, Y_3; Y_3, Y_2, Y_4 is collinear. Let us multiply the coordinates of the points Y_1, Y_2, Y_3 successively by u, v, w and take the following transformation:

$$y_1 = a_1ux_1+b_1vx_2+c_1wx_3,$$
$$y_2 = a_2ux_1+b_2vx_2+c_2wx_3,$$
$$y_3 = a_3ux_1+b_3vx_2+c_3wx_3.$$

It will be seen that by substituting successively the coordinates of the points $X_1(1, 0, 0)$, $X_2(0, 1, 0)$, $X_3(0, 0, 1)$, $X_4(1, 1, 1)$ we obtain the coordinates of the points $Y_1(a_1u, a_2u, a_3u)$, $Y_2(b_1v, b_2v, b_3v)$, $Y_3(c_1w, c_2w, c_3w)$, $Y_4(d_1, d_2, d_3)$.

We have thus proved that every projective transformation can be written in form (4.1).

4.2. The so-called *affine transformations*, i.e. *transformations through affinity*, are a particular case of projective transformations.

A projective transformation is called *affine* if every point at infinity has a corresponding point at infinity. As we see from formula (4.1) that occurs only if

$$a_{31} = a_{32} = 0 \quad \text{and} \quad a_{33} \neq 0.$$

Since triples of proportional numbers are coordinates of the same point, we can assume that $a_{33} = 1$. Then, dividing both sides of the first two equations of (4.1) by y_3 or by x_3 (which is equal to y_3) and replacing the fractions x_1/x_3, x_2/x_3, y_1/y_3, y_2/y_3 by the Cartesian coordinates x, y, ξ, η, we obtain formulas defining the affine transformation

$$\xi = a_{11}x+a_{12}y+a_{13}, \qquad \eta = a_{21}x+a_{22}y+a_{23}. \tag{4.3}$$

From the assumption that the determinant (4.2) is different from zero, we obtain, on expanding it according to the terms of the third line,

$$A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0. \tag{4.4}$$

It follows immediately from the definition of the affine transformation (or from formulas (4.3)) that every pair of parallel straight lines l_1 and l_2 is transformed into a pair of parallel lines l_1' and l_2'. Indeed, an affine transformation transforms a whole Euclidean plane into a whole Euclidean plane, and thus if the straight lines l_1' and l_2' had a point in common, then the straight lines l_1 and l_2 would also have a point in common, which is contrary to our assumption. Thus a parallelogram is transformed into a parallelogram.

Let us now choose three points A, B, C on a certain straight line l: the corresponding points A', B', C' lie on a straight line l'. As follows from the considerations of § 3, we have the equation

$$(A'B'C'D'^{\infty}) = (ABCD^{\infty}),$$

where D^{∞} and D'^{∞} are by hypothesis corresponding points. Since the division ratio for a point at infinity is equal to unity, we obtain from the last equation

$$\frac{A'C'}{B'C'} = \frac{AC}{BC}.$$

It will thus be seen that all segments on the straight line l (e.g. AC and BC) undergo the same contraction (or elongation) after an affine transformation.

We shall also prove that having three arbitrary non-collinear points M, N, P on one plane and three arbitrary points M', N', P', also non-collinear, on another plane, we can find such an affine transformation of one plane into the other that the points M, N, P will be transformed into the points M', N', P' respectively.

In order to find the coefficients a_{ik} of the required transformation we write formulas defining the transformation of the points $M(x_m, y_m)$, $N(x_n, y_n)$, $P(x_p, y_p)$ into the points $M'(\xi_m, \eta_m)$, $N'(\xi_n, \eta_n)$, $P'(\xi_p, \eta_p)$:

$$\begin{aligned} \xi_m &= a_{11}x_m + a_{12}y_m + a_{13}, & \eta_m &= a_{21}x_m + a_{22}y_m + a_{23}, \\ \xi_n &= a_{11}x_n + a_{12}y_n + a_{13}, & \eta_n &= a_{21}x_n + a_{22}y_n + a_{23}, \\ \xi_p &= a_{11}x_p + a_{12}y_p + a_{13}, & \eta_p &= a_{21}x_p + a_{22}y_p + a_{23}. \end{aligned} \tag{4.5}$$

It will be seen that these two systems of equations with unknowns a_{11}, a_{12}, a_{13} and a_{21}, a_{22} and a_{23} have a unique solution because the determinant (common to both systems)

$$W = \begin{vmatrix} x_m & y_m & 1 \\ x_n & y_n & 1 \\ x_p & y_p & 1 \end{vmatrix}$$

is different from zero, the points M, N, P not being collinear.

It must also be shown that the determinant of the affine transformation

$$\xi = a_{11}x+a_{12}y+a_{13}, \qquad \eta = a_{21}x+a_{22}y+a_{23},$$

i.e. the determinant

$$w = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix},$$

is different from zero. Since equations (4.5) imply

$$Wa_{11} = \begin{vmatrix} \xi_m & y_m & 1 \\ \xi_n & y_n & 1 \\ \xi_p & y_p & 1 \end{vmatrix} = W_1, \qquad Wa_{12} = \begin{vmatrix} x_m & \xi_m & 1 \\ x_n & \xi_n & 1 \\ x_p & \xi_p & 1 \end{vmatrix} = W_2,$$

$$Wa_{21} = \begin{vmatrix} \eta_m & y_m & 1 \\ \eta_n & y_n & 1 \\ \eta_p & y_p & 1 \end{vmatrix} = W_3, \qquad Wa_{22} = \begin{vmatrix} x_m & \eta_m & 1 \\ x_n & \eta_n & 1 \\ x_p & \eta_p & 1 \end{vmatrix} = W_4,$$

we obtain by substituting w in the determinant

$$\begin{aligned} W^2w &= \begin{vmatrix} W_1 & W_2 \\ W_3 & W_4 \end{vmatrix} = W_1W_4-W_2W_3 \\ &= (\xi_ny_p+\xi_py_m+\xi_my_n-\xi_py_n-\xi_my_p-\xi_ny_m)\times \\ &\qquad \times(x_n\eta_p+x_p\eta_m+x_m\eta_n-x_p\eta_n-x_m\eta_p-x_n\eta_m)- \\ &\quad -(x_n\xi_p+x_p\xi_m+x_m\xi_n-x_n\xi_m-x_p\xi_n-x_m\xi_p)\times \\ &\qquad \times(y_p\eta_n+y_m\eta_p+y_n\eta_m-y_n\eta_p-y_p\eta_m-y_m\eta_n) \\ &= (\xi_m\eta_n+\xi_p\eta_m+\xi_n\eta_p-\xi_p\eta_n-\xi_n\eta_m-\xi_m\eta_p)\times \\ &\qquad \times(x_my_n+x_py_m+x_ny_p-x_py_n-x_ny_m-x_my_p) = WW', \end{aligned}$$

where

$$W' = \begin{vmatrix} \xi_m & \eta_m & 1 \\ \xi_n & \eta_n & 1 \\ \xi_p & \eta_p & 1 \end{vmatrix}.$$

Since $W \neq 0$ and $W' \neq 0$ (the points M', N', P' not being collinear), the determinant w is different from 0.

4.3. We shall solve the following problem:

Write the formulas for a projective transformation changing given four points $A(a_1, a_2, a_3)$, $B(b_1, b_2, b_3)$, $C(c_1, c_2, c_3)$, and $D(d_1, d_2, d_3)$, no three of which are collinear, into four vertices of a rectangle.

In order to transform the quadrilateral $ABCD$ into the rectangle $A'B'C'D'$ (Fig. 16) of the plane β in such a way as to make the side $A'B'$ parallel to the side $C'D'$ and the side $A'D'$ parallel to the side $B'C'$ it is sufficient that the required transformation should assign to the diagonal points P and Q the points at infinity P' and Q' lying on the axes of the system, i.e. the points (1, 0, 0) and (0, 1, 0).

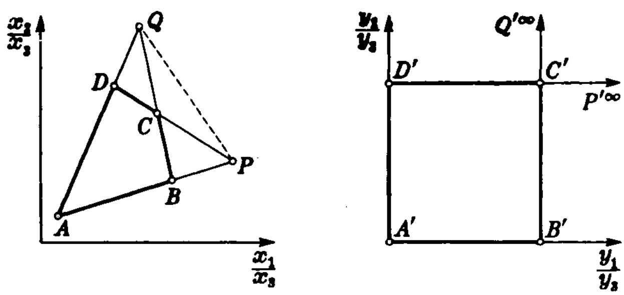

FIG. 16

We shall give a solution which satisfies one more condition: The point A is transformed into the point (0, 0, 1), which is the origin of the system of axes y_1/y_3 and y_2/y_3.

To begin with, it will be observed that the coefficients a_{ik} of transformation (4.1) should be determined in such a manner as to satisfy the following conditions:

a. Every point of the straight line AB, i.e. the straight line

$$\begin{vmatrix} a_1 & b_1 & x_1 \\ a_2 & b_2 & x_2 \\ a_3 & b_3 & x_3 \end{vmatrix} = 0,$$

should have a corresponding point (y_1, y_2, y_3) for which $y_2 = 0$.

b. Every point of the straight line AD, i.e. the straight line

$$\begin{vmatrix} a_1 & d_1 & x_1 \\ a_2 & d_2 & x_2 \\ a_3 & d_3 & x_3 \end{vmatrix} = 0$$

should have a corresponding point (y_1, y_2, y_3) for which $y_1 = 0$.

c. Every point of the straight line PQ, i.e. the straight line

$$\begin{vmatrix} p_1 & q_1 & x_1 \\ p_2 & q_2 & x_2 \\ p_3 & q_3 & x_3 \end{vmatrix} = 0,$$

where p_1, p_2, p_3 and q_1, q_2, q_3 are the coordinates of the diagonal points P and Q, should have a corresponding point (y_1, y_2, y_3) for which $y_3 = 0$.

It is obvious that conditions a, b and c will be satisfied if we assume

$$y_1 = \begin{vmatrix} a_1 & d_1 & x_1 \\ a_2 & d_2 & x_2 \\ a_3 & d_3 & x_3 \end{vmatrix}, \quad y_2 = \begin{vmatrix} a_1 & b_1 & x_1 \\ a_2 & b_2 & x_2 \\ a_3 & b_3 & x_3 \end{vmatrix}, \quad y_3 = \begin{vmatrix} p_1 & q_1 & x_1 \\ p_2 & q_2 & x_2 \\ p_3 & q_3 & x_3 \end{vmatrix}.$$

Example. Transform a plane by projection in such a manner as to have the points $A(1, 1)$, $B(4, 0)$, $C(0, 3)$, D^∞ (D^∞ being the point at infinity of the line $y = 2x$) transformed into the vertices of a rectangle (Fig. 17).

The homogeneous coordinates of the points A, B, C, D^∞ are, successively the threes 1, 1, 1; 4, 0, 1; 0, 3, 1; 1, 2, 0 (the point $D^\infty(1, 2, 0)$ being a point at infinity because its third coordinate is 0 and it lies on the straight line $y = 2x$ since the first two of its coordinates satisfy the equation of that line).

In order to determine the coordinates of the point P we must solve the system of equations

$$\begin{vmatrix} 1 & 4 & x_1 \\ 1 & 0 & x_2 \\ 1 & 1 & x_3 \end{vmatrix} = 0 \text{ (line } AB), \qquad \begin{vmatrix} 0 & 1 & x_1 \\ 3 & 2 & x_2 \\ 1 & 0 & x_3 \end{vmatrix} = 0 \text{ (line } CD).$$

By calculation we obtain numbers -5, 11, 7 as the coordinates of the point P.

Similarly, in order to find the coordinates of the point Q we must solve the system of equations

$$\begin{vmatrix} 4 & 1 & x_1 \\ 0 & 2 & x_2 \\ 1 & 0 & x_3 \end{vmatrix} = 0 \text{ (line } BD\text{)}, \qquad \begin{vmatrix} 1 & 0 & x_1 \\ 1 & 3 & x_2 \\ 1 & 1 & x_3 \end{vmatrix} = 0 \text{ (line } AC\text{)}.$$

Hence we obtain numbers 11, -10, 4 as the coordinates of the point Q.

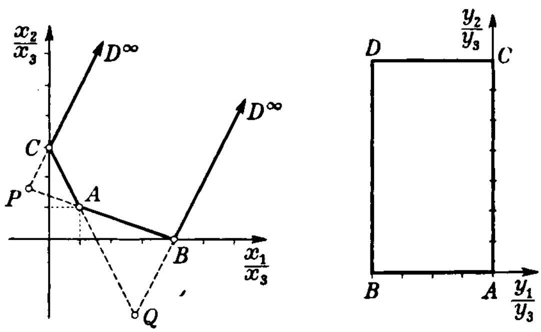

FIG. 17

We thus see that the formulas for the projective transformation have the form

$$y_1 = \begin{vmatrix} 1 & 0 & x_1 \\ 1 & 3 & x_2 \\ 1 & 1 & x_3 \end{vmatrix} = -2x_1 - x_2 + 3x_3,$$

$$y_2 = \begin{vmatrix} 1 & 4 & x_1 \\ 1 & 0 & x_2 \\ 1 & 1 & x_3 \end{vmatrix} = x_1 + 3x_2 - 4x_3,$$

$$y_3 = \begin{vmatrix} -5 & 11 & x_1 \\ 11 & -10 & x_2 \\ 7 & 4 & x_3 \end{vmatrix} = 114x_1 + 97x_2 - 71x_3.$$

Substituting the coordinates of the point D^∞, i.e. numbers 1, 2, 0, we obtain $y_1 = -4$, $y_2 = 7$, $y_3 = 308$ as the homogeneous coordinates of the point D'; the Cartesian coordinates are $-4/308$ and $7/308$.

Since the point A is transformed into the origin of the system, the given quadrilateral is transformed into a rectangle with sides 1/77 and 1/44.

4.4. The problem of a projective transformation of a plane which transforms a given quadrilateral into a rectangle is of fundamental importance for nomography. The method of solution which has been given here is not the shortest as regards calculation. The most convenient calculation for our purposes is that based on the properties of matrices.

Here is what we need to know about matrices:

Let m and n be natural numbers. Assume that every pair of natural numbers i, k, $i \leqslant m$, $k \leqslant n$, has a certain number a_{ik} corresponding to it: we thus have nm numbers a_{ik}. The system of numbers

$$\mathfrak{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = [a_{ik}] \tag{4.6}$$

written in the form of a rectangular table with m rows and n columns is called a *matrix.* If $m = n$, then matrix (4.6) is called. a *square matrix.*

Matrices $\mathfrak{A} = [a_{ik}]$ and $\mathfrak{B} = [b_{ik}]$ are regarded as equal if they are identical, i.e. $a_{ik} = b_{ik}$ for all pairs of indices i, k.

A matrix $\mathfrak{A}'$ which results from a matrix $\mathfrak{A}$ by changing rows into columns and columns into rows is called a *transposed matrix* E.g. numbers

$$\begin{bmatrix} 1 & 3 \\ 4 & 2 \\ 2 & -8 \end{bmatrix}$$

form a matrix with 3 rows and 2 columns, and the transposed matrix

$$\begin{bmatrix} 1 & 4 & 2 \\ 3 & 2 & -8 \end{bmatrix}$$

has 2 rows and 3 columns.

Square matrices for which $a_{ii} = 1$ and $a_{ik} = 0$ for $i \neq k$, i.e. matrices of the form

$$\mathfrak{J} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

are called *unity matrices*.

Consider two matrices

$$\mathfrak{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad \mathfrak{B} = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \cdot & \cdot & \cdot & \cdot \\ b_{n1} & b_{n2} & \dots & b_{np} \end{bmatrix}$$

of which the first has as many columns as the second has rows. The *scalar product* of these matrices is the name which we give to a new matrix with m rows and p columns,

$$\begin{bmatrix} d_{11} & d_{12} & \dots & d_{1p} \\ d_{21} & d_{22} & \dots & d_{2p} \\ \cdot & \cdot & \cdot & \cdot \\ d_{m1} & d_{m2} & \dots & d_{mp} \end{bmatrix}$$

in which

$$d_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \dots + a_{in}b_{nk}. \tag{4.7}$$

In the particular case where $m = 1$ and $p = 1$, i.e. where matrix $\mathfrak{A}$ has one row and matrix $\mathfrak{B}$ one column, the product $\mathfrak{A}\mathfrak{B}$ is a matrix composed of one number:

$$[a_{11}a_{12} \dots a_{1n}] \begin{bmatrix} b_{11} \\ b_{21} \\ \cdot \\ \cdot \\ \cdot \\ b_{n1} \end{bmatrix} = [a_{11}b_{11} + a_{12}b_{21} + \dots + a_{1n}b_{n1}];$$

this number is called the *scalar product* of the one-row matrix $\mathfrak{A}_1$ and the one-column matrix $\mathfrak{B}_1$.

In the general case, in the product $\mathfrak{A}\mathfrak{B}$ we have, in accordance with formula (4.6), at the place of i, k the scalar product of the row of matrix $\mathfrak{A}$ with the index i and the column of matrix $\mathfrak{B}$ with the index k. It will thus be seen that the definition of the product of matrices is meaningful only if the length of row in the first factor is the same as the length of column in the second factor.

It is easy to see that, on the whole, a product of two matrices is not commutative, i.e. $\mathfrak{A}\mathfrak{B} \neq \mathfrak{B}\mathfrak{A}$. For example if

$$\mathfrak{A} = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad \mathfrak{B} = \begin{bmatrix} 2 & 3 \\ 8 & 0 \end{bmatrix},$$

then

$$\mathfrak{A}\mathfrak{B} = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 8 & 0 \end{bmatrix} = \begin{bmatrix} 0.2+1.8 & 0.3+1.0 \\ 2.2+3.8 & 2.3+3.0 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 28 & 6 \end{bmatrix},$$

and

$$\mathfrak{B}\mathfrak{A} = \begin{bmatrix} 2 & 3 \\ 8 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 2.0+3.2 & 2.1+3.3 \\ 8.0+0.2 & 8.1+0.3 \end{bmatrix} = \begin{bmatrix} 6 & 11 \\ 0 & 8 \end{bmatrix}.$$

However, the law of associativity does hold for the product of matrices:

$$(\mathfrak{A}\mathfrak{B})\mathfrak{C} = \mathfrak{A}(\mathfrak{B}\mathfrak{C}).$$

We shall prove this for matrices having three rows and three columns (in the sequel only such matrices will be needed).

The term d_{ik} of the product $\mathfrak{A}\mathfrak{B}$ has by definition the form

$$d_{ik} = a_{i1}b_{1k}+a_{i2}b_{2k}+a_{i3}b_{3k},$$

whence the term x_{ij} of the product $(\mathfrak{A}\mathfrak{B})\mathfrak{C}$ has the form

$$\begin{aligned} x_{ij} &= d_{i1}c_{1j}+d_{i2}c_{2j}+d_{i3}c_{3j} \\ &= (a_{i1}b_{11}+a_{i2}b_{21}+a_{i3}b_{31})\,c_{1j}+(a_{i1}b_{12}+a_{i2}b_{22}+a_{i3}b_{32})\,c_{2j}+ \\ &\qquad +(a_{i1}b_{13}+a_{i2}b_{23}+a_{i3}b_{33})\,c_{3j}. \end{aligned}$$

Similarly, denoting by e_{ik} the terms of the product $\mathfrak{B}\mathfrak{C}$, we have

$$e_{ik} = b_{i1}c_{1k}+b_{i2}c_{2k}+b_{i3}c_{3k}$$

and denoting the general term of the product $\mathfrak{A}(\mathfrak{B}\mathfrak{C})$ by y_{ij}, we have

$$\begin{aligned} y_{ij} &= a_{i1}e_{1j}+a_{i2}e_{2j}+a_{i3}e_{3j} \\ &= a_{i1}(b_{11}c_{1j}+b_{12}c_{2j}+b_{13}c_{3j})+a_{i2}(b_{21}c_{1j}+b_{22}c_{2j}+b_{23}c_{3j})+ \\ &\qquad +a_{i3}(b_{31}c_{1j}+b_{32}c_{2j}+b_{33}c_{3j}). \end{aligned}$$

Obviously for every pair of indices i, j we have $x_{ij} = y_{ij}$, whence $(\mathfrak{A}\mathfrak{B})\mathfrak{C} = \mathfrak{A}(\mathfrak{B}\mathfrak{C})$.

If the matrix is square

$$\mathfrak{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix},$$

we can form a determinant

$$|\mathfrak{A}| = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix},$$

which is called the *determinant* of matrix $\mathfrak{A}$.

In the case where $|\mathfrak{A}| = 0$, matrix $\mathfrak{A}$ is called a *singular matrix*, and if $|\mathfrak{A}| \neq 0$ the matrix is called *non-singular*.

A matrix formed from matrix $\mathfrak{A}$ by deleting certain columns and certain rows in it is called a *submatrix* of matrix $\mathfrak{A}$. E.g. from the matrix

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{bmatrix}$$

we can form a submatrix

$$\begin{bmatrix} b & d \\ j & l \end{bmatrix}.$$

From a given matrix $\mathfrak{A}$ let us form all square submatrices. Let $\mathfrak{C}$ denote that singular submatrix of matrix $\mathfrak{A}$ which has the largest number of rows (columns). The number of rows (columns) of submatrix $\mathfrak{C}$ is called the *rank* of matrix $\mathfrak{A}$.

E.g. the matrix

$$\begin{bmatrix} 3 & 2 & 1 \\ 6 & 4 & 2 \end{bmatrix}$$

is of the first rank because the submatrix consisting of the term a_{11}, for instance, is non-singular but every submatrix with two rows is singular since its determinant is equal to zero.

If $\mathfrak{A}$ and $\mathfrak{B}$ are square matrices with the same number of rows (columns), then the *Cauchy formula* holds:

$$|\mathfrak{A}\mathfrak{B}| = |\mathfrak{A}|\,|\mathfrak{B}|,$$

i.e. the value of the determinant of the product of matrices $\mathfrak{A}\mathfrak{B}$ is equal to the product of the values of the determinants of matrices $\mathfrak{A}$ and $\mathfrak{B}$.

We shall prove the Cauchy formula for a matrix with three rows.

Let

$$\mathfrak{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \text{and} \quad \mathfrak{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}.$$

It follows from the definition of the product of matrices $\mathfrak{A}$ and $\mathfrak{B}$ that

$$\mathfrak{A}\mathfrak{B} = \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32} & a_{11}b_{13}+a_{12}b_{23}+a_{13}b_{33} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32} & a_{21}b_{13}+a_{22}b_{23}+a_{23}b_{33} \\ a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31} & a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32} & a_{31}b_{13}+a_{32}b_{23}+a_{33}b_{33} \end{bmatrix}.$$

Let us apply to the determinant $|\mathfrak{A}\mathfrak{B}|$ the formula for the addition of determinants differing only in the terms of one column, i.e. the formula

$$\begin{vmatrix} a_{11} & a_{12} & x_1 \\ a_{21} & a_{22} & x_2 \\ a_{31} & a_{32} & x_3 \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & y_1 \\ a_{21} & a_{22} & y_2 \\ a_{31} & a_{32} & y_3 \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & x_1+y_1 \\ a_{21} & a_{22} & x_2+y_2 \\ a_{31} & a_{32} & x_3+y_3 \end{vmatrix}.$$

Making repeated use of this formula we shall be able to write the determinant $|\mathfrak{A}\mathfrak{B}|$ as a sum of 27 determinants of the type

$$\begin{vmatrix} a_{1i}b_{i1} & a_{1j}b_{j2} & a_{1s}b_{s3} \\ a_{2i}b_{i1} & a_{2j}b_{j2} & a_{2s}b_{s3} \\ a_{3i}b_{i1} & a_{3j}b_{j2} & a_{3s}b_{s3} \end{vmatrix} = b_{i1}b_{j2}b_{s3} \begin{vmatrix} a_{1i} & a_{1j} & a_{1s} \\ a_{2i} & a_{2j} & a_{2s} \\ a_{3i} & a_{3j} & a_{3s} \end{vmatrix}.$$

The determinant on the right side of the equality is equal to zero if at least two of the numbers i, j, s are equal. Obviously in the remaining cases the determinant is equal to $|\mathfrak{A}|$ or to $-|\mathfrak{A}|$ according to whether the three natural numbers i, j, s are an even or an odd permutation. There are as many such determinants as there are permutations of numbers 1, 2, 3. We thus have

$$|\mathfrak{B}\mathfrak{A}| = (b_{11}b_{22}b_{33}+b_{21}b_{32}b_{13}+b_{31}b_{12}b_{23}-$$

$$-b_{11}b_{32}b_{23}-b_{31}b_{22}b_{13}-b_{21}b_{12}b_{33})\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = |\mathfrak{A}||\mathfrak{B}|.$$

If a square matrix $\mathfrak{A}$ is non-singular, then we denote by $\mathfrak{A}^{-1}$ a matrix which satisfies the equation $\mathfrak{A}\mathfrak{A}^{-1} = \mathfrak{J}$ where $\mathfrak{J}$ denotes a unity matrix. The matrix $\mathfrak{A}^{-1}$ is called *inverse* to matrix $\mathfrak{A}$.

Let

$$\mathfrak{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

be a given non-singular matrix, i.e. such that the value of the determinant $|\mathfrak{A}| = W$ is different from zero. Assume that

$$\mathfrak{A}^{-1} = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}.$$

Multiplying $\mathfrak{A}$ by $\mathfrak{A}^{-1}$ we obtain

$$\mathfrak{A}\mathfrak{A}^{-1} = \begin{bmatrix} a_{11}x_{11}+a_{12}x_{21}+a_{13}x_{31} & a_{11}x_{12}+a_{12}x_{22}+a_{13}x_{32} & a_{11}x_{13}+a_{12}x_{23}+a_{13}x_{33} \\ a_{21}x_{11}+a_{22}x_{21}+a_{23}x_{31} & a_{21}x_{12}+a_{22}x_{22}+a_{23}x_{32} & a_{21}x_{13}+a_{22}x_{23}+a_{23}x_{33} \\ a_{31}x_{11}+a_{32}x_{21}+a_{33}x_{31} & a_{31}x_{12}+a_{32}x_{22}+a_{33}x_{32} & a_{31}x_{13}+a_{32}x_{23}+a_{33}x_{33} \end{bmatrix}.$$

From the condition

$$\mathfrak{A}\mathfrak{A}^{-1} = \mathfrak{J} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

we have, by the definition of equality of two matrices, nine equations

$$\begin{aligned} a_{11}x_{11}+a_{12}x_{21}+a_{13}x_{31} &= 1, & a_{11}x_{12}+a_{12}x_{22}+a_{13}x_{32} &= 0, \\ a_{21}x_{11}+a_{22}x_{21}+a_{23}x_{31} &= 0, & a_{21}x_{12}+a_{22}x_{22}+a_{23}x_{32} &= 1, \\ a_{31}x_{11}+a_{32}x_{21}+a_{33}x_{31} &= 0, & a_{31}x_{12}+a_{32}x_{22}+a_{33}x_{32} &= 0, \end{aligned}$$

$$\begin{aligned} a_{11}x_{13}+a_{12}x_{23}+a_{13}x_{33} &= 0, \\ a_{21}x_{13}+a_{22}x_{23}+a_{23}x_{33} &= 0, \\ a_{31}x_{13}+a_{32}x_{23}+a_{33}x_{33} &= 1, \end{aligned}$$

whence we obtain uniquely

$$\begin{aligned} x_{11} &= A_{11}/W, & x_{12} &= A_{21}/W, & x_{13} &= A_{31}/W, \\ x_{21} &= A_{12}/W, & x_{22} &= A_{22}/W, & x_{23} &= A_{32}/W, \\ x_{31} &= A_{13}/W, & x_{32} &= A_{23}/W, & x_{33} &= A_{33}/W, \end{aligned}$$

where A_{ik} denotes the minor of the determinant $|\mathfrak{A}|$ corresponding to the term a_{ik}.

It should be observed that in the matrix $\mathfrak{A}^{-1}$ at the place of the term with indices i, k there is a number proportional to the minor of the term with indices k, i. Thus, in order to write the matrix $\mathfrak{A}^{-1}$, we must take the table of minors A_{ik}, change its rows into columns and its columns into rows and divide each term of the matrix thus obtained by the value of the determinant W.

4.5. Let us apply the matrix calculus to projective transformations. To begin with, it will be observed that the three formulas for a projective transformation of a plane (x_1, x_2, x_3) onto a plane (y_1, y_2, y_3),

$$\begin{aligned} y_1 &= a_{11}x_1+a_{12}x_2+a_{13}x_3, \\ y_2 &= a_{21}x_1+a_{22}x_2+a_{23}x_3, \\ y_3 &= a_{31}x_1+a_{32}x_3+a_{33}x_3 \end{aligned}$$

can be replaced with the aid of matrices by one formula

$$[y_1\ y_2\ y_3] = [x_1\ \ x_2\ x_3]\begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}.$$

If in a transformation defined by matrix $\mathfrak{A}$
point $X = (x_1, x_2, x_3)$ has a corresponding point $Y = (y_1, y_2, y_3)$,
point $X' = (x_1', x_2', x_3')$ has a corresponding point $Y' = (y_1', y_2', y_3')$,
point $X'' = (x_1'', x_2'', x_3'')$ has a corresponding point Y''
$= (y_1'', y_2'', y_3'')$,

then we can write this in the following way:

$$\begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \\ x_1' & x_2' & x_3' \\ x_1'' & x_2'' & x_3'' \end{bmatrix}\begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}. \tag{4.8}$$

On the grounds of the definition of the product of matrices and the Cauchy theorem on the product of determinants we can see that in the case where $\mathfrak{A} \neq 0$ the three points Y, Y', Y'' are collinear if and only if the three points X, X', X'' are collinear.

Let us return to the problem of a projective transformation of a plane which transforms a given (non-degenerated) quadrilateral $ABCD$ into a rectangle with sides parallel to the axes of coordinates.

If we want to transform the side AB into a segment of the axis y_1/y_3 and the side AC into a segment of the axis y_2/y_3, we must transform

point $P(p_1, p_2, p_3)$ into point Y_1^∞ (1, 0, 0),
point $Q(q_1, q_2, q_3)$ into point Y_2^∞ (0, 1, 0),
point $A(a_1, a_2, a_3)$ into point $A'(0, 0, 1)$.

According to formula (4.8) we must find such a matrix $\mathfrak{B}$ as would satisfy the equation

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ a_1 & a_2 & a_3 \end{bmatrix}\mathfrak{B}.$$

As we see, the matrix $\mathfrak{P}$ is inverse to the matrix composed of the coordinates of points P, Q and A.

Let us solve the following problem:

Find the projective transformation which transforms the four points $A(1, 3)$, $B(2, 2)$, $C(4, 3)$, $D(2, 4)$ into the four points $A'(0, 0)$, $B'(1, 0)$, $C'(1, 1)$ $D'(0, 1)$ and find the coordinates of a point M' into which the point $M(5, 1)$ will be transformed (Fig. 18).

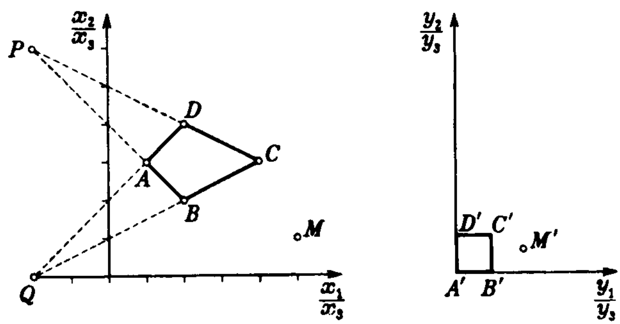

FIG. 18

The coordinates of point P will be obtained by solving the system of equations

$$\begin{vmatrix} 1 & 2 & x_1 \\ 3 & 2 & x_2 \\ 1 & 1 & x_3 \end{vmatrix} = 0, \qquad \begin{vmatrix} 4 & 2 & x_1 \\ 3 & 4 & x_2 \\ 1 & 1 & x_3 \end{vmatrix} = 0,$$

since it is the point of intersection of the straight line passing through the points $A(1, 3, 1)$ and $B(2, 2, 1)$ with the straight line passing through the points $C(4, 3, 1)$ and $D(2, 4, 1)$. Elementary calculation gives the numbers -2, 6, 1 as the coordinates of point P.

Similarly, the coordinates of point Q will be obtained by solving the system of equations

$$\begin{vmatrix} 1 & 2 & x_1 \\ 3 & 4 & x_2 \\ 1 & 1 & x_3 \end{vmatrix} = 0, \qquad \begin{vmatrix} 2 & 4 & x_1 \\ 2 & 3 & x_2 \\ 1 & 1 & x_3 \end{vmatrix} = 0,$$

since it is the point of intersection of the straight line passing through the points $A(1, 3, 1)$ and $D(2, 4, 1)$ with the straight line passing through the points $B(2, 2, 1)$ and $C(4, 3, 1)$. We obtain the three numbers $-2, 0, 1$ as the coordinates of point Q.

Let us arrange the homogeneous coordinates of the points P, Q and A in a matrix

$$\mathfrak{A} = \begin{bmatrix} -2 & -2 & 1 \\ 6 & 0 & 3 \\ 1 & 1 & 1 \end{bmatrix}.$$

It follows from our previous considerations that the matrix $\mathfrak{A}^{-1}$ should be found. The minors of the determinant $|\mathfrak{A}|$ have the values:

$$A_{11} = \begin{vmatrix} 0 & 3 \\ 1 & 1 \end{vmatrix} = -3, \qquad A_{21} = -\begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} = 3,$$

$$A_{31} = \begin{vmatrix} -2 & 1 \\ 0 & 3 \end{vmatrix} = -6,$$

$$A_{12} = -\begin{vmatrix} 6 & 3 \\ 1 & 1 \end{vmatrix} = -3, \qquad A_{22} = \begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} = -3,$$

$$A_{32} = -\begin{vmatrix} -2 & 1 \\ 6 & 3 \end{vmatrix} = 12,$$

$$A_{13} = \begin{vmatrix} 6 & 0 \\ 1 & 1 \end{vmatrix} = 6, \qquad A_{23} = -\begin{vmatrix} -2 & -2 \\ 1 & 1 \end{vmatrix} = 0,$$

$$A_{33} = \begin{vmatrix} -2 & -2 \\ 6 & 0 \end{vmatrix} = 12,$$

and since $|\mathfrak{A}| = 18$, we obtain the following inverse matrix:

$$\mathfrak{A}^{-1} = \begin{bmatrix} -1/6 & -1/6 & 1/3 \\ 1/6 & -1/6 & 0 \\ -1/3 & 2/3 & 2/3 \end{bmatrix}.$$

We thus have the projective transformation

$$[y_1 \ \ y_2 \ \ y_3] = [x_1 \ \ x_2 \ \ x_3]\,\mathfrak{A}^{-1}.$$

which transforms the quadrilateral $ABCD$ into the rectangle $A'B'C'D'$. Let us find the coordinates of the points B' and D'. Substituting the coordinates of the point B for x_1, x_2, x_3, we obtain

$$[2\ 2\ 1]\begin{bmatrix} -1/6 & -1/6 & 1/3 \\ 1/6 & -1/6 & 0 \\ -1/3 & 2/3 & 2/3 \end{bmatrix} = [-1/3\ 0\ 4/3],$$

and substituting the coordinates of the point D, we obtain

$$[2\ 4\ 1]\begin{bmatrix} -1/6 & -1/6 & 1/3 \\ 1/6 & -1/6 & 0 \\ -1/3 & 2/3 & 2/3 \end{bmatrix} = [0\ -1/3\ 4/3].$$

The point $M(5, 1, 1)$ is here transformed into the point

$$[5\ 1\ 1]\begin{bmatrix} -1/6 & -1/6 & 1/3 \\ 1/6 & -1/6 & 0 \\ -1/3 & 2/3 & 2/3 \end{bmatrix} = [-1\ \ -1/3\ \ 7/3].$$

The point B has been transformed into the point $(-1/4, 0, 1)$, and therefore its abscissa x_1/x_3 must be multiplied by -4 in order that the point B' have the abscissa 1. Similarly, the transformation of the point D into the point $(0, -1/4, 1)$ implies that the ordinate x_2/x_3 must be multiplied by -4 in order that the point D' have the ordinate 1. Consequently, we make an affine transformation multiplying every abscissa by -4 and every ordinate by -4. We thus finally obtain the pair of numbers $12/7$, $4/7$ as the coordinates of point M', which corresponds to point M.

§ 5. Rectilinear coordinates. Correlation

5.1. Consider three numbers

$$u_1,\ u_2,\ u_3 \tag{5.1}$$

which are not all equal to zero. Assign to these three numbers a straight line with the equation

$$u_1x_1+u_2x_2+u_3x_3 = 0, \tag{5.2}$$

i.e. an equation in which numbers u_1, u_2, u_3 are coefficients.

It will be observed that under this agreement every proportional three, i.e. a triple of numbers u_1', u_2', u_3' such that

$$u_1' : u_2' : u_3' = u_1 : u_2 : u_3, \tag{5.3}$$

has the same straight line corresponding to it because the equation

$$u_1'x_1 + u_2'x_2 + u_3'x_3 = 0$$

is, under the assumption (5.3), equivalent to equation (5.2). It will also be observed that non-proportional threes have different straight lines corresponding to them.

Owing to these properties threes of numbers can be regarded as the so called *rectilinear coordinates* of a straight line, more exactly: as the *homogeneous coordinates* of a straight line.

For example the straight line

$$y = ax + b \quad \text{or} \quad \frac{x_2}{x_3} = a\frac{x_1}{x_3} + b \quad \text{or} \quad ax_1 - x_2 + bx_3 = 0$$

has the coordinates a, -1, b.

The coordinates of a straight line passing through two points $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$ are, as follows from the equation

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0,$$

the minors of the matrix

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix},$$

i.e. the numbers

$$u_1 = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \qquad u_2 = -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \qquad u_3 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$$

Threes of numbers can thus denote both points and straight lines. If numbers a_1, a_2, a_3 are the coordinates of a point P and numbers l_1, l_2, l_3 the coordinates of a straight line l, then the point P lies on the straight line l if and only if the equation

$$l_1a_1 + l_2a_2 + l_3a_3 = 0$$

is satisfied.

The following theorem is obvious — it is analogous to a well-known theorem of analytic geometry in which numbers denote coordinates of a point:

Three straight lines

$$u_1 x_1 + u_2 x_2 + u_3 x_3 = 0,$$
$$u_1' x_1 + u_2' x_2 + u_3' x_3 = 0,$$
$$u_1'' x_1 + u_2'' x_2 + u_3'' x_3 = 0,$$

pass through one point (ordinary or at infinity) if and only if the coefficients of the equations of those straight lines satisfy the following equation:

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ u_1' & u_2' & u_3' \\ u_1'' & u_2'' & u_3'' \end{vmatrix} = 0.$$

Observe the geometrical significance of numbers u_1, u_2, u_3, which are the coordinates of the straight line l.

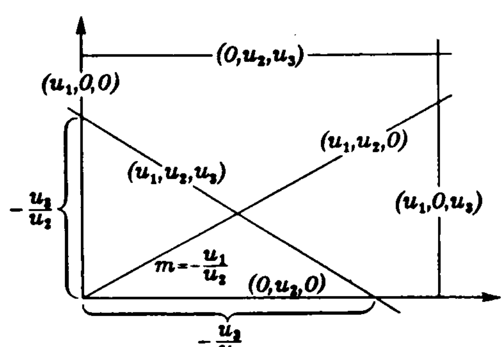

Fig. 19

If $u_1 \neq 0$, $u_2 \neq 0$ and $u_3 \neq 0$, then writing the equation in Cartesian coordinates we have

$$\frac{u_1}{u_3} \cdot \frac{x_1}{x_3} + \frac{u_2}{u_3} \cdot \frac{x_2}{x_3} + 1 = 0 \quad \text{or} \quad \frac{x}{-u_3/u_1} + \frac{y}{-u_3/u_2} = 1.$$

The straight line then intersects the axes x and y at the points with coordinates $-u_3/u_1$, 0 and 0, $-u_3/u_2$ (Fig. 19).

If $u_3 = 0$, i.e. $u_1x_1+u_2x_2 = 0$, then the equation of the straight line for $u_2 \neq 0$ in Cartesian coordinates has the form

$$y = -\frac{u_1}{u_2}x.$$

The straight line passes through the origin of the system and has a direction coefficient $-u_1/u_2$. If, moreover, $u_1 = 0$, then the line in question is the axis x.

If $u_3 \neq 0$ but at least one of the numbers u_1 and u_2 is equal to zero (e.g. $u_2 = 0$), then we have the straight line

$$u_1x_1+u_3x_3 = 0 \quad \text{or} \quad x = -u_3/u_1;$$

if $u_1 \neq 0$ it is a line parallel to the axis y, and if $u_1 = 0$ it is the straight line at infinity $x_3 = 0$.

5.2. *If $P'(a_1', a_2', a_3')$ and $P''(a_1'', a_2'', a_3'')$ are two different points, i.e. if*

$$a_1'' : a_2'' : a_3'' \neq a_1' : a_2' : a_3', \tag{5.4}$$

then the point $P(x_1, x_2, x_3)$ is a point of the straight line $P'P''$, if and only if there exist numbers λ, μ not equal to zero and such that the following equalities hold:

$$x_1 = \lambda a_1'+\mu a_1'', \quad x_2 = \lambda a_2'+\mu a_2'', \quad x_3 = \lambda a_3'+\mu a_3''. \tag{5.5}$$

P r o o f. a. If x_1, x_2, x_3 have values agreeing with formula (5.5), then of course the point P lies on the straight line $P'P''$ because the equation of the straight line $P'P''$ is satisfied:

$$\begin{vmatrix} \lambda a_1'+\mu a_1'' & \lambda a_2'+\mu a_2'' & \lambda a_3'+\mu a_3'' \\ a_1' & a_2' & a_3' \\ a_1'' & a_2'' & a_3'' \end{vmatrix} = 0.$$

b. Now let $\bar{P}(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ be a point lying on the straight line $P'P''$, i.e. let the following equation be satisfied:

$$\begin{vmatrix} \bar{x}_1 & \bar{x}_2 & \bar{x}_3 \\ a_1' & a_2' & a_3' \\ a_1'' & a_2'' & a_3'' \end{vmatrix} = 0. \tag{5.6}$$

At least one of the minors

$$\delta_{ik} = \begin{vmatrix} a_i' & a_k' \\ a_i'' & a_k'' \end{vmatrix}$$

is different from zero. E.g. if

$$\delta_{12} = \begin{vmatrix} a_1' & a_2' \\ a_1'' & a_2'' \end{vmatrix} \neq 0,$$

then let us consider numbers λ_0 and μ_0 (uniquely defined) satisfying the equations

$$a_1'\lambda_0 + a_2'\mu_0 = \bar{x}_1, \qquad a_2''\lambda_0 + a_2''\mu_0 = \bar{x}_2. \tag{5.7}$$

Let us substitute the left sides of equations (5.7) in equation (5.6):

$$\begin{vmatrix} a_1'\lambda_0 + a_2'\mu_0 & a_1''\lambda_0 + a_2''\mu_0 & \bar{x}_3 \\ a_1' & a_2' & a_3' \\ a_1'' & a_2'' & a_3'' \end{vmatrix} = 0.$$

Expanding the determinant on the left according to the last column we obtain

$$\delta_{12}\bar{x}_3 - a_3'\delta_{12}\lambda_0 + a_3''(-\delta_{12}\mu_0) = 0,$$

and consequently, in view of $\delta_{12} \neq 0$, we have the equation

$$\bar{x}_3 = \lambda_0 a_3' + \mu_0 a_3''.$$

Combining it with equations (5.7) we obtain a system of equations of form (5.5), q.e.d.

5.3. *If the straight lines $U'(u_1', u_2', u_3')$ and $U''(u_1'', u_2'', u_3'')$ are different from each other, i.e. if*

$$u_1' : u_2' : u_3' \neq u_1'' : u_2'' : u_3'',$$

then the straight line $U(u_1, u_2, u_3)$ passes through the point of intersection of the straight lines U' and U'' if and only if there exist numbers λ and μ for which the following equalities hold:

$$u_1 = \lambda u_1' + \mu u_1'', \qquad u_2 = \lambda u_2' + \mu u_2'', \qquad u_3 = \lambda u_3' + \mu u_3''. \tag{5.8}$$

P r o o f. a. The sufficiency of the condition is obvious since (as follows from the theorem of § 5.1) the straight lines U, U', U'' pass through one point.

b. Now let the straight line $U(u_1, u_2, u_3)$ pass through the point (x_1, x_2, x_3), i.e. let

$$u_1x_1+u_2x_2+u_3x_3 = 0;$$

since at the same time

$$u_1'x_1+u_2'x_2+u_3'x_3 = 0, \qquad u_1''x_1+u_2''x_2+u_3''x_3 = 0,$$

we have

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ u_1' & u_2' & u_3' \\ u_1'' & u_2'' & u_3'' \end{vmatrix} = 0.$$

Hence we infer, as in § 5.2, that equations (5.8) hold.

5.4. The definition of rectilinear coordinates allows us to define a certain special correspondence between points and straight lines in a plane. In order to simplify our considerations suppose that we have two planes π and π' (Fig. 20) on which the triples of numbers x_1, x_2, x_3 and x_1', x_2', x_3' denote coordinates of points and the triples of numbers u_1, u_2, u_3 and u_1', u_2', u_3' denote coordinates of straight lines.

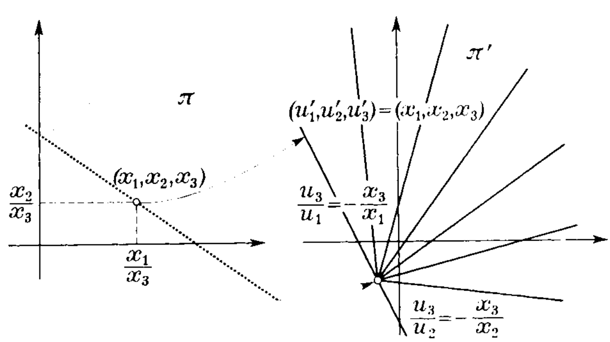

Fig. 20

Let us assign to every point $X(x_1, x_2, x_3)$ of the first plane a straight line $l(u_1', u_2', u_3')$ of the second plane if

$$u_1' = x_1, \qquad u_2' = x_2, \qquad u_3' = x_3.$$

Correspondence of this kind is called *correlation*.

We shall prove the following theorem.

In a correlation, points lying on a straight line of one plane correspond, on the other plane, to straight lines which pass through a fixed point, i.e. lines of a certain pencil.

The same can be expressed in a different way as follows: if a point runs along a straight line on a plane π, then the straight line corresponding to it rotates about a point.

This follows directly from the considerations of § 5.3; for every point of the straight line passing through points (x_1, x_2, x_3) and (x_1', x_2', x_3') has coordinates

$$\lambda x_1+\mu x_1', \quad \lambda x_2+\mu x_2', \quad \lambda x_3+\mu x_3'. \tag{5.9}$$

On the other hand, we know that the numbers u_1', u_2', u_3', equal to the numbers of (5.9) respectively, are rectilinear coordinates of straight lines of a certain pencil.

It will be observed that the following numbers are the coordinates of the straight line passing through the points (x_1, x_2, x_3) and (x_1', x_2', x_3'):

$$\begin{vmatrix} x_2 & x_3 \\ x_2' & x_3' \end{vmatrix}, \quad -\begin{vmatrix} x_1 & x_3 \\ x_1' & x_3' \end{vmatrix}, \quad \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}, \tag{5.10}$$

and the numbers

$$\begin{vmatrix} u_2 & u_3 \\ u_2' & u_3' \end{vmatrix}, \quad -\begin{vmatrix} u_1 & u_3 \\ u_1' & u_3' \end{vmatrix}, \quad \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix},$$

equal by correlation to numbers (5.10), are the coordinates of the point W' on the plane π', through which pass the straight lines with the coordinates u_1, u_2, u_3 and u_1', u_2', u_3'.

It will thus be seen that a straight line of the plane π has, in the correlation here assumed, a corresponding point on the plane π'. It is essential for our further considerations that if point P lies on a straight line l, then the straight line p' corresponding to point P passes through point L' assigned to the straight line l.

If a point runs over a curve C with parametric equations

$$x_1 = \varphi(t), \quad x_2 = \psi(t), \quad x_3 = \chi(t),$$

then the straight lines corresponding to those points

$$u_1' = \varphi(t), \quad u_2' = \psi(t), \quad u_3' = \chi(t)$$

form a certain one-parameter family of straight lines $\boldsymbol{R}$ which, if the first derivatives of the functions φ, ψ, χ, are continuous, is the set of tangents to a certain curve C'; that curve is called the *envelope* of the family of straight lines $\boldsymbol{R}$.

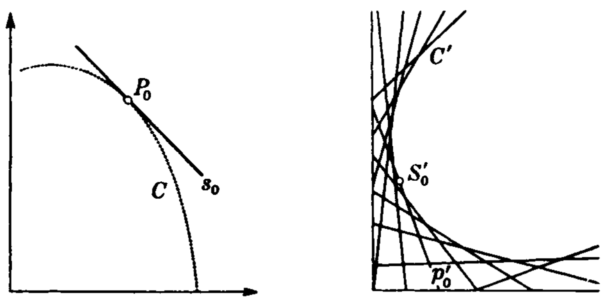

FIG. 21

It will be seen that, under our assumptions, the tangent s_0 to the curve C, which is the limit of the straight lines joining point P_0 with point P which tends to point P_0, has a corresponding point S_0', which is the limit of the points of intersection of the straight line p_0' (corresponding to point P_0) with the straight lines p' (corresponding to point P) (Fig. 21).

CHAPTER II

EQUATIONS WITH TWO VARIABLES

§ 6. Graph of a function

6.1. Consider a function $y = f(x)$ defined in an interval (a, b). Assume that it is continuous and monotone, i.e. either increasing in the interval (a, b) or decreasing in the interval (a, b).

The *graph of a function* is, as we know, the set of all points with the coordinates x and $f(x)$, x assuming all values in the interval (a, b).

If $f(x)$ is a continuous function, its graph is a curve (Fig. 22).

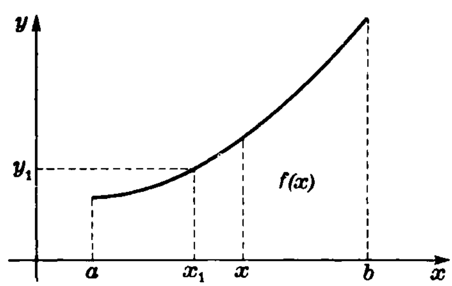

FIG. 22

In problems of natural science and technology continuous functions are the most frequent, the variability range of x being divisible into partial intervals in which the functions are monotone. In practice it is often necessary to find the values of the function $f(x)$ for a great many values of the argument. If the function $f(x)$ is expressed by a complex formula, the numerical calculation of the required values would be a lengthy and cumbersome business. We then execute the graph of the function, and, for reasons which will later become obvious, we usually choose different units on the axes of the system; thus points (0, 1) and (1, 0) need not be equally distant from the origin of the system

(0, 0). We find that for the required degree of accuracy the unit chosen in the figure can be so large as to make the graph replace the calculation of the values of the function. As follows from the definition of the graph of a function, from point x_1 on the x-axis we should draw a parallel to the y-axis and from the point of intersection of that line with the graph we should draw a parallel to the x-axis. The number y_1 obtained in this way is the required value of the function.

We can avoid drawing lines parallel to the axes of the system if we perform the drawing on square paper.

For example, suppose we are given the function

$$s = 25 - gt^2/2 = 25 - 0{\cdot}4905\, t^2,$$

defining the length of the path s (in metres) in relation to the time t (in seconds); according to the notation adopted in physics,

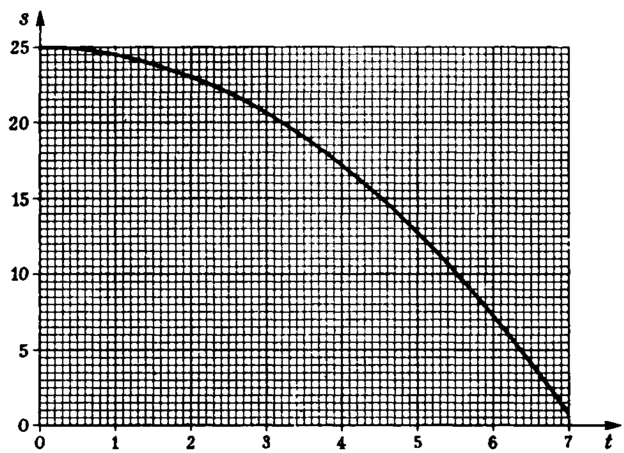

FIG. 23

we write t instead of x and a instead of y. At the time $t = 0$ the path is $s = 25$ m; thus it is the formula for the path in the case of a body falling freely under the influence of gravity from a height of 25 metres.

Taking the interval (0, 7), we obtain a graph like the one in Fig. 23. Assuming that in a drawing it is easy to distinguish points at a distance of 0·5 mm from one another, we can read from the graph the time with an accuracy of 1/20 second and the path with an accuracy of 0·25 metre. Obviously, if we needed to know the path s with greater accuracy, we should have to adopt larger units on the s-axis. E.g., for the accuracy of 20 cm we should have to mark the division 20 cm at a point 0·5 mm distant from the origin of the system; then the point on the s-axis which is now marked 5 m would be marked 4 m. Thus the drawing would have the dimensions 7 cm $\times$6·25 cm.

It can be read from the graph that in time $t = 4{\cdot}5$ sec the path s will be about 15 m, for example; similarly, we find that for $s = 9$ m we have $t = 5{\cdot}6$ sec.

A change of unit on one axis results in certain cases in a considerable drawing simplification. Thus for instance the graph of the function $y = \sqrt{16-0{\cdot}64x^2}$ for $0 < x < 5$ forms an arc of an ellipse (Fig. 24a) whose equation is

$$0{\cdot}64x^2+y^2 = 16;$$

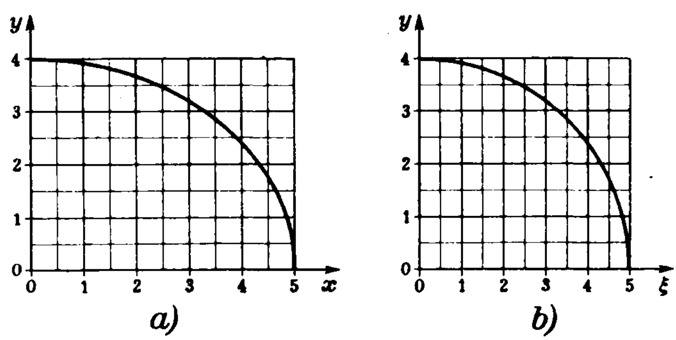

FIG. 24

assuming $\sqrt{0{\cdot}64x} = \xi$, i.e. adopting the unit $\lambda_x = \sqrt{0{\cdot}64} = 0{\cdot}8$ on the x-axis, we obtain a circle (Fig. 24b) whose equation is

$$\xi^2+y^2 = 16.$$

6.2. The *polar coordinates* of a point P on a plane are, as we know, a pair of numbers φ, r, $0 \leqslant \varphi < 2\pi$ and $0 \leqslant r < \infty$ of which the first denotes the angle between a constant half-line with the origin O and the vector $\overline{OP}$, and the other — the length of the segment OP.

Given a function $r = f(\varphi)$, we obtain its graph as the set of those points with the coordinates φ, r for which $r = f(\varphi)$ (Fig. 25).

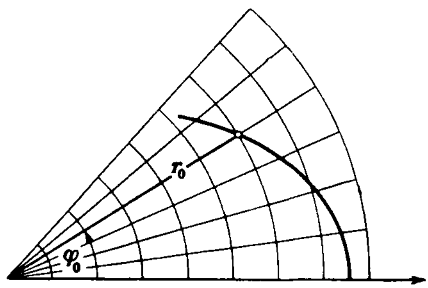

FIG. 25

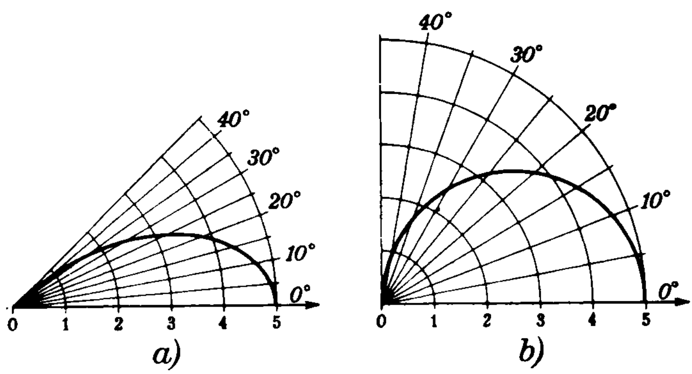

FIG. 26

As with the Cartesian coordinates, in some cases we can obtain a simpler drawing by changing the unit on one of the axes. E.g. the function $r = 5\cos 2\varphi$ for $0 \leqslant \varphi \leqslant 45°$, whose graph is the curve in Fig. 26a, can be represented through enlarging twice the unit of the angle φ (i.e. on substituting $2\varphi = \bar{\varphi}$) by the curve $r = 5\cos\bar{\varphi}$, which, as can easily be seen, is a semi-circle with diameter 5 (Fig. 26b).

We can use the graphs in Figs. 26a and 26b to read the values of r when φ is given and to read the values of φ when r is given (in the interval from 0 to 5).

6.3. Besides the orthogonal and the polar systems of coordinates there exist many others, for the notion of systems of coordinates can be generalized in the following way:

We are given a family of curves R_x and a family of curves R_y, different from R_x; each curve K_x of the family R_x has a number x corresponding to it in a bi-unique manner; each curve K_y of the family R_y has a number y corresponding to it in a bi-unique manner. If one curve K_x and one curve K_y pass through every point P of part E' of a plane E and if the curves K_x and K_y have no other point in common in part E', then the pair of curves K_x and K_y can be assigned in a bi-unique manner to point P; the pair of numbers corresponding to the pair of curves K_x and K_y are called the *pair of coordinates* of point P.

Let us take for instance points X_0 and $Y_0 \neq X_0$ (Fig. 27) and let K_x denote a circle with origin X_0 and radius x and K_y a circle with origin Y_0 and radius y.

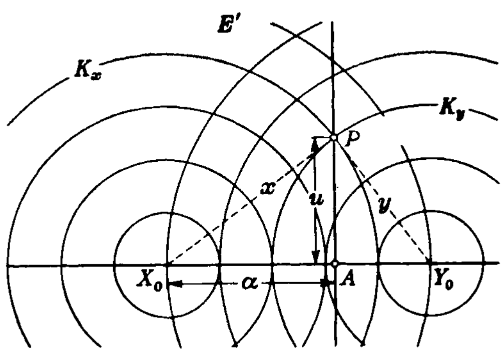

FIG. 27

If E' denotes one of the half-planes into which the straight line X_0Y_0 divides the plane, then every point P belonging to the half-plane E' has a corresponding pair of positive coordinates x and y satisfying the inequalities

$$|x-y| < X_0Y_0 < x+y;$$

they are the coordinates of the point P.

We find, for instance, that in this system of coordinates the graph of the function $y = \sqrt{x^2+a}$ is a straight line perpendicular to $X_0 Y_0$.

Indeed, for every point of a straight line perpendicular to $X_0 Y_0$ and passing through point A with coordinates a and $X_0 Y_0 - a$ we have

$$x^2 - a^2 = u^2 = y^2 - (X_0 Y_0 - a)^2, \quad \text{whence} \quad x^2 + X_0^2 Y_0^2 - 2aX_0 Y_0 = y^2;$$

thus it is sufficient to assume $X_0^2 Y_0^2 - 2aX_0 Y_0 = a$, whence $a = (X_0^2 Y_0^2 - a)/2X_0 Y_0$.

The function $y = c\sqrt{x^2+a}$ for $c \neq 1$ will of course have a curvilinear graph. Changing, as in the preceding examples, the unit on the y-axis, we can obtain, here also, a graph in the form of a straight line.

Another example of coordinates will be obtained if we substitute $\xi = x+y$, $\eta = x-y$ in the preceding example.

It can easily be seen that the curves K_ξ are ellipses with foci X_0 and Y_0 and the curves K_η are halves of hyperboles with the same foci.

In all the above-mentioned systems of coordinates the graph of the function $y = f(x)$ may serve for reading the values $f(x_i)$ of the function for given values of the argument x_i. The reading involves a certain error, but the drawing can be made on such a scale that the error will be less than a given number.

Drawings of this kind often replace very cumbersome calculations and are used in technology in cases where a function (or formula) occurs very frequently and the value of the function for a given argument ought to be found quickly. The commonest drawing for a function of two variables, however, is the so-called *functional scale*.

Exercises

1. Find the value k for which the substitution

$$r' = r, \qquad \varphi' = k\varphi$$

changes the graph of the function

$$r = \frac{a}{b \sin 3\varphi + c \cos 3\varphi}$$

in polar coordinates into a straight line.

2. Concentric circles K_y and straight lines K_x parallel to one another form a system of coordinates in which the graph of the function

$$y = \sqrt{x^2+(a-x\cot a)^2}$$

is a straight line if the point $y = 0$ lies on the straight line $x = 0$. Indicate the position of the straight line and find the units λ_x and λ_y in such a way that a given function

$$y = \sqrt{Ax^2+Bx+C}$$

have a rectilinear graph.

3. Two families of concentric circles K_x and K_y form a system of coordinates; show that the equation

$$\sqrt{x^2-a^2} - \sqrt{y^2-b^2} = c$$

is an equation of a straight line and indicate the system of coordinates in which the function

$$y = \sqrt{ax^2+b\sqrt{x^2-c^2}+d}$$

has a rectilinear graph.

§ 7. Functional scale

Suppose we are given a certain function, e.g.

$$y = 2\sqrt{x+4} \quad \text{for} \quad 0 \leqslant x \leqslant 5.$$

It will be observed that the function is increasing and, as x changes from 0 to 5, y changes from 4 to 6. Let us draw the so called *functional scale* of this function. Therefore, let us mark on a segment a part of the number axis from point 4 to point 6; we shall thus have points corresponding to the values of the variable y. We choose the unit according to the accuracy with which we want to read numbers y. E.g., if we require the error to be less than 0·01, the points of the number axis marked with numbers differing by 0·01 should be placed at a distance of about 0·5 mm from one another; since there are to be 200 such intervals between 4 and 6, the segment will be 100 mm long (Fig. 28). Let us mark the values of function y on the left side of the vertical segment. Let us then substitute for the argument x—in succession—numbers 0, 0·1, 0·2, ..., 5 and the intermediate numbers; after substituting the given number for x in the formula, let us mark that number on the axis at the point at which we have already marked the corresponding number y. For instance, taking $x = 1$ we find that $y = 2\sqrt{5} \approx 4{\cdot}47$; we then mark the number *1* for x on the right side of the segment at the point where we have already marked the number 4·47 for y.

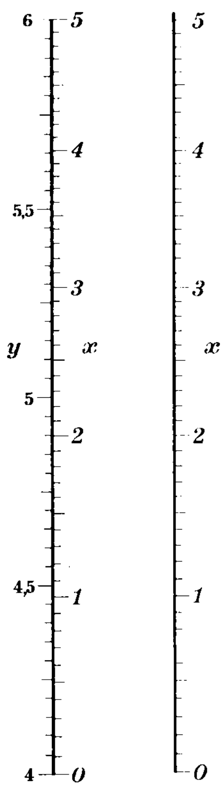

FIG. 28

A drawing thus made is called the *double scale* of the function $y = 2\sqrt{x+4}$. Erasing the markings on the left side we shall obtain the so-called *single scale* or the *scale* of function $f(x)$.

An essential feature of the double scale is the fact that the drawing is a part of the number axis y, i.e. that two numbers y_1 and y_2 are always represented by two points which are end-points of a segment whose length $y_1 y_2$ is proportional to the difference of those numbers; the scale for x need not have this property, of course. Another characteristic feature of the double scale is the fact that $\bar{x}$ and $\bar{y}$ determine one point only if they satisfy the equation $\bar{y} = f(\bar{x})$.

Double scales of functions are component elements of nomograms for functions of many variables, and consequently it is very important to discuss certain properties of functional scales.

To begin with it will be observed that the double scale of the function $y = f(x)$ can be obtained by means of a graph of that function. Indeed, drawing from point x_0 (Fig. 29) a line parallel to the y-axis as far as the point of intersection with the graph of the function, and then drawing a line parallel to the x-axis, we obtain on the y-axis a point which we also mark x_0; this point also corresponds to the value of y_0, and thus we have on the y-axis the same drawing as has been defined as the double scale of the function $y = f(x)$.

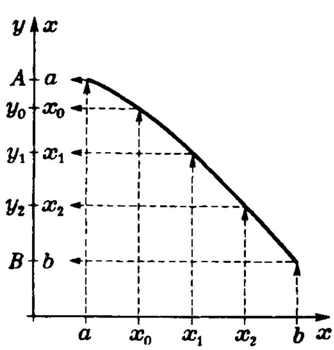

FIG. 29

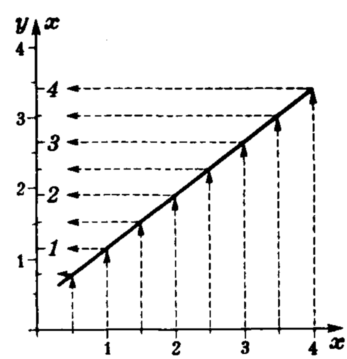

FIG. 30

In the special case where the function $f(x)$ is a linear function $f(x) = ax+b$ the scale of arguments x is of the same character as the scale of the values of y: points X_1 and X_2 determine segments X_1X_2 whose lengths are proportional to the difference X_2-X_1 (Fig. 30). Such scales are called *regular* or *uniform scales.*

If the scale is regular, then segments bounded by points differing by unity are equal. A segment like that is called the *unit of the scale.* Denoting by λ_x the unit of the scale on the x-axis, we can see that the length of the segment X_1X_2 determined by the points corresponding to numbers x_1 and x_2 on the scale is equal to $\lambda_x|x_2-x_1|$, i.e.

$$X_1X_2 = \lambda_x|x_2-x_1|.$$

It can easily be seen that only linear functions $y = ax+b$ have regular scales. If the scale is not regular (like the scale of x'es in Fig. 28 for instance), then we can consider a mean unit λ' between points X_1X_2 of the scale; it is a number λ' satisfying the equation

$$X_1X_2 = \lambda'|x_2-x_1|.$$

Reducing the interval (x_1, x_2) we come to the definition of the unit λ_{x_0} at the point x_0 of the scale as the limit of the quotient of positive numbers:

$$\lambda_{x_0} = \lim_{x\to x_0} \frac{XX_0}{|x-x_0|}.$$

In drawing functional scales it is interesting to observe the ratio of the unit λ_{x_0} at point x_0 of the scale of x'es to the unit λ_y for the scale of y's. As follows from the definition of the unit λ_y, we have

$$\lambda_{x_0} : \lambda_{y_0} = \lim_{x\to x_0} \frac{XX_0}{|x-x_0|} : \lim_{y\to y_0} \frac{YY_0}{|y-y_0|} = \lim_{\substack{x\to x_0\\ y\to y_0}} \left|\frac{y-y_0}{x-x_0}\right| = \left|\frac{dy}{dx}\right| = |f'(x_0)|$$

since the segment XX_0 is of course equal to (identical with) the segment YY_0.

We thus obtain the formula

$$\lambda_{x_0} = \lambda_{y_0}|f'(x_0)|. \tag{7.1}$$

The ratio of the unit λ_{x_0} of the scale of x'es at the point x_0 to the (constant) unit λ_y is equal to the absolute value of the derivative of the function at the point x_0.

Let us take for example the function

$$f(x) = 2\sqrt{x+4}.$$

We find the derivative of this function:

$$f'(x) = 2\cdot\tfrac{1}{2}(x+4)^{-1/2} = \frac{1}{\sqrt{x+4}}.$$

Substituting in formula (7.1) the value of the derivative of the given function at the point $x = 0$ we obtain

$$\lambda_0 = \lambda_y|1/\sqrt{4}\,| = \lambda_y/2,$$

and substituting the value of the derivative at the point $x = 5$, we have

$$\lambda_5 = \lambda_y|1/\sqrt{9}\,| = \lambda_y/3,$$

i.e., in the neighbourhood of the point $x = 0$ the unit λ_x is equal to one half of the unit λ_y and in the neighbourhood of the point $x = 5$ the unit λ_x is equal to one third of the unit λ_y. Thus the reading of the argument x in the neighbourhood of the point $x = 0$ is half as accurate as the reading of the value of y, and in the neighbourhood of the point $x = 5$ its accuracy is one third of the accuracy of reading the value of y.

EXAMPLE 1. Draw the double scale of the function

$$f(x) = (x^3-3x^2+6x)/50$$

for the values of x in the interval $0 < x < 1{\cdot}2$; the error of the reading of x should be less than $0{\cdot}01$ and the error of the reading of y should be less than $0{\cdot}001$.

To establish the units λ_x and λ_y let us find the derivative $f'(x)$

$$f'(x) = (3x^2-6x+6)/50 = [3(x-1)^2+3]/50.$$

The derivative has its minimum value at the point $x = 1$. We thus have

$$\lambda_x : \lambda_y \geqslant \lambda_1 : \lambda_y = f'(1) = 0{\cdot}06.$$

It follows from the terms of the problem that the unit λ_x should be equal to at least 5 cm ($\lambda_x \geqslant 0{\cdot}5$ mm/$0{\cdot}01 = 5$ cm), and the unit λ_y should be at least 50 cm ($\lambda_y \geqslant 0{\cdot}5$ mm/$0{\cdot}001$ $= 50$ cm); we thus have the inequalities

$$\lambda_x \geqslant 5\text{ cm},\quad \lambda_y \geqslant 50\text{ cm},\quad \lambda_x \geqslant \lambda_y\, 0{\cdot}006 \geqslant 50\text{ cm}\cdot 0\cdot 06 = 3\text{ cm}.$$

These inequalities will be satisfied if we assume that

$$\lambda_x \geqslant \lambda_1 = 5 \text{ cm}, \quad \text{i.e.} \quad \lambda_y = \lambda_1/0{\cdot}06 = 5/0{\cdot}06 \text{ cm} = 83{\cdot}3 \text{ cm}.$$

Since for the end-points of the interval we have

$$f(0) = 0 \quad \text{and} \quad f(1{\cdot}2) = 0{\cdot}09216,$$

with the limit $\lambda_y = 84$ cm, for example, the length of the scale for y will be slightly less than 84 cm (Fig. 31).

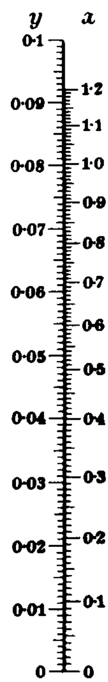

FIG. 31

Remark. In cases where the function for which we are to draw the functional scale is given only for a finite number of arguments and is monotone, we usually draw first the graph and then, with its aid, the functional scale. To understand this better let us consider the following example:

EXAMPLE 2. Draw a functional scale for a function defined for the following values of the argument:

x	0·3	1·6	2·2	3·7	4·4	5·7	7·1	8·4	9·1	10·7
y	0·9	2·6	3·2	5·1	5.6	6·6	7·4	8·1	8·4	8·8

Let us mark the points with given coordinates in the orthogonal system of axes (Fig. 32), and then, making use of the

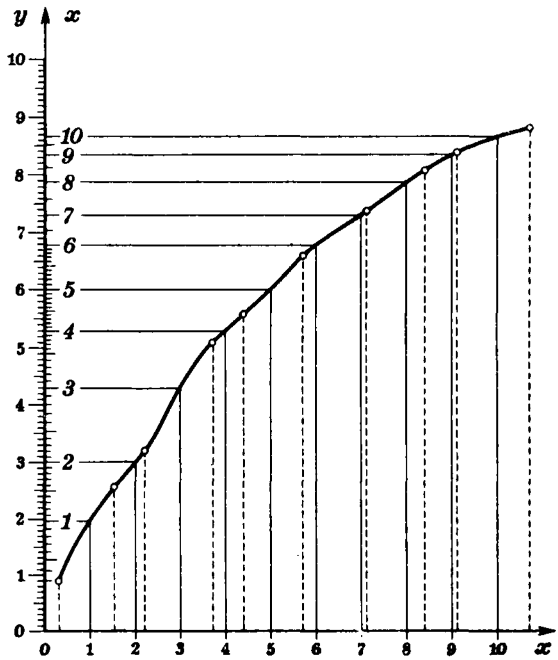

FIG. 32

assumption of monotonicity, let us draw an approximate graph. We now draw vertical lines from points 1, 2, ... and the intermediate points of the axis of abscissas, and horizontal lines from the points of their intersection with the graph as far as their intersection with the y-axis. Thus we have on that axis both the points corresponding to the values of x and the points corresponding to the values of the variable y, assigned to the values of x on the strength of the table and by agreement. Consequently, we have a double functional scale.

Exercises

1. Construct a double scale for the conversion of the degrees of Centigrade and Réaumur.

2. Construct a double scale for the conversion of the degrees of Centigrade and Fahrenheit.

3. Construct a double scale for the function $y = \pi x^2$ in the interval $1 \leqslant x \leqslant 3$ with reading accuracy up to two decimal places for x and y.

4. Construct a single scale for the functions a. $y = \sin x$ for angles from $0°$ to $90°$ assuming $\lambda_y = 10$ cm; b. $y = 0{\cdot}8 \sin z$ for the same λ_y. Combining the two drawings in such a way that the same points for y coincide, we obtain a double scale for the equation

$$0{\cdot}8 \sin z = \sin x, \quad \text{i.e.} \quad x = \text{arc}\sin(0{\cdot}8 \sin z)$$

whence we can read the values of x and z which correspond to each other.

5. Construct a double scale for a function $y(x)$ defined by the equation $y^5 - 3xy - 2x + 1 = 0$ for the interval $(0,\ 0{\cdot}5)$ of the variable x; the drawing should be made to a scale that would ensure the reading error for x to be less than $0{\cdot}001$.

Hint: Draw a graph of the inverse function first.

§ 8. Logarithmic scale

A scale of a linear function is regular, i.e. the unit λ_x is the same at each point x. This is an important property if reading accuracy is to be the same everywhere. However, in most problems of science and technology, the so called *relative accuracy*, given by the ratio of the error Δx to the value of x, i.e. the fraction $\Delta x/x = b_w$, is of greater importance. For it is obvious that if, for instance, we find the volume v of a sphere, then the same reading error Δv will have a different significance for different values of v; namely its significance is less for larger values of v; the only sensible measure of accuracy is then the ratio $\Delta v/v$.

Our problem is to find a scale which would be best in this respect, i.e. a scale for which the relative error $b_w = \Delta x/x$ would be constant at each point x of that scale.

In order to find that scale, let us observe that the smaller the unit λ_x at point x the greater the error Δx of reading the number x, i.e. that the product $\lambda_0 \Delta x$ has a constant value m for every scale. Thus the condition that the quotient $b_w = \Delta x/x$

or $x/\Delta x$ be constant is equivalent to the condition that the product $x\lambda_x = xm/\Delta x = m/b_w$ be constant. We thus have

$$\lambda_x x = c, \quad \text{i.e.} \quad \lambda_y |f'(x)|\, x = c, \quad \text{and thus} \quad f'(x) = c_1/x.$$

The last equality gives by integration

$$f(x) = c_1 \log_e x = x_1 \log x/\log e = C \log x.$$

Consequently, the only functional scale for which the same relative accuracy obtains for every point x is the *logarithmic scale.*

Let us construct a logarithmic scale for the logarithms with base 10 (Fig. 33a). For numbers y we have, as usual, a regular scale. Let us take for y the interval from 0 to 1. Knowing that

$$\log 1 = 0, \quad \log 2 = 0{,}301, \quad \ldots, \quad \log 10 = 1$$

we mark for x the values 1, 2, ..., 10 on the right side of the segment, opposite the corresponding points of the scale for y. We mark the fractional values of x in a similar way, obtaining thus a logarithmic scale for x.

If the scale length in Fig. 33a were 2·5 cm, we could distinguish on it the values of y with an accuracy up to three decimal places, since points differing by 0·001 would be spaced at a distance of

$$25 \text{ cm}/1000 = 1/4 \text{ mm}.$$

Increasing the scale length to 2·5 m we could obtain a double scale which would replace four-digit tables of logarithms. Figure 33b represents a 3/20 part of such a scale, namely the part for y contained between 0·5 and 0·65.

A characteristic feature of the logarithmic scale is the property of the logarithmic function defined by the formula

$$\log ab = \log a + \log b.$$

From this equation we draw the following conclusions:

a) In order to find on a (single) logarithmic scale the point representing a number ab, we must add the segment delimited by points *1* and a to the segment delimited by points *1* and b.

b) In order to find on a (single) logarithmic scale the point

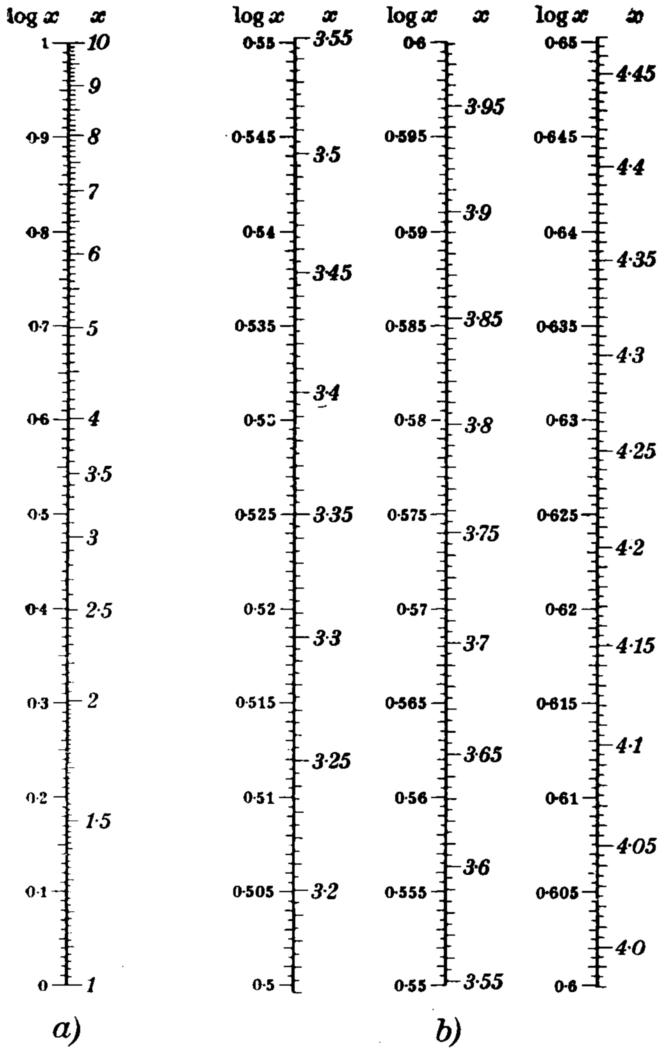

FIG. 33

representing a number a/b, we must subtract the segment delimited by points *1* and b from the segment delimited by points *1* and a.

Both operations can be performed on a slide rule by applying the so called movable scale to the so-called constant scale.

The multiplication result will be obtained on the constant scale only if it is contained between 1 and 10; the same applies to finding the quotient. The question arises what is to be done if the result is not contained between 1 and 10. Now, the formula

$$\log 10a = \log 10 + \log a = 1 + \log a$$

implies that the logarithms of numbers contained between 10 and 10.10 will be contained on an extension of the scale, between 1 and 2 of the y-scale; it should be observed here that the unit of the y-scale is the segment between 1 and 10 of the x-scale since $\log 10 - \log 1 = 1$. Therefore, disregarding the decimal point in the numbers denoting the values of the argument x, we shall have identical scales for x between 0 and 1 of the variable y and between 1 and 2 of the variable y. The same obtains of course for numbers x contained between 100 and 1000, etc. For the same reason the scale for x varying from 0·1 to 1, from 0·01 to 0·1, etc., is, if we disregard the decimal point, identical with the scale of arguments between 1 and 10. It will thus be seen that the logarithmic scale with base 10 is a periodic scale, the period being that part of the scale which corresponds to numbers x between 1 and 10.

On these grounds we can obtain the result of addition (multiplication or division) also in those cases for which no reading has so far been possible. Namely it is sufficient to shift the scale of the slide rule the unit of the logarithmic scale, i.e. to put 10 in place of 1 or *vice versa*.

Thus, seeking the product 8.3 for example we should put the 10, and not the 1, of the movable scale against 8, and we should read the result on the constant scale under the 3 of the movable scale.

Exercises

1. On a given segment AB draw a logarithmic scale for the interval from $x = 310$ to $x = 745$ by means of a parallel projection of the scale marked on the slide rule.

2. Construct two logarithmic scales with the same logarithmic unit and then combine them by putting the point log 1 of the first scale upon the point log 10 of the second scale and the point log 1 of the second scale upon the point log 10 of the first scale. What equation is satisfied by the numbers appearing at the combined points of the two scales?

3. Construct the scales $u = \log x$ and $v = \log y$, adopting for u a unit twice as large as the one for v; combine the scales putting the point $u = 0$ upon the point $v = 0$ and $u = 1$ upon $v = 2$. What relation is satisfied by the numbers x and y represented by the coinciding points?

4. Construct a nomogram for the equation $y = x\sqrt{3}$ consisting of two combined logarithmic scales.

5. Construct a nomogram for the formula of the volume of a sphere $v = 4\pi r^3/3$ consisting of two combined logarithmic scales, for the interval $3 \leqslant r \leqslant 50$; the relative error should not exceed 5%.

§ 9. Projective scale

Functions of the form

$$h(x) = \frac{ax+b}{cx+d} \tag{9.1}$$

are particularly important for the construction of nomograms.

If $c \neq 0$, then the function $h(x)$ is defined for all x except $x = -d/c$.

Denote by H the set of all functions of form (9.1) for which the coefficients a, b, c, d satisfy the condition

$$W = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0 .$$

Functions of the set H are called *homographic functions.* Obviously, if $W = 0$, then $h(x)$ is a constant.

To begin with, it will be observed that:

Every function belonging to the set H is monotone in every interval that does not contain the number $x_0 = -d/c$.

This follows directly from the fact that the derivative

$$h'(x) = \frac{a(cx+d)-c(ax+b)}{(cx+d)^2} = \frac{ad-bc}{(cx+d)^2} = \frac{W}{(cx+d)^2} \neq 0$$

for every x for which the function $h(x)$ is defined has the same sign. Thus if $W > 0$, then $h(x)$ is an increasing function, and if

$W < 0$, then $h(x)$ is a decreasing function. Scales of homographic functions for which $c \neq 0$ are called *projective scales.*

Before we proceed to the construction of projective scales, let us list certain properties of homographic functions.

THEOREM 1. *The set of all homographic functions forms a group if by the group operation we understand the formation of a compound function. (The unit element of that group is the identity function $h(x) = x$.)*

P r o o f. Let

$$h_1(x) = \frac{a_1 x + b_1}{c_1 x + d_1} \quad \text{and} \quad h_2(x) = \frac{a_2 x + b_2}{c_2 x + d_2},$$

for

$$W_1 = \begin{vmatrix} a_1 & b_1 \\ c_1 & d_1 \end{vmatrix} \neq 0, \qquad W_2 = \begin{vmatrix} a_2 & b_2 \\ c_2 & d_2 \end{vmatrix} \neq 0,$$

be functions belonging to the set H. We shall prove that the compound function $h_1\big(h_2(x)\big) = h(x)$ also belongs to the set H. Indeed

$$h_1\big(h_2(x)\big) = \frac{a_1 \dfrac{a_2 x + b_2}{c_2 x + d_2} + b_1}{c_1 \dfrac{a_2 x + b_2}{c_2 x + d_2} + d_1} = \frac{a_1(a_2 x + b_2) + b_1(c_2 x + d_2)}{c_1(a_2 x + b_2) + d_1(c_2 x + d_2)}$$

$$= \frac{(a_1 a_2 + b_1 c_2) x + (a_1 b_2 + b_1 d_2)}{(a_2 c_1 + c_2 d_1) x + (b_2 c_1 + d_1 d_2)},$$

and

$$W = \begin{vmatrix} a_1 a_2 + b_1 c_2 & a_1 b_2 + b_1 d_2 \\ a_2 c_1 + c_2 d_1 & b_2 c_1 + d_1 d_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ c_1 & d_1 \end{vmatrix} \begin{vmatrix} a_2 & b_2 \\ c_2 & d_2 \end{vmatrix} = W_1 W_2 \neq 0.$$

We shall prove, moreover, that for every function $h(x)$ of the set H there exists a function $h_0(x)$ such that $h_0\big(h(x)\big) = h\big(h_0(x)\big) = x$.

It will easily be observed that if

$$h(x) = \frac{ax + b}{cx + d}, \quad \text{and} \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0,$$

then

$$h_0(x) = \frac{dx-b}{-cx+a}, \quad \text{where} \quad \begin{vmatrix} d & -b \\ -c & a \end{vmatrix} \neq 0.$$

The theorem is thus proved.

THEOREM 2. *If $h_1(x)$ and $h_2(x)$ are different homographic functions, then there exist at most two different values x' and x'' such that*

$$h_1(x') = h_2(x') \quad \textit{and} \quad h_1(x'') = h_2(x'').$$

Proof. Let

$$h_i(x) = \frac{a_i x+b_i}{c_i x+d_i} \quad \text{for} \quad i = 1, 2.$$

Equation $h_1(x) = h_2(x)$ implies

$$\frac{a_1 x+b_1}{c_1 x+d_1} = \frac{a_2 x+b_2}{c_2 x+d_2}, \tag{9.2}$$

whence

$$(a_1 c_2 - a_2 c_1) x^2 + (a_1 d_2 + b_1 c_2 - b_2 c_1 - a_2 d_1) x + (b_1 d_2 - b_2 d_1) = 0. \tag{9.3}$$

This equation generally has two roots; in the case where the coefficient $a_1 c_2 - a_2 c_1$ is equal to zero, one of them tends to infinity.

Suppose now that equation (9.3) has three different roots. As we know, it is satisfied for every value of x; in that case, however, equation (9.2) is also satisfied for every value of x. If neither of the numbers c_1 and d_1 is equal to zero, then also neither of the numbers c_2 and d_2 is equal to zero, and we obtain the proportion

$$a_1 : b_1 : c_1 : d_1 = a_2 : b_2 : c_2 : d_2. \tag{9.4}$$

If $c_1 = 0$, then $d_1 \neq 0$ and $c_2 = 0$ and also $d_2 \neq 0$; equation (9.4) is then satisfied. The same applies to the case $d_1 = 0$.

Thus equation (9.4) always holds, i.e. $h_1 = h_2$, contrary to our assumption.

The theorem is thus proved.

Suppose we are given two regular scales with arbitrary units and a point P belonging to neither of them. Projecting one scale upon the other by means of straight lines passing through the point P, we obtain on the second scale a double scale of a homographic function. In order to prove this let us place one scale on the y-axis of a system of orthogonal coordinates (Fig. 34) and let us express the second scale by means of parametric equations

$$x = \alpha t+\beta, \qquad y = \gamma t+\delta .$$

Obviously, points with coordinates x, y corresponding to the same values of the parameter t form a regular scale; the unit of that scale is the distance between the points assigned, for example, to the numbers $t = 0$ and $t = 1$, i.e. $\sqrt{\alpha^2+\gamma^2}$; we thus assume that $\alpha^2+\gamma^2 > 0$.

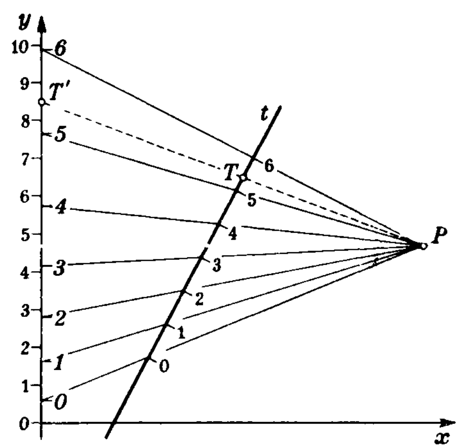

Fig. 34

Point T', which is a projection of point T with coordinates $\alpha t+\beta$ and $\gamma t+\delta$ from point $P(r, s)$ $(r \neq 0)$, will have coordinates 0, y satisfying the equation

$$\begin{vmatrix} 0 & y & 1 \\ \alpha t+\beta & \gamma t+\delta & 1 \\ r & s & 1 \end{vmatrix} = 0,$$

i.e.

$$-y(\alpha t+\beta-r)+s(\alpha t+\beta)-r(\gamma t+\delta) = 0,$$

whence

$$y = \frac{(s\alpha-r\gamma)t+s\beta-r\delta}{\alpha t+\beta-r} . \qquad (9.5)$$

This is a function of form (9.1). In order to prove that it is homographic let us compute the determinant

$$W = \begin{vmatrix} s\alpha - r\gamma & s\beta - r\delta \\ \alpha & \beta - r \end{vmatrix} = r(r\gamma - \alpha s + \alpha\delta - \beta\gamma) = r\begin{vmatrix} \alpha & \beta - r \\ \gamma & \delta - s \end{vmatrix}.$$

It follows from the assumption that the abscissa r from point P is different from zero.

If $W = 0$, i.e. if the determinant

$$\begin{vmatrix} \alpha & \beta - r \\ \gamma & \delta - s \end{vmatrix}$$

were equal to zero, then the system of equations

$$r = \alpha t + \beta, \quad s = \gamma t + \delta$$

would have a solution with respect to t, and thus point $P(r, s)$ would lie on the straight line t, which is not the case. Thus we have $W \neq 0$, i.e. the relation between t and y is a homographic function. This implies that a projection of a regular scale is a projective scale or a regular scale; as can be seen from (9.5), the projection is a regular scale if $\alpha = 0$. This can be generalized by theorem 1 to projections of projective scales.

THEOREM 3. *A projection of a regular scale and a projection of a projective scale upon a straight line are projective scales or regular scales.*

By theorems 1–3 we can construct a projective scale if we have three points corresponding to three given numbers x_1, x_2, x_3.

For example, suppose we are given points *6*, *8*, *11* on a straight line (Fig. 35). By theorem 2 there exists only one homographic function which assigns to those points the values 6, 8, 11. In order to draw its scale, let us consider any regular scale whose point marked 6 coincides with the given point *6*.

Connecting by straight lines the points *8* and 8 and the points *11* and 11 of the two scales, we shall obtain at their intersection a certain point P. A projection of the regular scale from point P upon a given straight line is a projective scale (Theorem 3); since, however, that projection coincides with the required scale at points *6*, *8* and *11*, it is identical with that scale (Theorem 2).

In connection with the fact that, having fixed the end-points of the projective scale (in Fig. 35: $x_0 = 6$ and $x_2 = 11$), we can choose an arbitrary point in the figure to represent a certain intermediate value of the argument, we are confronted with the following two questions concerning the accuracy of the scale:

a) How can we construct a projective scale which would give the best possible absolute accuracy of reading?

b) How can we construct a projective scale with the best possible relative accuracy of reading?

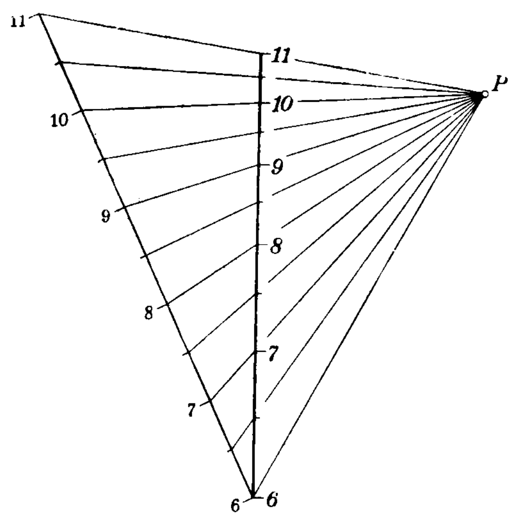

FIG. 35

The first question can be answered immediately. Since the regular scale is the best scale for absolute readings, the projective scale will be the more suitable the better approximation it is of the regular scale. Thus the arithmetic mean of the end-points $(x_0+x_2)/2$ should be taken as close as possible to the geometric mid-point of the scale segment in question.

In case b we should select a projective scale which would approximate a logarithmic scale as accurately as possible. This, as we know, is equivalent to the condition that the product $|x|\lambda_x$ of the absolute value of the argument and the unit for scale x

be, in the interval (x_0, x_2), a function of as little variation as possible. Seeking a homographic function satisfying the required conditions involves very cumbersome, though elementary, computations. In practice we obtain a satisfactory approximation if we connect the mid-point of the interval $(x_0+x_2)/2$ on a regular scale (point r_0 in Fig. 36) with point $(x_0+x_2)/2$ of such a lo-

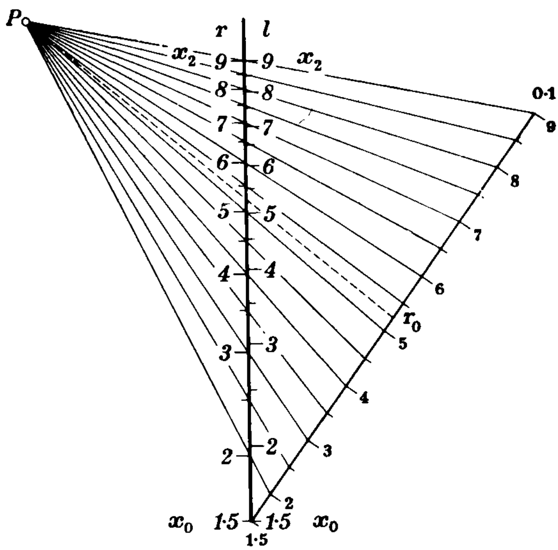

FIG. 36

garithmic scale l on a given straight line that points x_0 and x_2 of that scale coincide with points x_0 and x_2 of the required projective scale. Projecting from point P (of intersection of that straight line with the line joining points x_2 of the scales) we obtain the projective scale r on the given straight line. This scale will coincide with the logarithmic scale at three points: x_0, $(x_0+x_2)/2$ and x_2. It can easily be shown that both in one and in the other half of the interval (x_0, x_2) there are points at which the unit of scale r is larger than the unit of scale l and points at which the unit of scale r is smaller than the unit of scale l. Hence in the interval (x_0, x_2) there exist also at least two points at which the units of the two scales are equal.

In practice we often have to solve problems approaching the one under discussion and concerning projections of logarithmic scales. We shall explain this by means of an example:

Suppose we are given a one-parameter family of functions:

$$f(x) = \frac{3\log x}{4+a\log x}$$

defined in the interval $1{\cdot}5 \leqslant x \leqslant 8{\cdot}5$. Find a function of that family which, in the given interval, approaches a linear function in the sense that the arithmetic mean of the end-points of interval y corresponds to the arithmetic mean of the end-points of interval x.

This condition can be expressed as follows:

$$x_s = \frac{1{\cdot}5+8{\cdot}5}{2} = 5, \qquad 2f(5) = f(1{\cdot}5)+f(8{\cdot}5).$$

We thus have the equation

$$2\frac{3\log 5}{4+a\log 5} = \frac{3\log 1{\cdot}5}{4+a\log 1{\cdot}5} + \frac{3\log 8{\cdot}5}{4+a\log 8{\cdot}5},$$

i.e.

$$2(4+a\log 1{\cdot}5)(4+a\log 8{\cdot}5)\log 5$$
$$= (4+a\log 5)(4+a\log 8{\cdot}5)\log 1{\cdot}5 +$$
$$+(4+a\log 5)(4+a\log 1{\cdot}5)\log 8{\cdot}5\,.$$

Performing elementary calculations we find that a^2 vanishes and we obtain approximately the equation

$$0{\cdot}444a+1{\cdot}172 = 0, \qquad \text{whence} \qquad a = -2{\cdot}64.$$

The required function thus has the form

$$f(x) = \frac{3\log x}{4-2{\cdot}64\log x}.$$

For $x = 1{\cdot}5$ we have

$$f(1{\cdot}5) = 0{\cdot}5283/3{\cdot}532 = 0{\cdot}15,$$

for $x = 8{\cdot}5$

$$f(8{\cdot}5) = 2{\cdot}788/1{\cdot}54 = 1{\cdot}81.$$

Drawing a regular scale on y with end-points 0·15 and 1·81 at the points corresponding to the values $x = 1{\cdot}5$ and $x = 8{\cdot}5$, we finally obtain the required scale of a function, compounded from a homographic function and a logarithmic function, which in the given interval hardly differs from a linear function.

If we wished to find only a single scale, the result could be obtained more simply by a geometrical method. For, as follows from our considerations in this section, it is sufficient to draw an ordinary logarithmic scale AB (i.e. the scale of the function $y = \log x$) in the interval $1{\cdot}5 \leqslant x \leqslant 8{\cdot}5$ (Fig. 37), then draw an

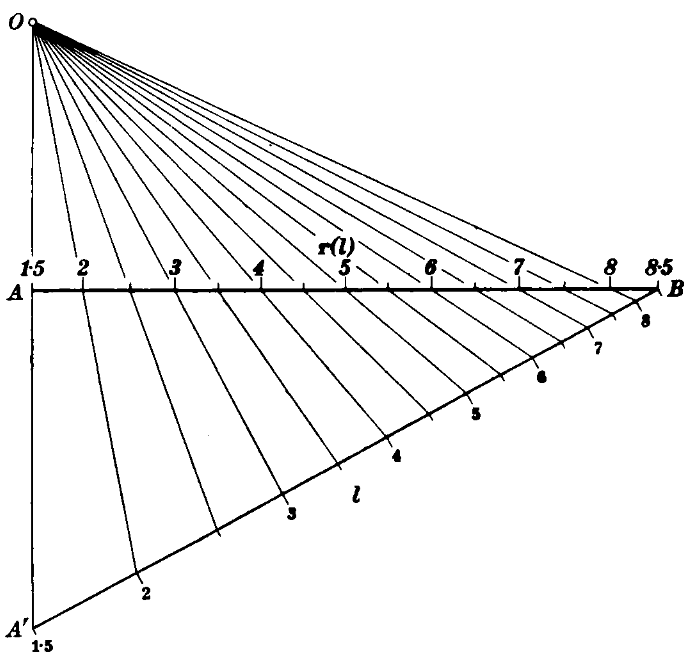

FIG. 37

arbitrary segment $A'B$ and finally make a projection of the scale AB upon the segment $A'B$ from point O, at which the straight line AA' intersects the straight line joining the point $x = 5$ of the scale $A'B$ and the mid-point of the segment AB.

Exercises

1. Construct a scale for the function $\varphi(x) = \sin x$ for the interval $(0°, 90°)$ and then, by projecting from a point, construct a scale for the compound function

$$\varphi(x) = h(\varphi(x)) = \frac{a \sin x + b}{c \sin x + b}$$

in such a manner as to make the point 45° the mid-point of the scale.

2. Approximate by means of a projective scale a segment of a logarithmic scale for arguments from 1 to 10 in such a manner as to obtain the least relative error. (A homographic function which assumes the same values for $x = 1$ and $x = 10$ as a logarithmic function is of the form

$$y = \frac{10-c}{9} \cdot \frac{x-1}{x-c}.)$$

Hint: We should find a c for which the variation of function $\varphi(x) = x\lambda_x$ in the interval $(1, 10)$, i.e. $\varphi(10) - \varphi(1)$, has its minimum value.

Solution. $c = -5{\cdot}64$; the mid-point $5{\cdot}5$ of the scale is below the point $5{\cdot}5$ of the logarithmic scale at point $x = 5{\cdot}2$.

3. Prove that the homographic function $y = h(x)$ which for x_0, x_1, x_2 assumes the values y_0, y_1, y_2 respectively can be written in the implicit form:

$$\begin{vmatrix} xy & x & y & 1 \\ x_0 y_0 & x_0 & y_0 & 1 \\ x_1 y_1 & x_1 & y_1 & 1 \\ x_2 y_2 & x_2 & y_2 & 1 \end{vmatrix} = 0.$$

4. Prove that by assuming for $x_1 = \dfrac{x_2 - x_0}{\ln x_2 - \ln x_2}$, $h(x_1) = \ln x_1$ we obtain a better approximation of a logarithmic scale in the interval (x_0, x_2) than by assuming

$$h\left(\frac{x_0 + x_2}{2}\right) = \ln \frac{x_0 + x_2}{2}.$$

Hint: Prove that

$$x_0 x_2 < \frac{x_2 - x_0}{\ln x_2 - \ln x_0} < \frac{x_0 + x_2}{2}$$

for $0 < x_0 < x_2$ ($\ln x$ denotes a logarithm with base e).

CHAPTER III

EQUATIONS WITH THREE VARIABLES

I. COLLINEATION NOMOGRAMS

§ 10. Equations of the form $f_1(u)+f_2(v)+f_3(w) = 0$. Nomograms with three parallel scales

10.1. The relation

$$w = u+v \tag{10.1}$$

can be represented graphically by means of a very simple drawing composed of three regular scales on parallel lines (Fig. 38). Namely it is sufficient to take equal units for u and v and a unit half as

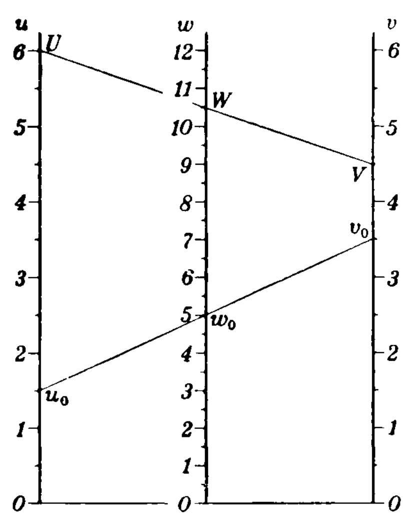

Fig. 38

large for w, and then place the scale for w half-way between the scales u and v, the zero points of the three scales being in line. A straight line intersecting the scales at points U, V, W delimits segments OU, OV, OW satisfying the condition

$$2\,OW = OU+OV.$$

If we denote by λ the common unit on scale u and scale v, then the unit on w will be $\lambda/2$ in length and the equation will be of the form

$$2\lambda w_0/2 = \lambda u_0+\lambda v_0, \quad \text{i.e.} \quad w_0 = u_0+v_0.$$

Figure 38 represents a frequent type of nomogram for a relation of three variables. It is very convenient to read because for given two values of u and v (or u and w, or v and w) we can find the corresponding value for w (or v, or u) by drawing one straight line. It will be observed, however, that the drawing is useful only if there is not much difference either in the variability ranges of u and v or in the required reading accuracies for those variables.

10.2. Consider the case where the variability range for u is several times less than the variability range for v, while the required accuracy is several times greater for u than it is for v.

The nomogram would form a trapeze with v as its greater base and u as its smaller base. In order to enlarge the unit λ_u we should have to enlarge the whole trapeze, which in many cases might prove practically impossible.

The question arises whether it is possible to deform the nomogram so as to enlarge the unit for u considerably with respect to the unit for v, retaining nevertheless the character of the drawing, i.e. the collinearity of the three points representing the three numbers u, v, w, which satisfy equation (10.1).

Obviously, we can admit all those—and only those—transformations of a plane in which straight lines are transformed into straight lines. As follows from our considerations in § 3, Chapter I, those transformations form the set of all projective transformations of a plane. The problem which we are discussing at present, however, is of a very simple nature and can be solved by direct elementary considerations.

The task can be reduced to finding a projective transformation of a plane in which a given trapeze UU_0V_0V is transformed into a rectangle with a given ratio of sides or into another trapeze with a given ratio of parallel sides.

For this purpose, let us take a plane β passing through scale u and on that plane a rectangle (or trapeze) $UU_0V_0'V'$ having a side in common with a given trapeze lying on another plane (Fig. 39).

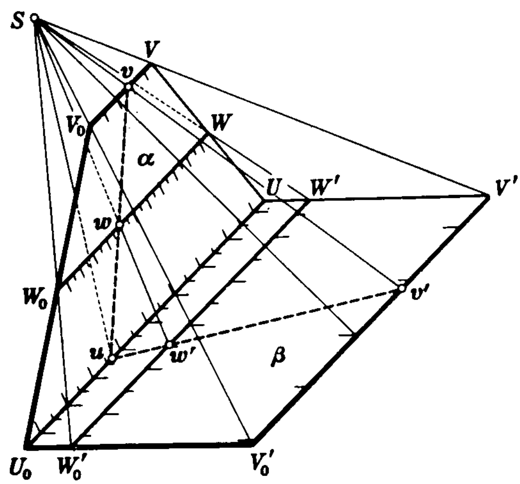

FIG. 39

Since the straight lines VV_0 and $V'V_0'$ are parallel, the straight lines V_0V_0' and VV' intersect. Let the point S of intersection of the lines V_0V_0' and VV' be the projection centre. Obviously, by projecting from centre S upon plane β we obtain a new nomogram, in which numbers u, v, w satisfying the equation $w = u+v$ are also represented by collinear points. This follows from the fact that a projection of three collinear points gives also three collinear points.

It will be observed that the scale $V_0'V'$, which is a projection of the regular scale V_0V, is also a regular scale, because the straight lines V_0V and $V_0'V'$ (the bases of the scales) are parallel to each other. However, if the projection centre S is an ordinary point, then of course the units of those scales differ from each other.

We have proved that there exists a transformation of the nomogram of Fig. 38 into a nomogram with parallel scales in which the unit for u remains unchanged and the unit for v can be chosen arbitrarily. The w-scale will be observed to have moved from its central position towards the scale with the smaller unit.

Owing to the existence of transformations of this kind, we shall make the drawing directly on plane β, drawing on the parallel lines u and v regular scales with arbitrary units and arbitrary senses. As we shall see, the construction of the w-scale can be founded on properties of the transformed nomogram, i.e. without the use of projections.

Nomograms in which readings are taken on a straight line, i.e. those in which points satisfying a given equation are collinear, are called *collineation nomograms.*

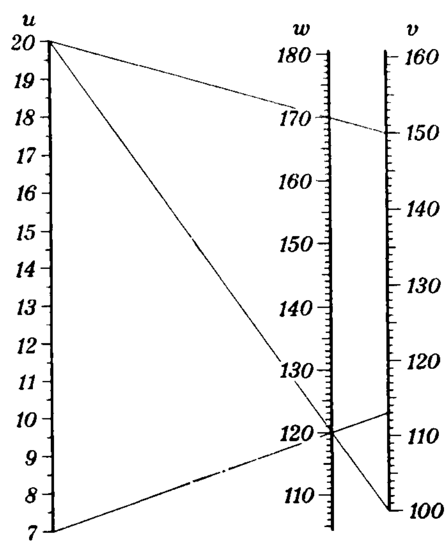

FIG. 40

EXAMPLE 1. Let us draw a nomogram for equation (10.1) if u varies from 7 to 20 and v—from 100 to 160: the unit for v should be one-fifth of the unit for u.

On two parallel lines, u and v, we draw scales with a unit satisfying the required condition (Fig. 40). Knowing that the w-scale will lie on a line parallel to the lines u and v, we first determine one point of the w-scale, e.g. $w = 120$, by two substitutions: $u_0 = 20$, $v_0 = 100$, and $u_1 = 7$, $v_1 = 113$. Joining points u_0 and v_0, and points u_1 and v_1, we obtain at the intersection the point *120* of the w-scale. Joining point *150* on v with point *20* on u we shall obtain point *170* on w. Other points can be obtained either in the same manner as point *120* or by completing a regular scale for which two points are known.

Remark. The nomogram constructed in Fig. 40 is convenient for finding the sum w when the components u and v are given since the w-scale is situated inside the scales for u and v and the reading results from interpolation. If the aim of the nomogram is to find the difference u of given numbers w and v, we must transform it so as to have the u-scale placed inside the scales for w and for v. Accordingly, we pass a plane β through the straight line u of the given nomogram (Fig. 41) and select

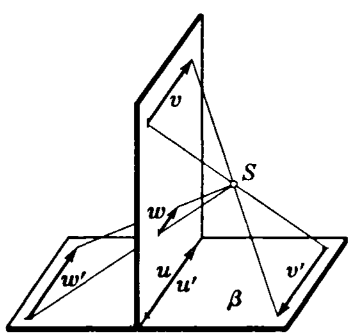

FIG. 41

on the plane a certain straight line v' parallel to line u. We choose the projection centre S on the plane containing the lines v and v' (in order that v' be the projection of v) between those two lines. It can be seen in the figure that the senses of the axes u' and v' will be different if they were identical on the axes u and v, and that the scale $u = u'$ will lie between the scales w' and v'.

Owing to this we can draw the nomogram for the difference $u = w-v$ in the same way as for the sum, adopting arbitrary units for the parallel scales w and v and choosing opposite senses (Fig. 42).

10.3. On the grounds of what we have so far considered we can construct a nomogram consisting of three parallel scales, as in Figs. 38, 40 and 42, for every relation of the form

$$f_1(u)+f_2(v)+f_3(w) = 0\,. \tag{10.2}$$

Indeed, on the strength of the substitutions

$$u' = f_1(u), \quad v' = f_2(v), \quad w' = -f_3(w) \tag{10.3}$$

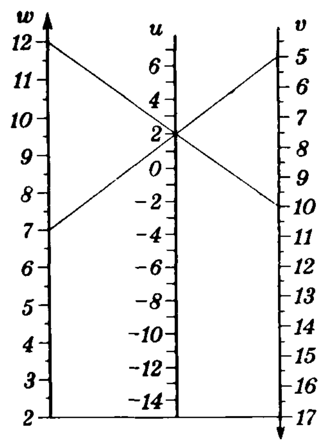

Fig. 42

it is sufficient to draw first a nomogram for the relation $w' = u'+v'$ (taking into account the limits for u', v', w'), and then replace the regular scales u', v', w' by functional scales (10.3).

Example 2. Construct a nomogram for the formula

$$z = \pi x^2 y/3$$

for the volume of a cone with height y and base radius x, adopting for x the interval from 275 cm to 320 cm and for y the interval from 360 cm to 600 cm.

The given formula is equivalent to the equation

$$\log z = 2\log x + \log y + \log \pi - \log 3\,.$$

Substituting

$$u = 2\log x, \quad v = \log y + \log \pi - \log 3 \quad \text{and} \quad w = \log z$$

we obtain for u and v the intervals

$$4{\cdot}88 = 2 \log 275 < u < 2 \log 320 = 5{\cdot}01\,,$$

$$2{\cdot}576 = \log 360 + 0{\cdot}02 < v < \log 600 + 0{\cdot}02 = 2{\cdot}8\,.$$

We first construct a nomogram for the formula $w = u+v$ (Fig. 43), adopting for u a unit about twice as large as the one for v, because every error in determining the logarithm of x is doubled on account of the coefficient 2. Point 7·6 of the

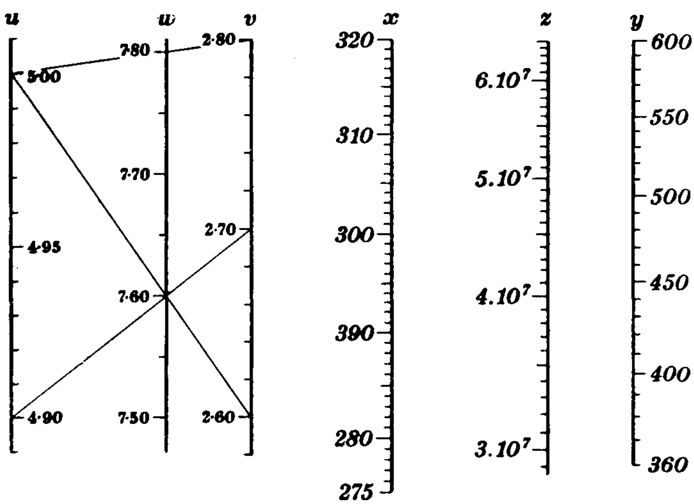

FIG. 43

w-scale is obtained by joining the points $u_0 = 5$ and $v_0 = 2{\cdot}6$ and the points $u_1 = 4{\cdot}9$ and $v_1 = 2{\cdot}7$; then drawing a straight line through the points $u_0 = 5$ and $v_1 = 2{\cdot}8$ we find the point $w = 7{\cdot}8$.

Having determined the regular scales for u, v and w, we replace them by logarithmic scales for x, y and z substituting for x successively numbers 275, 276, 277, ..., 320, for y successively numbers 360, 361, ..., 600 and finally for z the numbers between $z_0 = (\pi/3)\,275^2 \,.\, 360 = 28600000$ and $z_2 = (\pi/3)\,320^2 \,.\, 600 = 61400000$.

It will be observed that the scale for z, just as the other scales of this nomogram, has the same relative accuracy at every point. Since, in the neighbourhood of point $z = 40\,000\,000$ for example, we can distinguish strokes differing by $100\,000$, the relative error does not exceed $100\,000/40\,000\,000 = 1/4\,000$, i.e. $0{\cdot}25\%$.

Example 3. Let us construct a nomogram for the relation

$$\Delta = 3160\,\frac{G^{1\cdot 85}}{d^{4\cdot 97}}$$

considered in the intervals $40 < d < 350$ and $1000 < G < 10000$ with the required reading accuracy of 4%.

Introducing logarithms on both sides of the equation, we have

$$\log\Delta = 1{\cdot}85\log G - 4{\cdot}97\log d + \log 3160.$$

Substituting

$$u = 1{\cdot}85\log G, \qquad v = -4{\cdot}97\log d, \qquad w = \log\Delta - \log 3160$$

we obtain the equation

$$w = u+v;$$

we have the following intervals for the new variables:

$$5{\cdot}55 = 1{\cdot}85\log 1000 < u < 1{\cdot}85\log 10\,000 = 7{\cdot}4,$$

$$-12{\cdot}6 = -4{\cdot}97\,.\,2{\cdot}55 = -4{\cdot}97\log 350 < v \tag{10.4}$$

$$< -4{\cdot}97\log 40 = -4{\cdot}97\,.\,1{\cdot}6 = -8.$$

The accuracy of the logarithmic scale is 4% if the logarithmic unit, i.e. the distance between the points corresponding to numbers $\log 1$ and $\log 10$ is at least $2{\cdot}9$ cm. For, by formula (2.1) of § 2, Chapter II, we have

$$\lambda_x : \lambda_y = |f'(x)| = |(\log x)'| = (\log e \ln x)' = \frac{\log e}{x},$$

and thus

$$\lambda_x x = \lambda_y \log e \approx 0{\cdot}43\,\lambda_y.$$

For instance, if $x = 100$, then in order to retain the accuracy of 4% we must distinguish 96 from 100, i.e. we must have $4\lambda_{100} \geqslant 0{\cdot}5$ mm. We then obtain

$$\lambda_{100}\,.\,100 \geqslant \frac{0{\cdot}5}{4} = 12{\cdot}5,$$

i.e.

$$0{\cdot}43\,\lambda_y \geqslant 125\text{ mm}, \qquad \text{whence} \qquad \lambda_y \geqslant 29\text{ mm}.$$

Accordingly, we must construct a nomogram in which the logarithmic unit for Δ will be at least 2·9 cm. Let us first sketch a nomogram (the scale being 1:2) for the equation $w = u+v$ in

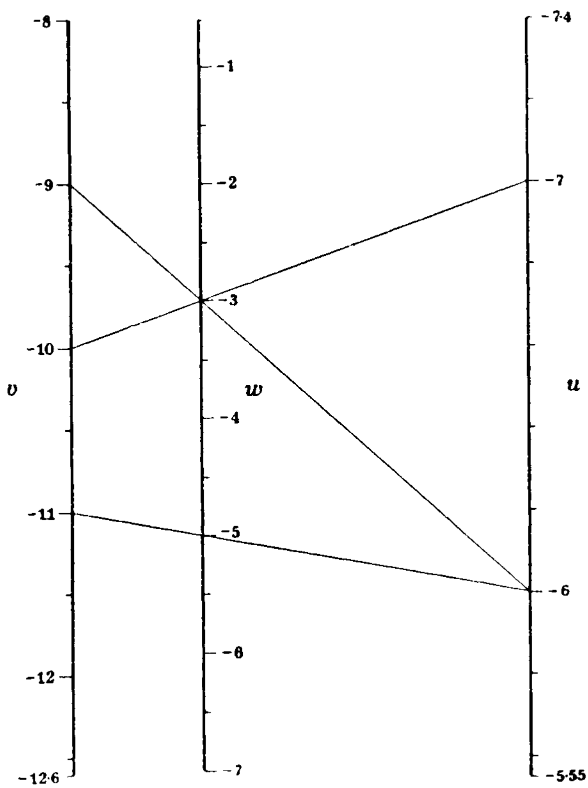

FIG. 44

the intervals (10.4). Adopting for u the unit $10/1{\cdot}85$ cm and for v the unit $10/4{\cdot}97$ cm, we obtain Fig. 44, which shows that the unit for w is more than 3 cm. The ultimate shape of the nomogram will be obtained by drawing instead of u, v, and w the corresponding logarithmic scales; this is shown in Fig. 1.

10.4. In many cases a nomogram of the three parallel scale type requires further transformations in order that it should assume a form satisfying the required conditions.

For example, suppose we are given the equation

$$z^2 = 2x^2+y^3. \tag{10.5}$$

We construct a nomogram for the intervals $0 \leqslant x \leqslant 5$, $0 \leqslant y \leqslant 4$ in the same way as before.

We substitute

$$u = 2x^2, \quad v = y^3, \quad w = z^2$$

and draw regular scales for u between 0 and 50, for v between 0 and 64, and for w exactly in the middle if the units are equal and the senses identical (Fig. 45).

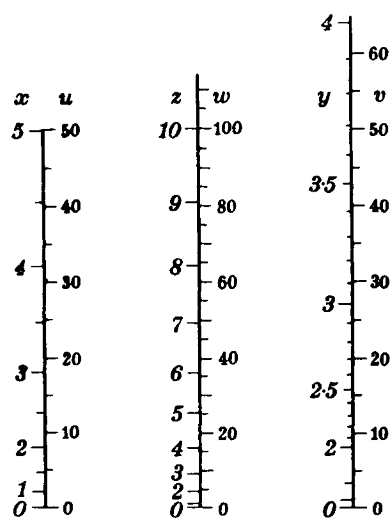

FIG. 45

Replacing the regular scales for u, v, w by functional scales, we find that the units λ_x, λ_y, λ_z increase very fast as numbers x, y, z move away from 0. The reading error would thus be very great in the neighbourhood of zero, while the accuracy for large x, y, z would be excessive. A proportional enlargement of the drawing in order to obtain greater accuracy in the neighbourhood of the zeros would result in dimensions practically unattainable. In that case it is advisable to deform the nomogram in such a way as to enlarge the neighbourhood of zero of the scales x, y, z and at the same time diminish the units for large x, y, z.

This can be achieved by means of projection upon another plane.

We choose plane β, upon which we shall project, in such a way that the edge k of its intersection with plane α intersects the scales x, y, z (Fig. 46). Projecting from a certain point S the

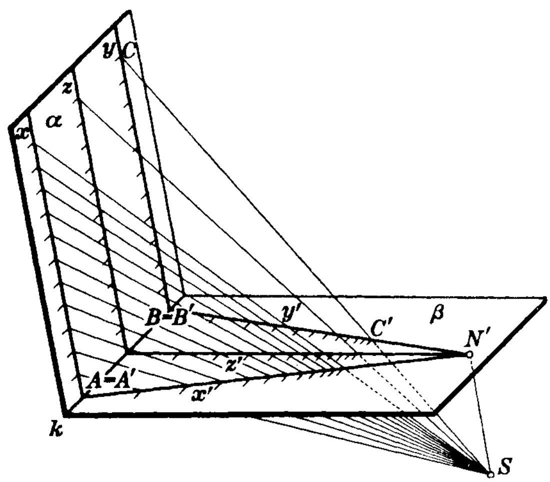

FIG. 46

parallel lines x, y, z, we shall see that the straight lines x', y', z' corresponding to them will intersect at one point. The regular scales u, v, w will be transformed into projective scales with a common point N' constituting the image of the point at infinity of the straight lines under consideration.

Let B and C be the end-points of the v-scale. As we know, by a suitable selection of the position of the projection centre S we can obtain a nomogram in which the points N', B' and C' will be given *a priori*; on this ground we shall draw the transformed nomogram for equation (10.5) in the following way (Fig. 47):

a. Adopting on the projective scale points *0*, *10*, *60* for the variable u, we complete it by means of the regular b-scale. The point at infinity lying on the b-scale is then transformed into point N lying on the u-scale.

b. We draw the transformed scale for v through point N and select on it point *100* and point *0* for example since it is already known that N will be a point corresponding to the point at infinity. The projective scale for v is obtained by a projection of the regular a-scale from point P_1; if we choose $P_1 = P$, then we must also have $a || NP$.

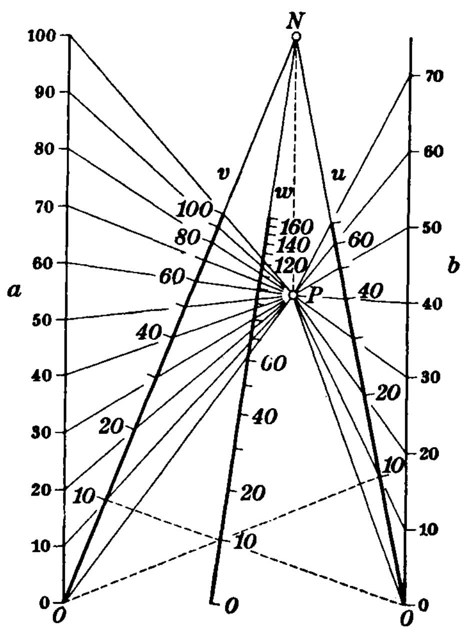

FIG. 47

c. Knowing that the line w must pass through point N, we find only one point of that line in the same way as before (in the case of a nomogram in the previous form), i.e. by determining straight lines passing for instance through point *0* on the u-scale and point *10* on the v-scale and through point *10* on the u-scale and point *0* on the v-scale; other points on the w-scale are obtained also by projection, e.g. by projecting the u-scale from a point on the v-scale.

d. Replacing the projective scales for u, v, w by the scales for x, y, z by the same method as that applied to the regular scales, we obtain the ultimate form of the nomogram, in which the scales have undergone the required deformations.

The above method of constructing nomograms is complicated and inexact because the ultimate forms of the scales x and y are obtained as a result of two drawing operations. We shall regard those nomograms as exact which are obtained directly by replacing a regular scale by a corresponding functional scale.

Such direct construction of a nomogram on the basis of regular scales will be discussed in the following section.

Exercises

1. Construct a nomogram for the relation between vibration frequency f, induction L and electrical capacity C

$$f = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}$$

for the intervals $0{\cdot}2 \leqslant L \leqslant 30$ microhenrys and $2 \leqslant C \leqslant 300$ microfarads.

2. Construct a nomogram for the equation

$$x+y+z = 100$$

taking for each variable the interval from 0 to 100.

3. Construct two nomograms for the equation

$$z = x^2+y^2$$

for the intervals $0 \leqslant x \leqslant 8$, $0 \leqslant y \leqslant 10$, absolute accuracy being required in the first interval and relative accuracy in the second.

4. Construct a nomogram for the equation

$$z = 2\pi/(x+3y^2)$$

taking for x the interval $(0, 3)$ and for y the interval $(1, 5)$.

5. Construct a nomogram for the equation

$$z = x/\sqrt{x^2+y^2}$$

where $10 \leqslant x \leqslant 100$ and $5 \leqslant y \leqslant 10$.

Hint: Write the equation in the form $y/x = \sqrt{1-x^2/z}$.

6. Construct a nomogram of the equation

$$a = \frac{4\pi r}{T^2}$$

for $1 \leqslant r \leqslant 10$, $1 \leqslant T \leqslant 100$.

§ 11. Equations of the form $1/f_1(u)+1/f_2(v)+1/f_3(w) = 0$. Nomograms with three scales passing through a point

11.1 Let us draw three regular scales u, v, w with a common unit (Fig. 48). Assume that both the axes u and w and the axes w and v form angles of 60°, i.e. the axes u and v form an angle of 120°.

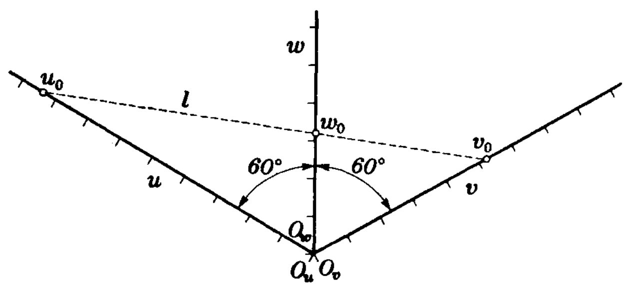

FIG. 48

A straight line l which does not pass through the zero point ntersects the axes at points u_0, v_0 and w_0; the sum of the areas of the triangles Ou_0w_0 and Ow_0v_0 is equal to the area of the triangle Ou_0v_0, i.e.

$$u_0w_0 \sin 60° + w_0v_0 \sin 60° = u_0v_0 \sin 120°\,,$$

and after reduction

$$u_0w_0+w_0v_0 = u_0v_0 \qquad \text{or} \qquad 1/u_0+1/v_0 = 1/w_0\,.$$

Consequently points u, v and w lying on a straight line which does not pass through the zero point satisfy the equation

$$1/u+1/v = 1/w\,. \tag{11.1}$$

If we changed the units, taking λ_u, λ_v and λ_w instead of 1, the drawing would be a nomogram for the relation

$$\frac{1/\lambda_u}{u}+\frac{1/\lambda_v}{v} = \frac{1/\lambda_w}{w} \qquad \text{or} \qquad \frac{a}{u}+\frac{b}{v} = \frac{c}{w}\,.$$

Let us subject this nomogram to an affine transformation (Chapter I, § 4). Accordingly, let us pass a plane β through a straight line k intersecting the scales u and v at certain points U and V (Fig. 49). On that plane let us select an arbitrary point O' and project our nomogram from the point at infinity of the straight line OO'. Obviously, in this manner regular scales will be transformed into regular scales with units proportional to the segments UO', VO' and WO'; it will also be seen that, given the ratio $UO'\colon VO'$, we can select the point O' in such a way as to make the angle $UO'V$ equal to an arbitrarily chosen angle φ. Using the fact that the angle $UO'V$ and the units on the scales u and v are arbitrary, in practical problems we make

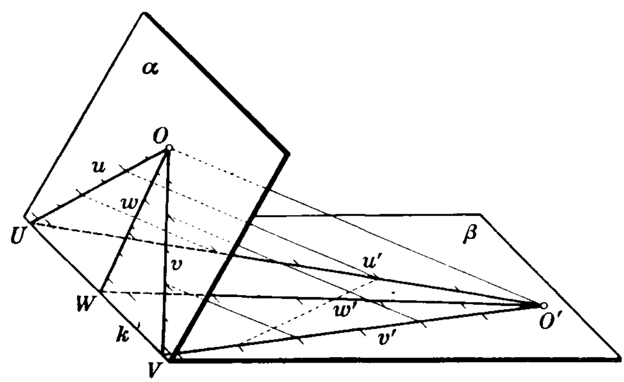

FIG. 49

a direct drawing of the nomogram lying on plane β: on arbitrary two straight lines passing through the point O' we determine regular scales with the zero point O' and arbitrary units, and then find the w-scale (also regular) by locating one of its points as in 10.1, § 10, i.e. by drawing lines u_0v_0 and u_1v_1 selected so as to give $1/u_0+1/v_0 = 1/u_1+1/v_1$.

EXAMPLE 1. Let us represent by a nomogram the equation

$$3/u+2/v = 4/w$$

for $5 \leqslant u \leqslant 10,\ 10 \leqslant v \leqslant 20$.

We draw arbitrary regular scales u and v with a common zero point (Fig. 50). Then, in order to find the point $w = 10$ of the w-scale, we find two parts of corresponding values of u_0, v_0, u_1, v_1, so as to have our equation satisfied:

$$3/u_0+2/20 = 4/10, \quad \text{whence} \quad u_0 = 10,$$
$$3/u_1+2/10 = 4/10, \quad \text{whence} \quad u_1 = 15.$$

Having located the point $w = 10$ of intersection of the line joining the points u_0, v_0 with the line joining the points u_1, v_1, we draw the regular w-scale with zero at point $0_u = 0_v$.

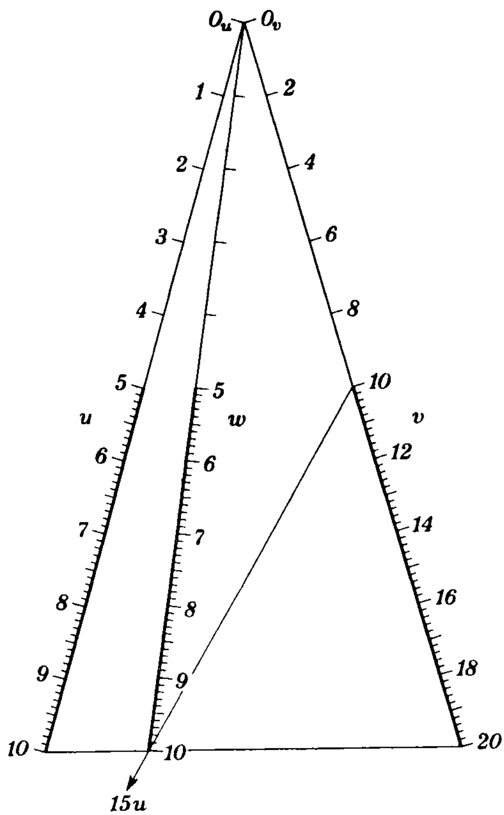

FIG. 50

11.2. Basing ourselves on equation (11.1) we can represent by a nomogram with three scales starting from one point any relation of the form

$$1/f_1(u)+1/f_2(v) = 1/f_3(w). \tag{11.2}$$

Namely, it suffices to substitute

$$u' = f_1(u), \qquad v' = f_2(v), \qquad w' = f_3(w) \tag{11.3}$$

and draw a nomogram for the equation $1/u'+1/v' = 1/w'$. The intervals on u', v' and w' are determined by the given limits of the variables u, v and w; finally the regular scales u', v', w' should be replaced by the functional scales of the functions $f_1(u)$, $f_2(v)$ and $f_3(w)$.

Every equation (11.2) is of course reducible to the form (10.1), i.e., to the form $w'' = u''+v''$, and represented by a nomogram with parallel scales; for that purpose it suffices to assume

$$u'' = 1/f_1(u), \qquad v'' = 1/f_2(v), \qquad w'' = 1/f_3(w)\,. \tag{11.4}$$

The selection of substitution depends on the shape of the function f_1 in the given intervals and on the required degree of accuracy. E.g. if the variability interval of u contains the zero point u_0 of function f_1, then we should of course use substitution (11.3); then the point of intersection of the scales will correspond to number u_0. If the variability interval of u contains the point u_1 at which function f_1 tends to infinity, substitution (11.3) is impossible because the scale of function f_1 would then be unlimited. We then choose formulas (11.4), because number u_1 will be marked at an ordinary point at the spot where $u'' = 0$.

The same remarks can be applied to the cases where the variability intervals contain points of very small values of the function and to the cases where the variability intervals contain points of very large values of the function.

EXAMPLE 2. Let us construct a nomogram for the function

$$z^2 = 2x^2+y^3$$

for the intervals $0 \leqslant x \leqslant 5$, $0 \leqslant y \leqslant 4$.

Let us write our equation in the form

$$z^2+2a+b = 2(x^2+a)+y^3+b$$

and assume

$$u = 1/(x^2+a), \qquad v = 1/(y^3+b), \qquad w = 1/(z^2+2a+b);$$

we shall obtain an equation in which u and v will vary in the intervals $1/(25+a) \leqslant u \leqslant 1/a$, $1/(64+b) \leqslant v \leqslant 1/b$. (An analogical substitution $u = 1/x^2$ and $v = 1/y^3$ would not give the desired results because the scales for u and v would be unlimited.)

The numbers a, b are arbitrary; taking a small number for a we shall see that the length ratio of the interval $\big(1/(25+a), 1/a\big)$ to the interval $(0, 1/a)$ tends to unity as $a \to 0$ since we have

$$\left(\frac{1}{a} - \frac{1}{25+a}\right) : \frac{1}{a} = 1 - \frac{a}{25+a} \underset{a \to 0}{\to} 1 .$$

For large numbers a this ratio tends to zero. Thus for small numbers the point O would be very close to one end of the scale and for large numbers it would be very far from it. Let us take intermediate values, e.g., $a = 20$ and $b = 50$. We shall obtain the intervals $1/(25+20) \leqslant u \leqslant 1/20$, $1/114 \leqslant v \leqslant 1/50$.

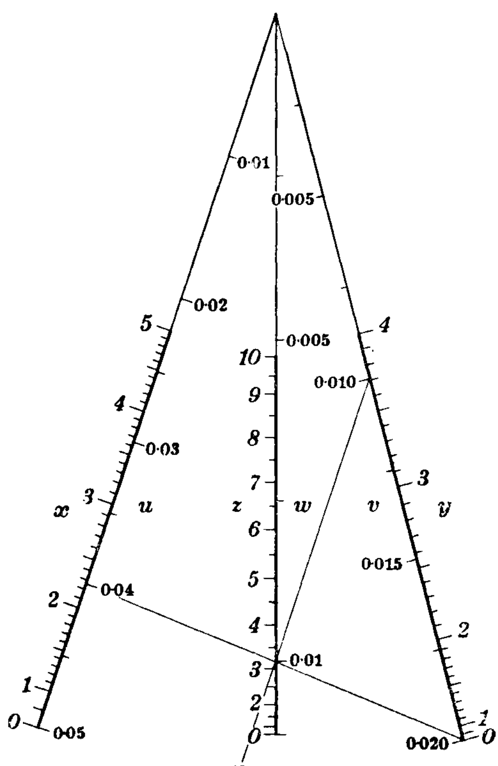

FIG. 51

The nomogram (Fig. 51) is constructed as in Example 1. Equation

$$2/u+1/v = 1/0{\cdot}01$$

will be satisfied for $v_0 = 1/50$ and $u_0 = 1/25$ and for $u_1 = \infty$ and $v_1 = 0{\cdot}01$.

We now replace the regular scale u by the functional scale of the function $u = 1/(x^2+20)$ in the interval $0 \leqslant x \leqslant 5$, the v-scale by the functional scale of the function $v = 2/(y^3+50)$ in the interval $0 \leqslant y \leqslant 4$, and finally the w-scale by the functional scale of the function $w = 1/(z^2+90)$ in the interval $0 \leqslant z < \sqrt{114}$.

Example 3. Let us represent by a nomogram the relation $\mu = \mu_1^n$ if the variables are contained in the intervals $0{\cdot}9 \leqslant \mu \leqslant 1{\cdot}5$, $0{\cdot}2 \leqslant n \leqslant 1$.

The required numbers are the values of μ_1.

Let us first take μ in the interval from 1 to 1·5 and write

$$\log \mu = n \log \mu_1, \qquad \log \log \mu = \log n + \log \log \mu_1. \qquad (11.5)$$

It will be observed that the first term is contained in the interval

$$-\infty = \log \log 1 \leqslant \log \log \mu < \log \log 1{\cdot}5 = -0{\cdot}75,$$

and the second term in the interval

$$-0{\cdot}7 = \log 0{\cdot}2 \leqslant \log n \leqslant \log 1 = 0$$

in view of which the third term is in the interval $(-\infty,\ 0{\cdot}25)$.

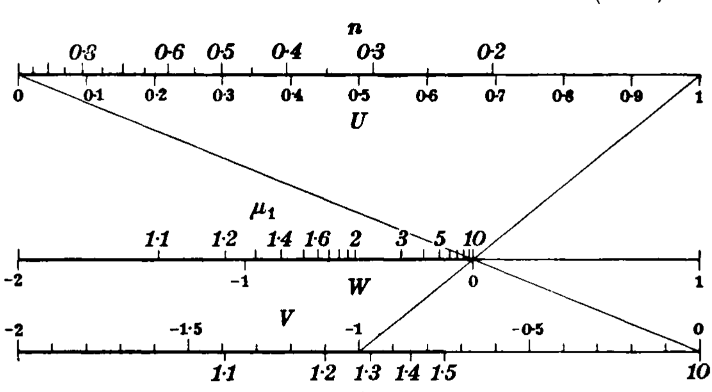

Fig. 52

If we drew for the relation

$$\log\log\mu_1 = -\log n + \log\log\mu$$

a nomogram with three parallel scales, we should have to substitute

$$U = -\log n, \qquad V = \log\log\mu, \qquad W = \log\log\mu_1.$$

This nomogram is represented in Fig. 52. The scales μ and μ_1 would be unlimited, and both of them would have too small units in the neighbourhood of their greatest values. In order to extend these scales in the neighbourhood of those values and at the same time draw nearer the points 1_μ and 1_{μ_1} situated in infinity, we shall make a transformation changing the line passing through the points $-0{\cdot}7_V$ and $0{\cdot}5_W$ into a line at infinity; it intersects the U-scale at the point $U = W-V = 0{\cdot}5+0{\cdot}7 = 1{\cdot}2$. This means that we use the substitution $U' = U-1{\cdot}2$, $V' = V+0{\cdot}7$ and $W' = W-0{\cdot}5$ in order that the points $0{\cdot}7_V, 0{\cdot}5_W, 1{\cdot}2_U$ should form a new zero axis, and then assume $u = 1/U'$, $v = 1/V'$, $w = 1/W'$ in order that this axis be transformed into a straight line at infinity. We thus finally have

$$u = \frac{1}{-1{\cdot}2-\log n}, \qquad v = \frac{1}{0{\cdot}7+\log\log\mu}, \qquad w = \frac{1}{-0{\cdot}5+\log\log\mu_1} \tag{11.6}$$

in the intervals $-2 \leqslant u \leqslant 0{\cdot}3$, $-18{\cdot}2 \leqslant v \leqslant 0$.

We first determine a nomogram for the equation $1/u+1/v = 1/w$, drawing the scale for u from 0 to -2 and the scale for v from 0 to -20, and finding the point -2 on the w-scale by intersecting the line joining the points -2_v, ∞_u by the line joining the points ∞_v, -2_u.

Finally we replace the scales u, v and w by functional scales according to substitution (11.6). We obtain the nomogram given in Fig. 53, which has finite scales for μ and μ_1 and can serve for reading the values of μ_1.

It will be observed that for numbers μ contained in the interval $0{\cdot}9 \leqslant \mu \leqslant 10$ the equation can be written in the form

$$\frac{1}{\mu} = \left(\frac{1}{\mu_1}\right)^n$$

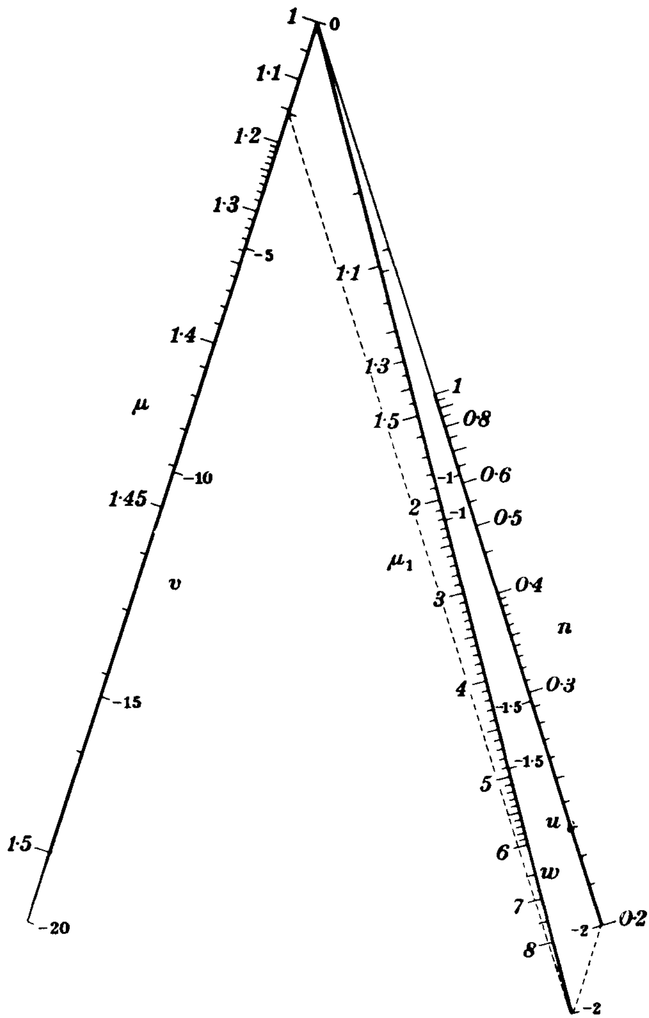

FIG. 53

$1/\mu$ and $1/\mu_1$ being numbers already marked on the nomogram. Thus to complete the drawing we add the numbers $\mu = 0{\cdot}9$, $0{\cdot}91$ ... and $\mu_1 = 0{\cdot}9$ ... using a different type in order to avoid errors.

11.3. The method of transforming an equation so as to obtain the required nomogram can be generalized.

Suppose we are given an equation $W = U + V$ and let a nomogram of the three parallel scale type have the form given in Fig. 54. Assume that none of the scales extends from $-\infty$ to $+\infty$

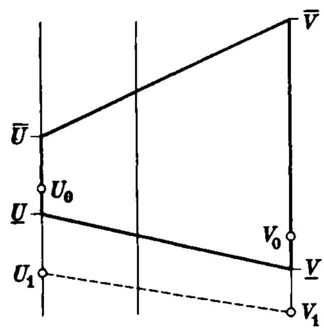

FIG. 54

and that in order to improve the accuracy of readings in the neighbourhood of point $\underline{U}$ and of the point V we want to obtain scales in which the point U_0 will pass into the mid-point of the transformed segment $\underline{U}\overline{U}$ and, similarly, the point V_0 will pass into the mid-point of corresponding segment $\underline{V}\overline{V}$. It follows from the considerations of § 4 (Chapter 1) that we can find points U_1 and V_1 such that the fours $(\underline{U}\overline{U}U_0U_1) = -1$ and $(\underline{V}\overline{V}V_0V_1) = -1$ will be harmonic. Making a transformation in which the straight line U_1V_1 is transformed into a straight line at infinity, we obtain scales for u and w in which the points U_0 and V_0 are the mid-points of the scale segments under consideration. The method of doing that is the following:

We write the given equation in the form $W - U_1 - V_1 = U - U_1 + V - V_1$, whence by substituting $u = 1/(U - U_1)$, $v = 1/(V - V_1)$, $w = 1/(W - U_1 - V_1)$ we obtain the equation $1/w + 1/u = 1/v$.

The scales for u, v and w will be regular; replacing them according to the substitutions by the projective scales U, V and W, we shall easily observe that the values of U_1 and V_1 will be represented by points at infinity. Hence numbers U_0 and V_0 will be the mid-points of segments U and V on the scale.

According to the above observations let us modify the transformation of the nomogram of the equation $\mu = \mu_1^n$ for

$$1 \leqslant \mu \leqslant 1{\cdot}5, \qquad 0{\cdot}2 \leqslant n \leqslant 1.$$

Let us set ourselves the task of transforming the scale μ so as to have the point $1{\cdot}25_\mu$ situated at the mid-point of the segment with end-points 1_μ and $1{\cdot}5_\mu$ and the point $0{\cdot}4_n$ at the mid-point of the segment $1_n 0{\cdot}2_n$.

To begin with, it will be observed that if, instead of exact numbers, we took their approximations, then the selected points $1{\cdot}25_\mu$ and $0{\cdot}4_n$ would not lie at the mid-points of the corresponding scales but in the neighbourhood of their mid-points.

Since $\underline{V} = -\infty$ and $\overline{V} = \log\log 1{\cdot}5 = -0{\cdot}75$ and $V_0 = \log\log 1{\cdot}25 = -1$, the harmonic point V_1 is $-0{\cdot}5$ because

$$(\underline{V}\overline{V}V_0V_1) = (V_0V_1\underline{V}\overline{V}) = \frac{V_0\underline{V}}{V_1\underline{V}} : \frac{V_0\overline{V}}{V_1\overline{V}}$$

$$= 1 : \frac{-0{\cdot}75+1{\cdot}5}{-0{\cdot}75+0{\cdot}5} = -1\,.$$

Since $\underline{U} = -\log 1 = 0$, $\bar{u} = \ln 0{\cdot}2 = 0{\cdot}7$ and $U_0 = -\ln 0{\cdot}4 = 0{\cdot}4$, the harmonic point U_1 corresponds to the value of U_1 satisfying the equation:

$$(\underline{U}\bar{U}U_0U_1) = \frac{\underline{U}U_0}{\bar{U}U_0} : \frac{\underline{U}U_1}{\bar{U}U_1} = \frac{0{\cdot}4-0}{0{\cdot}4-0{\cdot}7} : \frac{U_1-0}{U_1-0{\cdot}7} = -1$$

whence we obtain $U_1 = 2{\cdot}8$.

We now write the equation $W = U + V$ in the form

$$U - 2{\cdot}8 + V + 0{\cdot}5 = W - 2{\cdot}8 + 0{\cdot}5$$

and then substitute

$$u = \frac{1}{U-2{\cdot}8}, \qquad v = \frac{1}{V+0{\cdot}5},$$

$$w = \frac{1}{W-2{\cdot}3},$$

i.e.

$$u = \frac{1}{-\log n - 2{\cdot}8}, \qquad v = \frac{1}{\log\log\mu + 0{\cdot}5},$$

$$w = \frac{1}{\log\log\mu_1 - 2{\cdot}3}.$$

The variability intervals are

$$-0{\cdot}5 = \frac{1}{-\log 0{\cdot}2 - 2{\cdot}8} \leqslant u \leqslant \frac{1}{-\log 1 - 2{\cdot}8} = -0{\cdot}35,$$

$$-4 = \frac{1}{-\log\log 1{\cdot}5 + 0{\cdot}5} \leqslant v \leqslant \frac{1}{\log\log 1 + 0{\cdot}5} = 0.$$

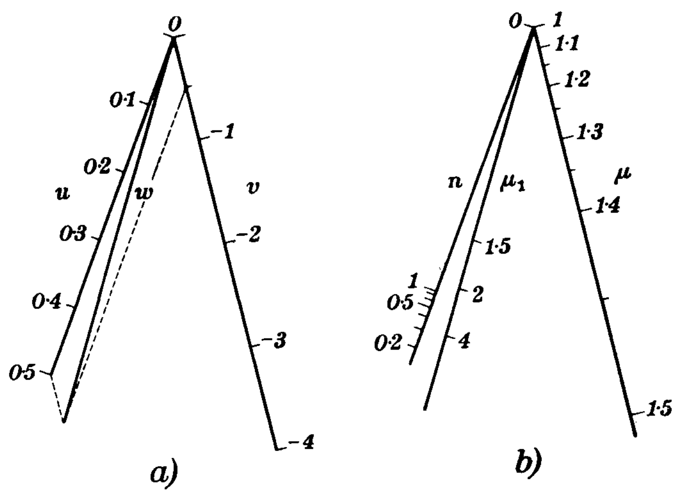

FIG. 55

We thus draw a nomogram from the limits determined for n and v (Fig. 55a), and then, using the substitutions, we draw the scales n, μ and μ_1 (Fig. 55b).

Exercises

1. Construct nomograms with three convergent scales for the following relation: $1/R = 1/R_1 + 1/R_2$ for R_1 and R_2 in the interval from 1 to 1000; R, R_1 and R_2 denote electrical resistance in ohms.

2. Construct a nomogram for the relation

$$E = \frac{9KG}{G+3K}$$

between the Young module E, the rigidity module G and the compression module K for $0{\cdot}2 \leqslant G \leqslant 0{\cdot}8$ and $0{\cdot}3 \leqslant E \leqslant 5{\cdot}2$.

§ 12. Equations of the form $f_1(u)\,f_2(v) = f_3(w)$. Nomograms of the letter N type

Take two regular scales with equal units and different senses on parallel lines u and v and a straight line w passing through their zero points (Fig. 56). Draw on line w a scale with the same unit, assuming $w = 0$ at point 0_u and $w = a$ at point 0_v, where a is the distance of the points 0_u and 0_v.

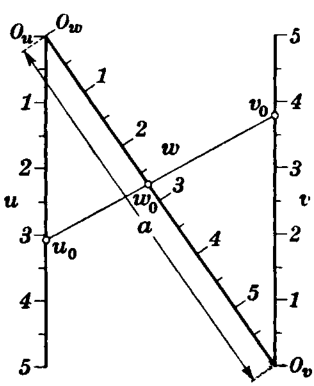

FIG. 56

It will be seen that the points u, v and w of the corresponding scales lie on a straight line if

$$u/v = w/(a-w). \tag{12.1}$$

Thus, by aid of a nomogram of this type, we can represent graphically a relation between u, v and w of form (12.1).

The variables u and v, occurring in the numerator and in the denominator, are marked on regular scales parallel to each other, and the variable w is marked on a projective scale on the line joining the zeros of the scales u and v. The assumed equality of the units is inessential because the constant factor which would have to be written in the numerator or in the denominator if the units λ_u and λ_v were different can be transferred to the w-scale without altering the character of the homographic function.

In practice we usually need only a part of the nomogram corresponding to given intervals on u and v. If that part does not contain points 0_u and 0_v, then it is necessary to find other points of the w-scale. This proves very simple. E.g., suppose we are given the intervals $5 \leqslant u \leqslant 7$ and $15 \leqslant v \leqslant 20$ (Fig. 57).

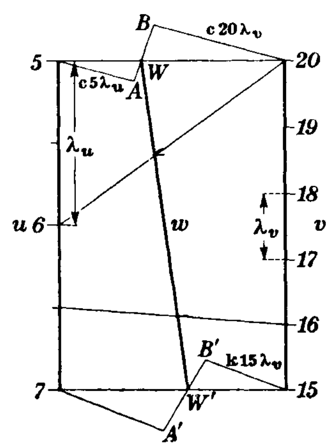

FIG. 57

In order to find point W of the w-scale on the line joining points 5_u and 20_v, let us observe that the ratio of the segments $5_u W : 20_v W$ is equal to the ratio $5\lambda_u : 20\lambda_v$ where λ_u and λ_v are unit segments on the regular scales u and v. Drawing from points 5_u and 20_v segments $5_u A$ and $20_v B$ parallel and proportional to $5\lambda_u$ and $20\lambda_u$ we obtain at the intersection with the line $5_u 20_v$ the point W of the w-scale. We can also divide the segment $5_u 20_v$ in the ratio

$$5_u W : 20_v W = 5\lambda_u : 20\lambda_v.$$

Similarly we can find point W' on the line $7_u 15_v$.

We can obtain points of the scale on the straight line WW' by joining together points of the scales u and v.

By means of nomograms of the type under discussion we represent functional relations of the form

$$f_1(u)\,f_2(v) = f_3(w); \tag{12.2}$$

as in the case of the sum of functions, it suffices to assume

$$u' = f_1(u), \quad v' = f_3(w) \quad \text{and} \quad (a-w)/w = f_2(v),$$

i.e.

$$w = \frac{a}{1+f_2(v)}.$$

R e m a r k. Relation (12.2) gives us by applying logarithms

$$\log f_1(u) + \log f_2(v) = \log f_3(w),$$

i.e. an equation which can be represented by a nomogram with three parallel scales, or by a nomogram with three convergent scales; in many cases, however, it is more convenient to retain the product form and draw a nomogram in the shape of letter N.

EXAMPLE 1. Let us draw a nomogram for the equation

$$T^{\infty} = T_1^2/(2T_1 - T_2)$$

in which we could read the variable T^{∞} given T_1 and the ratio T_2/T_1. The intervals are given by the inequalities $0 \leqslant T_1 \leqslant 30$, $1 \leqslant T_2/T_1 \leqslant 2$, $5 \leqslant T^{\infty} \leqslant 100$.

The equation in question gives the relation between the growth of temperature of a body under the influence of a supply of heat constant with regard to quantity. Let T_0 denote the temperature of the environment. If from the instant 0 a body receives in every unit of time increments of heat constant with regard to quantity, then, by well-known physical laws, we have

$$dT = a\,dt - bT\,dt;$$

a is a coefficient dependent on the quantity of the heat supplied in a unit of time and on the physical properties of the body, T is the difference between the body temperature and the temperature of environment a, and b is a positive coefficient which is the measure of the speed of cooling (as we know, cooling is proportional to the difference T of the body and the environment temperatures). Integrating this equation we obtain

$$\int \frac{dT}{a-bT} = \int dt, \qquad -\frac{1}{b} \ln (a-bT) = t-c, \qquad a-bT = e^{-bt-bc}$$

and finally

$$T = (a-e^{bc-bt})/b \qquad \text{or} \qquad T = A-Be^{-bt}$$

where A and B denote constants. Since at the instant $t = 0$ we should have $T = 0$,

$$0 = A-Be^{0}, \qquad \text{i.e.} \qquad A = B,$$

we thus have $T = A(1-e^{-bt})$.

If $t \to \infty$, then $T^{\infty} = A$; therefore we can write

$$T = T^{\infty}(1-e^{-bt}).$$

Let T_1 denote the temperature at the instant $t = t_1$ and T_2 the temperature at the instant $t_2 = 2t_1$; we then have

$$T_1 = T^{\infty}(1-e^{-bt_1}), \qquad T_1 = T^{\infty}(1-e^{-bt_2}),$$

whence

$$(1-T_1/T^{\infty})^2 = (e^{-bt_1})^2 = e^{-2bt_1} = 1-T_2/T^{\infty}$$

or

$$-2T_1/T^{\infty}+(T_1/T^{\infty})^2 = -T_2/T^{\infty}$$

i.e.

$$T^{\infty} = T_2^2/(2T_1-T_2).$$

Assuming $T^{\infty} = w$, $T_1 = u$ and $T_2/T_1 = v$ we obtain an equation

$$w = u^2/(2u+uv) \qquad \text{or} \qquad u/(2-v) = w \qquad \text{or} \qquad u/v' = w$$

where $v' = 2-v$.

The variables u and v' should be drawn on regular scales parallel to each other; since v' varies in the interval from $2-2 = 0$ to $2-1 = 1$, we draw (Fig. 58) two regular scales:

1. the u-scale in the interval from 0 to 30,
2. the v'-scale in the interval from 0 to 1.

We now find the points of the w-scale through the intersection by lines joining such points of the scales u and v' that the equation $u = wv'$ is satisfied; in our case this is obtained by projecting the u-scale from point $1_{v'}$ of the v'-scale (or from point $1{\cdot}5_{v'}$ but inscribing numbers w twice as large as the corresponding numbers u).

Replacing the scales u, v' and w by the values T_1, $2-T_2/T_1$ and T^∞ we obtain the required nomogram.

EXAMPLE 2. Let us draw a nomogram for the relation

$$z = 0{\cdot}85y^x$$

for $1{\cdot}2 \leqslant x \leqslant 1{\cdot}4$ and $12 \leqslant y \leqslant 20$ retaining a regular scale for x.

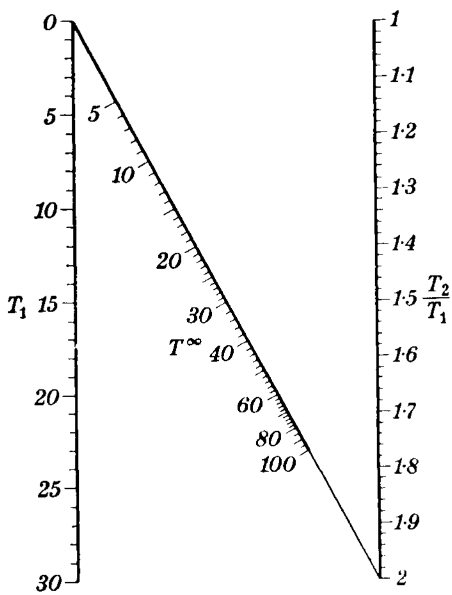

FIG. 58

This equation can be written in the form

$$\frac{\log z - \log 0{\cdot}85}{x} = \log y.$$

Assume

$$u = \log z - \log 0{\cdot}85 = \log z + 0{\cdot}0706, \qquad v = x, \qquad w = \log y;$$

we then have an equation $u/v = w$ in which v varies in the interval $1{\cdot}2 \leqslant v \leqslant 1{\cdot}4$ and w in the interval $\log 12 \leqslant w \leqslant \log 20$, i.e. $1{\cdot}079 \leqslant w \leqslant 1{\cdot}301$, and therefore u varies in the interval $1{\cdot}2 \,.\, 1{\cdot}079 \leqslant u \leqslant 1{\cdot}4 \,.\, 1{\cdot}301$, i.e., $1{\cdot}294 \leqslant u \leqslant 1{\cdot}821$.

Proceeding to the construction of the nomogram, we begin by drawing the regular scales of u and v on parallel lines (Fig. 59) and then, by means of the construction given in Fig. 57, we locate the points W and W' of the line w.

The scale of the variable w is usually obtained either by projecting the u-scale from a point of the v-scale or by projecting the v-scale from a point of the u-scale. The value w_0 which arises by the projection of the point u_0 from the point v_0 (or *vice versa*) is found from the equation $u_0 = v_0 w_0$.

The ultimate form of the nomogram (Fig. 60) is obtained by replacing the scales u, v and w by the scales x, y and z according to the substitutions.

EXAMPLE 3. Let us construct a nomogram for the equation

$$T = 2\pi\sqrt{l/g} \tag{12.3}$$

assuming the variables in the intervals $978 \leqslant g \leqslant 983$, $80 \leqslant l \leqslant 100$.

Equation (12.3) can be written in the form $l/g = T^2/4\pi^2$. We draw the scales for l and g on parallel lines (Fig. 61) and the scale for $w = T^2/4\pi^2$ on the straight line WW', the points W and W' being found as in Fig. 57. Having determined the w-scale by means of projecting from point *980* on the g-scale, we replace it by the T-scale in virtue of the formula $w = T^2/4\pi^2$.

Exercises

1. Construct a nomogram for the function

$$xyz = 10$$

or the intervals $5 \leqslant x \leqslant 6$, $1 \leqslant y \leqslant 2$.

2. Construct a nomogram for the function

$$z = 1/2xy$$

for the intervals $10 \leqslant x \leqslant 20$, $1 \leqslant y \leqslant 4$.

3. Construct a nomogram for the function

$$z = xy/(2x+3y)$$

for the intervals $4 \leqslant x \leqslant 5$, $2 \leqslant y \leqslant 4$.

4. Illustrate by means of a nomogram the Poisson law $p^{x-1}/T^{x} = 0{\cdot}0002$ for the exponents x between 1·3 and 1·5 and the absolute temperatures between 280 and 350.

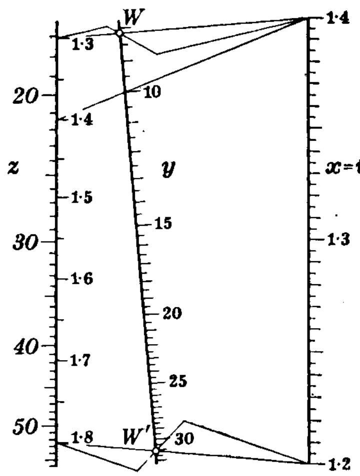

FIG. 59

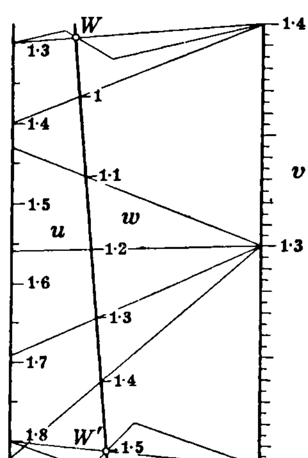

FIG. 60

5. Construct a nomogram for the formula $m = m_0/\sqrt{1-v^2/c^2}$ for the initial masses m_0 between 100 kg and 1000 kg and velocities v between 8000 km/sec and 14000 km/sec ($c = 300000$ km/sec).

6. Construct a nomogram for the motion of satellites around planets

$$r^3/T^2 = 1{\cdot}672M$$

or masses from $3 \cdot 10^{23}$ kg to $2 \cdot 10^{26}$ kg and distances r from 160 km to 1000 km. T denotes the time of a rotation around a planet measured in seconds.

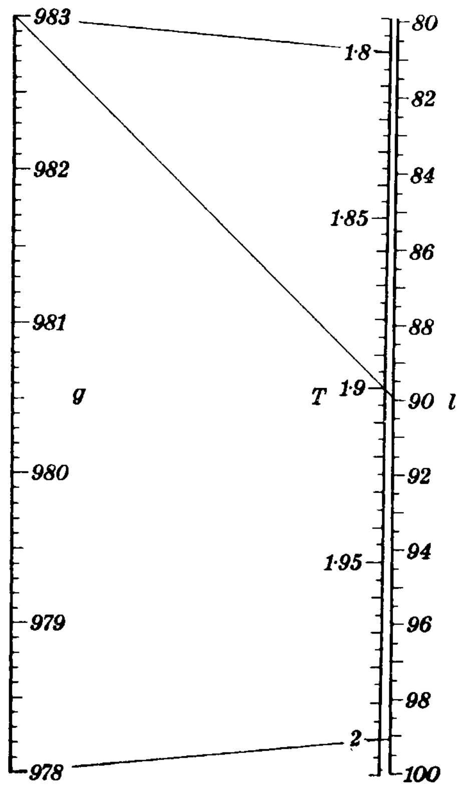

FIG. 61

§ 13. Equations of the form $f_1(x)f_2(y)f_3(z) = 1$. Nomograms with scales on the sides of a triangle

13.1. Let us draw on the sides of an equilateral triangle three regular scales with units equal to the side; let the zero points 0_u, 0_v and 0_w and the points 1_u, 1_v and 1_w be the vertices of the triangle and let them be situated in such a way as to observe the cyclic order, i.e. to have $0_u = 1_w$, $1_u = 0_v$, $1_v = 0_w$ (Fig. 62).

Let us take a straight line l intersecting the scales at points U, V and W different from 0_u, 0_v and 0_w and different from the points at infinity and find the relation between the numbers u, v and w. Drawing from point 1_u a straight line l' parallel to l and denoting by w_0 the number ascribed to the point of intersection of l' and the w-scale, we have

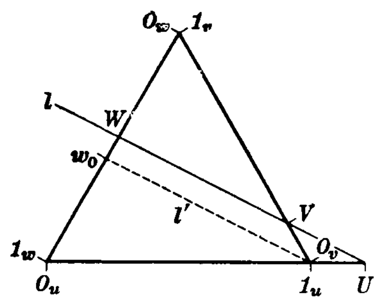

FIG. 62

$$w/(1-v) = w_0/1 \quad \text{and} \quad u/(1-w) = 1/(1-w_0),$$

whence we obtain

$$w/(1-v)+(1-w)u = 1,$$

$$uw+1-v-w+vw = u-uv,$$

$$uw+uv+vw-u-v-w+1 = 0 \tag{13.1}$$

or

$$(1-1/u)\,(1-1/v)\,(1-1/w) = 1. \tag{13.2}$$

If the line l passes through a vertex of the triangle, then, as can be seen from the form (13.1), the equation is satisfied; similarly, if l is parallel to one of the sides, the equation will be satisfied in the form (13.2); e.g., if $l \| u$, then we have

$$(1-1/v)\,(1-1/w) = 1 \quad \text{or} \quad v+w = 1,$$

which accords with the agreement concerning the cyclicity of senses on the scales v and w.

Changing the unit of the scales u, v and w, i.e. substituting

$$u' = c_1 u, \quad v' = c_2 v, \quad w' = c_3 w,$$

we obtain the equation

$$(1-a/u)(1-b/v)(1-c/w) = 1.$$

Our nomogram, consisting of three regular scales (which now have arbitrary units), can be changed by an affine transformation into a new nomogram, in which, on the grounds of our considerations of § 4 (Chapter I), the scales are situated on the sides of an arbitrary triangle. Now it is only the location of the zero points at the vertices of the triangle and the regularity of the scales that are essential.

EXAMPLE 1. Let us construct a nomogram for the equation

$$uvw = (u+2)(v-3)(w+1) \tag{13.3}$$

for the intervals $0 \leqslant u \leqslant 1$, $1 \leqslant v \leqslant 2$.

This equation can be written in the form

$$(1+2/u)(1-3/v)(1+1/w) = 1$$

or in the form

$$\left(1-\frac{1}{-u/2}\right)\left(1-\frac{1}{v/3}\right)\left(1-\frac{1}{-w}\right) = 1.$$

We shall thus have a nomogram consisting of three regular scales, and, according to the agreement concerning the position of the points U_u, 0_v and 0_w

1. The point 0_v will coincide with the point -2_u,
2. The point 0_w will coincide with the point 3_v,
3. The point 0_u will coincide with the point -1_w.

The shape of the nomogram is shown diagrammatically in Fig. 63a; the intervals of the variables u and v which are marked on it explain the construction of the ultimate form (Fig. 63b):

1. We draw an arbitrary regular scale on v for the values from 1 to 2,

2. We select an arbitrary point 0_u and then connect it with point 0_v and with point 3_v,

3. On the straight line $0_u 0_v$ we draw an interval (0, 1) of the u-scale, which has its point -2 at the point 0_v,

4. On the straight line $0_w 0_u$ we draw the w-scale, where point -1 coincides with the point 0_u.

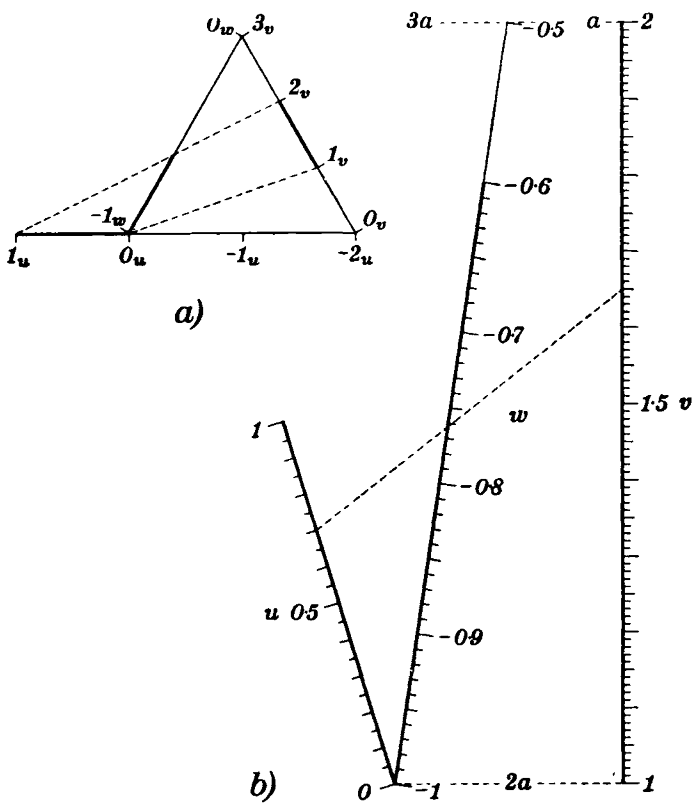

Fig. 63

13.2. By means of a nomogram with three functional scales on the sides of a triangle it is possible to represent any relation of the form

$$f_1(x)\,f_2(y)\,f_3(z) = 1; \tag{13.4}$$

it is sufficient to substitute

$$1-1/u = f_1(x), \quad \text{i.e.} \quad u = 1/[1-f_1(x)],$$
$$1-1/v = f_2(y), \quad \text{i.e.} \quad v = 1/[1-f_2(y)],$$
$$1-1/w = f_3(z), \quad \text{i.e.} \quad w = 1/[1-f_3(z)],$$

to draw a nomogram for the equation

$$(1-1/u)\,(1-1/v)\,(1-1/w) = 1$$

and finally to replace the regular scales of the variables u, v, w by the functional scales $f_1(x)$, $f_2(y)$ and $f_3(z)$.

For equation (13.4) it is also possible to construct a nomogram of the letter N type (§ 12)—by means of other substitutions of course. It must then be assumed that

$$u' = f_1(x), \quad v' = 1/f_2(y) \quad \text{and} \quad w/(a-w) = 1/f_3(z).$$

We can thus suppose that there exists a close connection between the nomograms which we have been considering. Indeed, by a projective transformation of a plane in which a vertex of a triangle becomes a point at infinity we obtain for equation (13.4) a nomogram with two parallel scales.

Example 2. Let us draw a nomogram for the equation

$$z = 0{\cdot}85\, y^x \tag{13.5}$$

for the intervals $0{\cdot}1 \leqslant x \leqslant 1$, $0{\cdot}01 \leqslant y \leqslant 0{\cdot}1$.

Transforming the equation we obtain

$$\log z - \log 0{\cdot}85 = x \log y,$$

$$\frac{1}{x} \cdot \frac{1}{\log y} (\log z - \log 0{\cdot}85) = 1.$$

Substitute

$$1-1/u = 1/x, \quad \text{i.e.} \quad u = x/(x-1),$$
$$1-1/v = 1/\log y, \quad \text{i.e.} \quad v = \log y/(\log y-1),$$
$$1-1/w = \log z - \log 0{\cdot}85, \quad \text{i.e.} \quad w = \frac{1}{1-(\log z - \log 0{\cdot}85)}.$$

Thus if $x \to 1$ then $u \to \infty$; therefore, the regular scale on u, and consequently the projective scales on x, would be unlimited.

Accordingly, let us write the equation in the form

$$\frac{1}{ax}\cdot\frac{1}{b\log y}\,(ab\log z - ab\log 0{\cdot}85) = 1$$

and let us try to find such numbers a and b that the nomogram be contained in a finite domain; let us add the condition that the deformations of the scales of the variables x and y should be as small as possible, i.e. that the values of the unit λ_x (or λ_z) for all x (or z) in the intervals under consideration should differ very little.

Obviously we now have

$$u = \frac{ax}{ax-1}, \qquad v = \frac{b\log y}{b\log y - 1}, \qquad w = \frac{1}{1-ab\,(\log z - \log 0{\cdot}85)}.$$

Let us deal first with the condition that the scale on z should be deformed as little as possible.

As follows from equation (13.5), the function $z/0{\cdot}85$ assumes the least value for $x = 1$ and $y = 0{\cdot}01$ and the greatest value for $x = 0{\cdot}1$ and $y = 0{\cdot}1$ (since the function y_0^x is decreasing for every y_0 from the interval $0{\cdot}01 \leqslant y_0 \leqslant 1$ and the function y^{x_0} is also decreasing for every $x_0 < 1$). We thus have the inequalities

$$y^x < y^{0\cdot 1} < 0{\cdot}1^{0\cdot 1} \qquad \text{and} \qquad y^x > 0{\cdot}01^x > 0{\cdot}01^1,$$

from which

$$0{\cdot}01^1 \leqslant z/0{\cdot}85 \leqslant 0{\cdot}1^{0\cdot 1}, \qquad -2 \leqslant \log z - \log 0{\cdot}85 \leqslant -1/10.$$

Since z varies in the interval from $z_1 = 0{\cdot}0085$ to $z_2 = 0{\cdot}675$, we shall obtain for z a scale similar to a regular scale if we assign the mid-point of the scale to the mean value, i.e. to $z_s = (0{\cdot}675 + 0{\cdot}0085)/2 = 0{\cdot}34175$. Let

$$w_1 = \frac{1}{1-ab\,(\log z_1 - \log 0{\cdot}85)},$$

$$w_2 = \frac{1}{1-ab\,(\log z_2 - \log 0{\cdot}85)},$$

$$w_s = \frac{1}{1-ab\,(\log z_s - \log 0{\cdot}85)};$$

we demand therefore that

$$w_1+w_2 = 2w_s, \quad \text{i.e.} \quad \frac{1}{1+2ab}+\frac{1}{1+ab/10} = \frac{2}{1+0{\cdot}4055\,ab}.$$

Performing the operations and solving the appropriate quadratic equation, we obtain approximately

$$ab = 3.$$

An analogous postulate for the variable x leads, on following the same procedure as for z, to the result

$$a = -0{\cdot}3.$$

We thus obtain ultimately the substitutions

$$u = \frac{0{\cdot}3x}{0{\cdot}3x+1}, \quad v = \frac{10\log y}{1+10\log y}, \quad w = \frac{1}{1-3(\log z - \log 0{\cdot}85)}$$

u, v and w varying in the intervals

$$0{\cdot}029 = \frac{0{\cdot}3 . 0{\cdot}1}{0{\cdot}3\cdot 0{\cdot}1+1} \leqslant u \leqslant \frac{0{\cdot}3 . 1}{0{\cdot}3+1} = 0{\cdot}23,$$

$$1{\cdot}05 = \frac{-2 . 10}{1-2 . 10} \leqslant v \leqslant \frac{-1 . 10}{1-1 . 10} = 1{\cdot}11,$$

$$0{\cdot}14 = \frac{1}{1+3 . 2} \leqslant w \leqslant \frac{1}{1+3 . 1/10} = 0{\cdot}77.$$

Let us make a diagrammatic drawing of our nomogram, marking the intervals for the variables u, v and w. Since the equation is of the form (13.2), the units of the scales are equal to the sides of a triangle (Fig. 64a).

This figure shows that in its final form the nomogram will consist of a part of side u from point 0_u to point $0{\cdot}25_u$, almost the whole side w and a part of the extension of side v from point $1{\cdot}05_v$ to point $1{\cdot}11_v$.

Proceeding to the construction of the nomogram (Fig. 64b) we transform the triangle in an affine manner as follows:

1. We draw an arbitrary regular scale u marking on it points from 0_u to $0{\cdot}25_u$,

2. Selecting an arbitrary point 0_w we draw the scale $0_w 1_w$, taking $1_w = 0_u$,

3. We draw a straight line v through the points 0_w and 1_u, i.e. we draw through 0_w a line parallel to the straight line joining points $0{\cdot}25_u$ and $0{\cdot}75_w$,

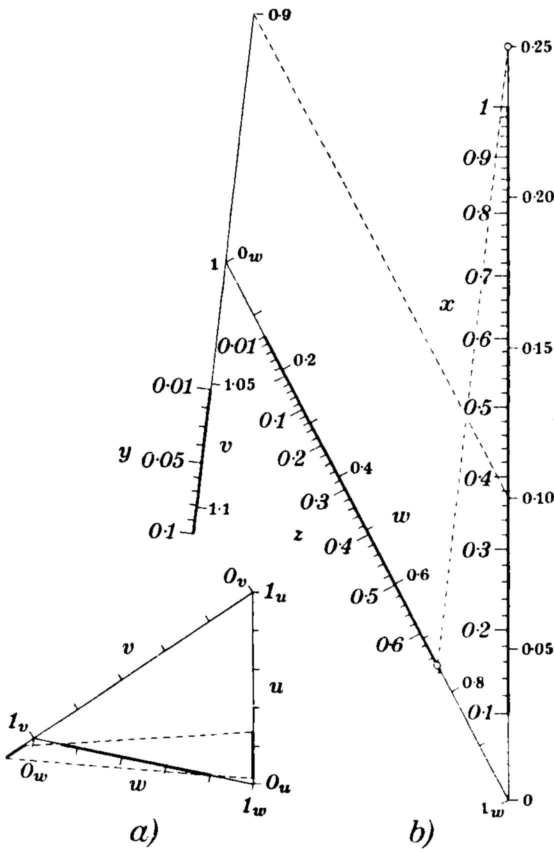

FIG. 64

4. We mark the point $0{\cdot}9_v$ at the intersection of the line v with a line parallel to w and passing through point $0{\cdot}1_u$.

Having obtained a nomogram for u, v and w, we replace the regular scales by the functional scales on the strength of substitutions.

Exercises

Construct nomograms for the equations:

1. $(1-1/u)(1-1/v)(1-1/w) = 1$ for u and v varying in the intervals $0{\cdot}5 \leqslant u \leqslant 0{\cdot}8$, $0{\cdot}7 \leqslant v \leqslant 1$.

2. $(u-1)(v-2)(w-3)-(u-4)(v-5)(w-6) = 0$ for u and v varying in the intervals $2 \leqslant u \leqslant 3$, $0 \leqslant v \leqslant 1$.

3. $\sin x \sin y \sin z + (1-\sin x)(1-\sin y)(1-\sin z) = 0$ for x and y varying in the intervals $0 \leqslant x \leqslant 30°$, $45° \leqslant y \leqslant 90°$.

4. $\tan^2 x \tan^2 y \tan^2 z + 1 = 0$ for x and y varying in the intervals $0 \leqslant x \leqslant 30°$, $60° \leqslant y \leqslant 90°$.

5. $z = 3{\cdot}2\, x^{3y}$ for x and y varying in the intervals $1 \leqslant x \leqslant 1{\cdot}5$, $0 \leqslant y \leqslant 0{\cdot}5$.

§ 14. Nomograms with three rectilinear scales

The nomograms which have been dealt with so far consist of rectilinear scales. By their means we can represent graphically the following relations:

(I) $f_3(z) = f_1(x)+f_2(y)$ (the scales of functions $u = f_1(x)$, $v = f_2(y)$ and $w = f_3(z)$ are parallel),

(II) $1/f_3(z) = 1/f_1(x)+1/f_2(y)$ (the scales of functions $u = f_1(x)$, $v = f_2(y)$ and $w = f_3(z)$ have a point in common),

(III) $f_3(z) = f_1(x)\, f_2(y)$ (the scales of functions $u = f_1(x)$ and $w = f_3(z)$ are parallel; the scale $v = f_2(y)/[a-f_2(y)]$ is situated on a straight line intersecting u and w),

(IV) $f_1(x)\, f_2(y)\, f_3(z) = 1$ (the scales $u = 1/[1-f_1(x)]$, $v = 1/[1-f_2(y)]$ and $w = 1/[1-f_3(z)]$ intersect in pairs at three different ordinary points).

It is not difficult to write the general form of an equation containing there variables which can be represented by means of nomograms with three rectilinear scales.

For this purpose let us take three scales, l_1', l_2' and l_3', on a plane α. If the lines l_1' and l_2' are not perpendicular, we transform the plane α by projection in such a manner as to make the corresponding scales l_1 and l_2 intersect at right angles. Let l_1 be the x-axis and l_2 the y-axis of an orthogonal system (Fig. 65).

The scale of the function $X = f_1(x)$ is marked on the X-axis, and the scale of the function $Y = f_2(y)$ is marked on the Y-axis. The scale l_3 is defined by the equations

$$X = \varphi(z), \qquad Y = \psi(z)$$

and

$$a\varphi(z)+b\psi(z)+c = 0 \tag{14.1}$$

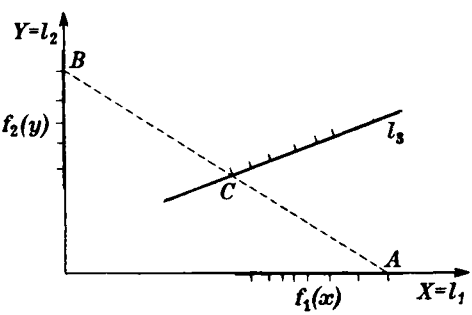

FIG. 65

for every value of z because the points with the coordinates $\varphi(z)$, $\psi(z)$ are situated on a straight line. Now if points A, B and C lie on a straight line, then coordinates X, 0, 0, Y and $\varphi(z)$, $\psi(z)$ must satisfy the equation

$$\begin{vmatrix} X & 0 & 1 \\ 0 & Y & 1 \\ \varphi(z) & \psi(z) & 1 \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} f_1(x) & 0 & 1 \\ 0 & f_2(y) & 1 \\ \varphi(z) & \psi(z) & 1 \end{vmatrix} = 0. \tag{14.2}$$

Three cases must be distinguished here:

1. $c = 0$,
2. $a = 0$ (or $b = 0$),
3. a, b and c are different from 0.

In the first case equation (14.1) gives

$$\psi(z) = k\varphi(z)$$

and consequently equation (14.2) assumes the form

$$\begin{vmatrix} f_1 & 0 & 1 \\ 0 & f_2 & 1 \\ \varphi & k\varphi & 1 \end{vmatrix} = f_1 f_2 - f_2\varphi - kf_1\varphi = 0$$

or

$$1/k\varphi = 1/kf_1 + 1/f_2$$

i.e. form (II).

In the second case, taking for example $a = 0$, we have

$$\psi(z) = m$$

and consequently

$$\begin{vmatrix} f_1 & 0 & 1 \\ 0 & f_2 & 1 \\ \varphi & m & 1 \end{vmatrix} = f_1 f_2 - f_2\varphi - mf_1 = 0$$

or

$$\varphi = \frac{f_1 f_2 - mhf_1}{f_2} = f_1\frac{f_2 - m}{f_2} = f_1 F_2 \quad \text{where} \quad F_2 = \frac{f_2 - m}{f_2}$$

i.e. form (III).

In the third case we find from equation (14.1)

$$\psi(z) = m\varphi(z) + n$$

and substitute this in equation (14.2); we obtain

$$\begin{vmatrix} f_1 & 0 & 1 \\ 0 & f_2 & 1 \\ \varphi & m\varphi + n & 1 \end{vmatrix} = f_1 f_2 - f_2\varphi - mf_1\varphi - nf_1 = 0,$$

which can also be written in the form

$$(mf_1 + n) f_2\varphi = f_1(f_2 - n)(m\varphi + n)$$

or

$$\frac{mf_1 + n}{f_1} \cdot \frac{f_2}{f_2 - n} \cdot \frac{\varphi}{m\varphi + n} - 1 = 0.$$

Assuming

$$\frac{mf_1+n}{f_1} = F_1(x)\,, \quad \frac{f_2-n}{f_2} = F_2(y)\,, \quad \frac{\varphi}{m\varphi+n} = F_3(z)$$

we obtain formula (IV).

We have thus proved that only those relations can be represented by collineation nomograms which can be written in one of the forms (I)–(IV).

This criterion can be expressed in another (equivalent) form.

A necessary and sufficient condition for the equation

$$F(x,\ y,\ z) = 0$$

to be representable by a collineation nomogram is that it should be of the form

$$\begin{vmatrix} \varphi_1(x) & \varphi_2(x) & \varphi_3(x) \\ \psi_1(y) & \psi_2(y) & \psi_3(y) \\ \chi_1(z) & \chi_2(z) & \chi_3(z) \end{vmatrix} = 0, \tag{14.3}$$

in which the functions $\varphi_1(x)$, $\varphi_2(x)$, $\varphi_3(x)$ are linearly dependent, functions $\psi_1(y)$ $\psi_2(y)$, $\psi_3(y)$ are linearly dependent and functions $\chi_1(z)$, $\chi_2(z)$, $\chi_3(z)$ are linearly dependent, i.e. that there should exist constants a_{ik} such that

$$a_{11}\varphi_1(x)+a_{12}\varphi_2(x)+a_{13}\varphi_3(x) = 0 \quad \text{for every } x, \tag{14.4}$$

$$a_{21}\psi_1(y)+a_{22}\psi_2(y)+a_{23}\psi_3(y) = 0 \quad \text{for every } y, \tag{14.5}$$

$$a_{31}\chi_1(z)+a_{32}\chi_2(z)+a_{33}\chi_3(z) = 0 \quad \text{for every } z \tag{14.6}$$

and

$$a_{i1}^2+a_{i2}^2+a_{i3}^2 \neq 0 \quad \text{for} \quad i = 1,\ 2,\ 3,$$

and

$$a_{i1} : a_{i2} : a_{i3} \neq a_{k1} : a_{k2} : a_{k3} \quad \text{for} \quad k \neq i. \tag{14.7}$$

This is obvious because:

1. Satisfying equation (14.4) is a necessary and sufficient condition for a point with homogeneous coordinates

$$x_1 = \varphi_1(x), \quad x_2 = \varphi_2(x), \quad x_3 = \varphi_3(x)$$

to lie on a certain straight line l_1;

2. Satisfying equation (14.5) is a necessary and sufficient condition for a point with homogeneous coordinates

$$y_1 = \psi_1(y), \quad y_2 = \psi_2(y), \quad y_3 = \psi_3(y)$$

to lie on a straight line l_2;

3. Satisfying equation (14.6) is a necessary and sufficient condition for a point with homogeneous coordinates

$$z_1 = \chi_1(z), \quad z_2 = \chi_2(z), \quad z_3 = \chi_3(z)$$

to lie on a straight line l_3.

Condition (14.7) means that l_1, l_2 and l_3 are three different lines.

§ 15. Nomograms with curvilinear scales

15.1. Let the equation

$$x = \varphi(t), \quad y = \psi(t), \quad a \leqslant t \leqslant b \tag{15.1}$$

define functions which assign to every value t_0 of the interval (a, b) a point of a plane with the coordinates $x_0 = \varphi(t_0)$, $y_0 = \psi(t_0)$. If the functions $\varphi(t)$ and $\psi(t)$ are continuous and if for two different values t_1 and t_2 the point $\big(\varphi(t_1), \psi(t_1)\big)$ is always different from the point $\big(\varphi(t_2), \psi(t_2)\big)$, then equations (15.1) represent a certain line L, called an *arc*. The correspondence between the values of the parameter t of the interval (a, b) and the points of the arc is then one-to-one. If the arc is a segment, i.e. if there exist three numbers a, b, c (with $a^2+b^2 > 0$) such that

$$a\varphi(t)+b\psi(t)+c = 0 \quad \text{for every value of } t,$$

then the segment in question can be regarded both as the scale of the function

$$y = -\frac{a\varphi(t)+c}{b} \quad \text{if} \quad b \neq 0$$

and as the scale of the function

$$x = -\frac{b\psi(t)+c}{a} \quad \text{if} \quad a \neq 0.$$

If the arc L is not a segment, we shall call it a *curvilinear scale*. Thus a curvilinear scale is an arc whose points correspond in a one-to-one manner to the values of the parameter t of the interval (a, b) by formulas (15.1).

EXAMPLE 1. Draw a curvilinear scale defined by the equations

$$x = t^2, \qquad y = 3/t^2$$

for t belonging to the interval $(1, 3)$.

Here the arc is a part of the hyperbola $xy = 3$ (Fig. 66), and the coordinates of the points corresponding to the value $t_0 = 1$, $t_1 = 1{\cdot}1, \ldots, t_n = 3$ are obtained by substituting those numbers in formulas $x = t^2$ and $y = 3/t^2$.

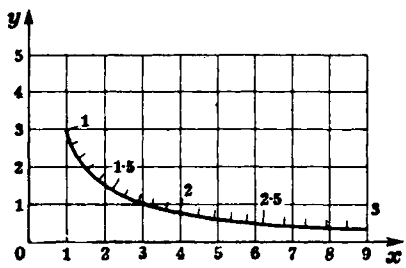

FIG. 66

15.2. Consider three curvilinear scales,

$$\begin{aligned} x_1 &= \varphi_1(u), & y_1 &= \psi_1(u) & &\text{for} & a_1 &\leqslant u \leqslant b_1, \\ x_2 &= \varphi_2(v), & y_2 &= \psi_2(v) & &\text{for} & a_2 &\leqslant v \leqslant b_2, \\ x_3 &= \varphi_3(w), & y_3 &= \psi_3(w) & &\text{for} & a_3 &\leqslant w \leqslant b_3. \end{aligned}$$

The question arises what relation is satisfied by the three numbers u, v and w if the points corresponding to them lie on a straight line.

As we know, the determinant formed from the coordinates of these points must then be equal to zero; we thus have the equation

$$\begin{vmatrix} \varphi_1(u) & \psi_1(u) & 1 \\ \varphi_2(v) & \psi_2(v) & 1 \\ \varphi_3(w) & \psi_3(w) & 1 \end{vmatrix} = 0. \tag{15.2}$$

Conversely, if equation (15.2) is satisfied for three numbers u, v, w, then the corresponding points with the coordinates $\varphi_1(u)$, $\psi_1(u)$, $\varphi_2(v)$, $\psi_2(v)$, $\varphi_3(w)$, $\psi_3(w)$ are collinear (Fig. 67).

A drawing consisting of three curvilinear scales u, v and w is thus, by definition, a nomogram for relation (15.1).

By means of a nomogram of this type we can represent any relation between three variables which can be written in the form of determinant (15.2); it is an essential condition that there should be functions of one variable only in each row of the determinant.

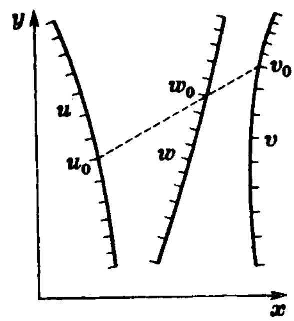

FIG. 67

EXAMPLE 2. Draw a nomogram for the equation

$$\begin{vmatrix} -\sqrt{u} & u & 1 \\ v & -5\log v & 1 \\ 2w & 3/w & 1 \end{vmatrix} = 0 \qquad (15.3)$$

where u and w vary in the intervals $0 \leqslant u \leqslant 9$, $0{\cdot}3 \leqslant w \leqslant 2$.

The nomogram (Fig. 68) consists of three curvilinear scales:

$$x_1 = -\sqrt{u}, \qquad y_1 = u, \qquad \text{(u)}$$

$$x_2 = v, \qquad y_2 = -5\log v, \qquad \text{(v)}$$

$$x_3 = w, \qquad y_3 = 3/w. \qquad \text{(w)}$$

The u-scale is obtained by substituting for u in equations (u) numbers from the interval $(0, 9)$; similarly the curvilinear scale v is obtained by substituting in equations (v) numbers from the interval (0·3, 2).

On the basis of equation (15.3) and of the given limits for u and w it would be possible to find the limits for the variable v; in simple cases, however, it is more profitable, after drawing

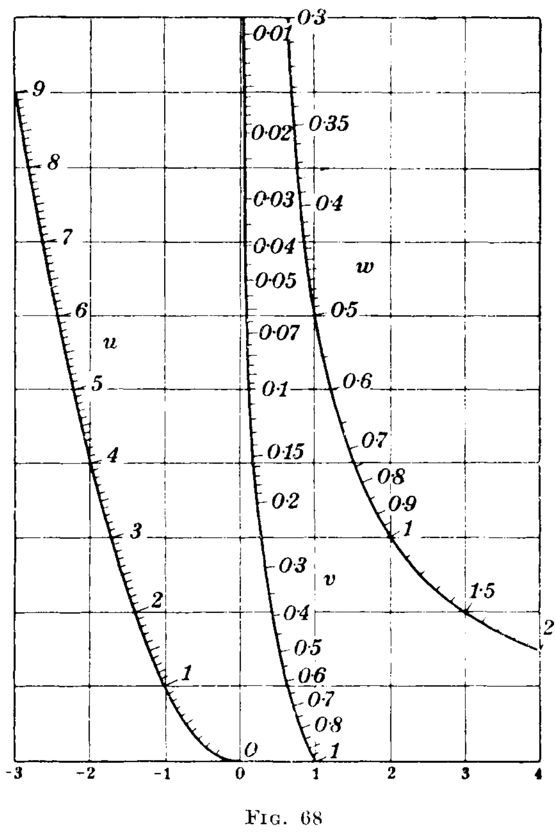

FIG. 68

the curve v, i.e. the graph of the function $y = -5 \log x$, to obtain the limits by joining the end-points of the scales u and w. In our case, proceeding in this way, we obtain for v approximately the interval $0{\cdot}01 \leqslant v \leqslant 0{\cdot}85$.

EXAMPLE 3. Draw a nomogram for the equation

$$\begin{vmatrix} u^2 & 4 & u \\ \sin^2 v & \cos^2 v & \sin v \cos v \\ 0 & 5w^2 & 1 \end{vmatrix} = 0$$

for the intervals $1 \leqslant u \leqslant 2$, $14° \leqslant v \leqslant 26°$.

The result can be obtained through reducing the equation to the essential form by division in such a manner as to have the last column consist of three unities. There are two ways of achieving this:

1. Dividing both sides of the equality by $u \sin v \cos v$ we obtain

$$\begin{vmatrix} u & 4/u & 1 \\ \tan v & \cot v & 1 \\ 0 & 5w^2 & 1 \end{vmatrix} = 0 ;$$

2. Dividing both sides of the equality by $20\, w^2 \cos^2 v$ and interchanging column two with column three, we have

$$\begin{vmatrix} u^2/4 & u/4 & 1 \\ \tan^2 v & \tan v & 1 \\ 0 & 1/5w^2 & 1 \end{vmatrix} = 0 .$$

In the first case the equations of the curvilinear scales have the form

$$x_1 = u , \qquad y_1 = 4/u ,$$

$$x_2 = \tan v , \qquad y_2 = \cot v ,$$

$$x_3 = 0 , \qquad y_3 = 5w^2 .$$

Proceeding to the execution of a nomogram for the equation in form 1. we determine the curve (u) for the values from 1 to 2; we obtain an arc AB of a hyperbola (Fig. 69).

The v-scale also lies on a hyperbola because

$$x_2 y_2 = \tan v \cot v = 1 ,$$

and for the angles from 14° to 26° we have

$$0{\cdot}2493 \leqslant x_2 \leqslant 0{\cdot}4877 , \qquad 4{\cdot}011 \leqslant y_2 \leqslant 2{\cdot}050 .$$

We thus have an arc CD of the hyperbola $xy = 1$.

The scale of the variable w lies on the y-axis because $x_3 = 0$. In order to find the limits let us observe that every straight line joining a certain point of the arc AB with a certain point of the arc CD should hit the w-scale. Thus joining point A with point D and point B with point C we obtain points E and F which

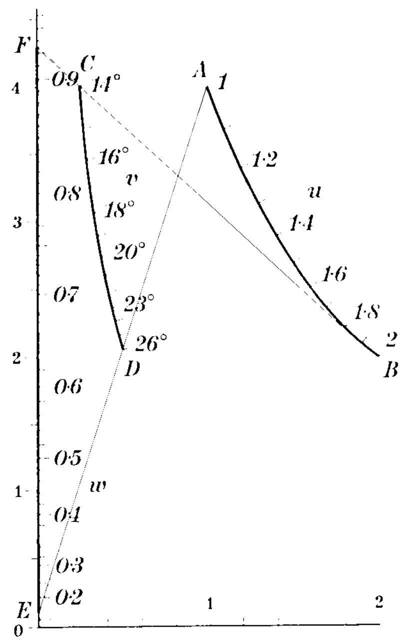

FIG. 69

are end-points of the v-scale. Point E is seen to be near the origin of the system and point F has an ordinate of about 4·3. We thus have $0 \leqslant 5w^2 \leqslant 4{\cdot}5$, i.e. $0 \leqslant w \leqslant 0{\cdot}927$.

Let us construct one more nomogram, when the given equation is of form 2. We obtain the following scale equations:

$$\bar{x}_1 = u^2/4, \quad \bar{y}_1 = u/4, \quad \bar{x}_2 = \tan^2 v, \quad \bar{y}_2 = \tan v,$$

$$\bar{x}_3 = 0, \quad \bar{y}_3 = 1/5w^2.$$

The u-scale lies on the parabola

$$\bar{x}_1 = 4\bar{y}_1^2,$$

the v-scale lies on the parabola

$$\bar{x}_2 = \bar{y}_2^2,$$

and the w-scale is a part of the y-axis (Fig. 70).

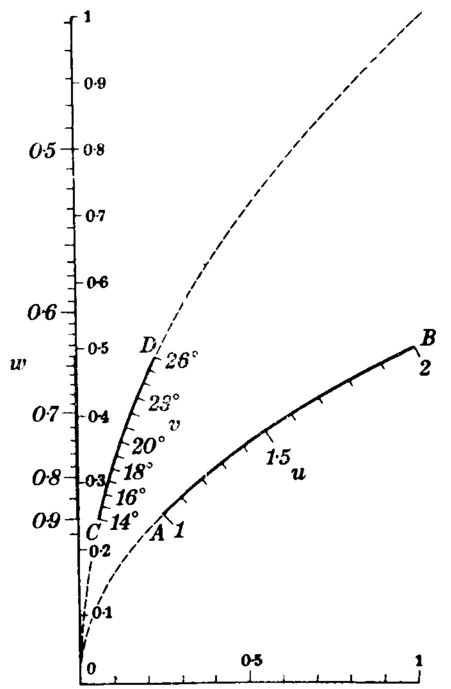

FIG. 70

On drawing the straight lines BC and AD it will be observed that the w-scale has extended to infinity in the positive direction of the y-axis from point E, whose ordinate is about 0·24.

EXAMPLE 4. Draw a nomogram for the equation of the second degree

$$w^2+uw+v = 0 \qquad (15.4)$$

if the coefficients u and v vary in the intervals

$$0 \leqslant u \leqslant 3, \quad -2 \leqslant v \leqslant 0.$$

It will be observed that equation (15.4) can be written in the form

$$\begin{vmatrix} -u & 1 & 1 \\ v & 0 & 1 \\ w^2 & w & w-1 \end{vmatrix} = 0.$$

Let us divide both sides of this equation by $w-1$: we shall obtain

$$\begin{vmatrix} -u & 1 & 1 \\ v & 0 & 1 \\ w^2/(w-1) & w/(w-1) & 1 \end{vmatrix} = 0,$$

i.e. the scale equations are

$$x_1 = -u, \qquad y_1 = 1, \tag{u}$$

$$x_2 = v, \qquad y_2 = 0, \tag{v}$$

$$x_3 = w^2/(w-1), \qquad y_3 = w/(w-1), \tag{w}$$

where (u) is a regular scale on the straight line $y = 1$, (v) is a regular scale on the y-axis, and (w) is a curvilinear scale on a hyperbola, since by eliminating w from equations (w) we obtain in succession

$$\frac{x}{y} = w \quad \text{and} \quad y = \frac{x/y}{x/y-1} = \frac{x}{x-y},$$

or also

$$x = \frac{1}{y-1} + 1 + y.$$

The equation

$$x = y+1+1/(y-1)$$

represents, as we know, a hyperbola with the asymptotes $x = y+1$ and $y - 1 = 0$.

Drawing the functional scales (u), (v) and (w) we obtain a nomogram (Fig. 71) which permits us to determine with a high degree of accuracy the positive root w_1 of an equation of the second degree. The other root, w_2, can be obtained with the same accuracy from the well-known formula $w_1+w_2 = u$.

Changing the units on the x-axis we can construct a nomogram for a wider range of coefficients u and v (Fig. 72).

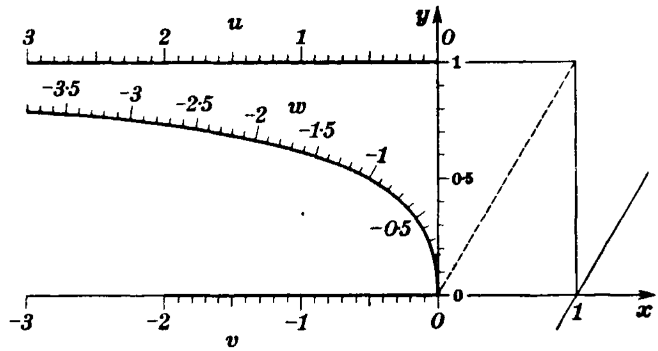

Fig. 71

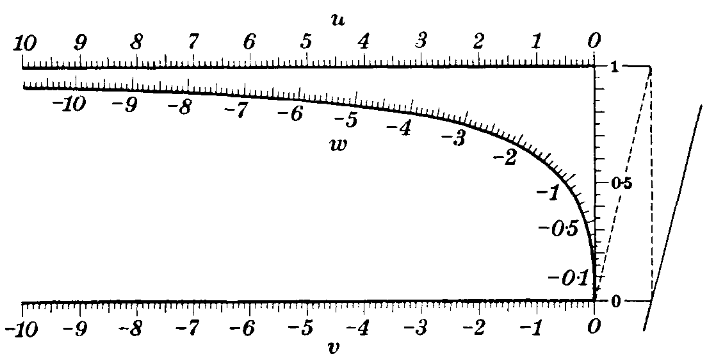

Fig. 72

15.3. Reducing an equation to form (15.2) is in many cases a difficult task; we deal with it in the last chapter of this textbook. The difficulty lies in the circumstance that in each row of the determinant appearing on the left side of equation (15.2) only one variable occurs. In practice this problem is usually solved in two stages. We first try to reduce the equation to one of the following canonical forms:

$$f_1(x)\,g_3(z)+f_2(y)\,h_3(z)+1 = 0 \quad \text{(the Cauchy equation)},$$

$$f_1(x)\,f_2(y)\,g_3(z)+[f_1(x)+f_2(y)]\,h_3(z)+1 = 0 \quad \text{(the Clark equation)},$$

$$f_1(x) = \frac{f_2(y)+f_3(z)}{g_2(y)+g_3(z)} \quad \text{(the Soreau equation I)},$$

$$\frac{f_1(x)+f_2(y)}{g_1(x)+g_2(y)} = \frac{f_1(x)+f_3(z)}{g_1(x)+g_3(z)} \quad \text{(the Soreau equation II)}.$$

Secondly we reduce each of these equations to form (15.2). The manner of doing this will be discussed in § 24.

Exercises

1. On the basis of Example 4 give the determinant form and draw a nomogram for the equation $s = rw\pi + 2r^2\pi$ for the intervals $0 \leqslant r \leqslant 10$, $0 \leqslant w \leqslant 20$.

2. Verify whether the equation $w = uv$ can be written in the form

$$\begin{vmatrix} 1 & 0 & -u \\ 0 & v & v-1 \\ -1 & w & w \end{vmatrix} = 0$$

and draw nomograms for the equations

a. $$V = 4a^2b\pi/3$$

for the intervals $0 \leqslant a \leqslant 5$, $0 \leqslant b \leqslant 20$,

b. $$\gamma = \frac{1 \cdot 293}{760} H \frac{273}{273+t}$$

for the intervals $0 \leqslant t \leqslant 35$, $630 \leqslant H \leqslant 800$,

c. $$v = c\sqrt{2gz} \quad (g = 9 \cdot 81)$$

for the intervals $0 \cdot 1 \leqslant z \leqslant 1 \cdot 5$, $0 \cdot 85 \leqslant c \leqslant 0 \cdot 97$,

d. $$J = V^2/R$$

for the intervals $10 \leqslant R \leqslant 100$, $110 \leqslant V \leqslant 220$,

e. $$B = ab^3/12$$

for the intervals $0 \leqslant a \leqslant 100$, $0 \leqslant b \leqslant 5$,

f. $$u+v = u/w$$

for the intervals $0 \leqslant u \leqslant 6$, $0 \leqslant v \leqslant 30$,

g. $$f = \frac{3 \cdot 3}{1+0 \cdot 1\, L^2/r^2}$$

for the intervals $0 \leqslant r \leqslant 5$, $40 \leqslant L \leqslant 50$.

3. Verify whether the equation $w = uv$ can be written in the form

$$\frac{1}{u-v}\begin{vmatrix} u & 1 & 1+u^2 \\ v & 1 & 1+v^2 \\ 0 & 1 & 1-w \end{vmatrix} = 0$$

and draw a nomogram for the intervals $0 \leqslant u \leqslant 5$, $-5 \leqslant v \leqslant 0$.

4. Verify whether the equation $w = u+v$ can be written in the form

$$\begin{vmatrix} 1+u^2 & u & u^2 \\ 1+v^2 & v & v^2 \\ w & 1 & w \end{vmatrix} = 0$$

and draw a nomogram of the equation

$$\log B = \log a + 3\log b - \log 12$$

for the intervals $0 \leqslant a \leqslant 10$, $10 \leqslant b \leqslant 100$.

§ 16. The Cauchy equation

16.1. The *Cauchy equation*

$$f_1(u)\,g_3(w) + f_2(v)\,h_3(w) + 1 = 0$$

can be reduced to form (15.2) by the identity

$$\begin{vmatrix} 1 & 0 & -f_1 \\ 0 & 1 & -f_2 \\ g_3 & h_3 & 1 \end{vmatrix} = 1 + f_1 g_3 + f_2 h_3. \tag{16.1}$$

If $f_1(u) \neq 0$ for u belonging to the interval under consideration and $f_2(v) \neq 0$ for v belonging to the variability interval of the second variable, then dividing both sides of the equation by the product $f_1(u) f_2(v)$ we have

$$\begin{vmatrix} -1/f_1 & 0 & 1 \\ 0 & -1/f_2 & 1 \\ g_3 & h_3 & 1 \end{vmatrix} = 0. \tag{16.2}$$

The nomogram consists of three scales:

1. A rectilinear scale on the x-axis —

$$x_1 = -1/f_1(u), \qquad y_1 = 0,$$

2. A rectilinear scale on the y-axis —

$$x_2 = 0, \qquad y_2 = -1/f_2(v),$$

3. Generally a curvilinear scale defined by the parametric equation

$$x_3 = g_3(w), \qquad y_3 = h_3(w).$$

If functions $f_1(u)$ and $f_2(v)$ had a zero place in the intervals in question or in their vicinity, another procedure could be followed:

Let a and b be arbitrary numbers different from 0 and satisfying the condition that the function $a g_3(w) + b h_3(w) = f_3(w)$

should be different from 0 in the whole variability interval of the variable w. Let us multiply the terms of the first column of determinant (16.1) by a and the terms of the second column by b, and let us then add them; we obtain the equation

$$\begin{vmatrix} a & 0 & -f_1 \\ b & b & -f_2 \\ ag+bh & bh & 1 \end{vmatrix} = 0.$$

Dividing both sides of this equation by $ab(ag_3+bh_3) \neq 0$ and interchanging the first and the third column we obtain the equivalent equation

$$\begin{vmatrix} f_1/a & 0 & 1 \\ -f_2/b & 1 & 1 \\ 1/(ag+bh) & bh/(ag+bh) & 1 \end{vmatrix} = 0. \tag{16.3}$$

On the basis of this equation we can construct a new nomogram for the same equation (16.1), in which the scales will have the equations

$$x_1 = -f_1(u)/a \qquad y_1 = 0, \tag{u}$$

$$x_2 = -f_2(v)/b, \qquad y_2 = 1, \tag{v}$$

$$x_3 = \frac{1}{ag_3(w)+bh_3(w)}, \qquad y_3 = \frac{bh_3(w)}{ag_3(w)+bh_3(w)}. \tag{w}$$

The first two scales lie on parallel lines, the third is generally curvilinear.

16.2. The method described above can be generalized:

Let $\mathfrak{A}$ be an arbitrary non-singular matrix (Chapter I, § 4)

$$\mathfrak{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad (|\mathfrak{A}| \neq 0)$$

and, for brevity, let $\mathfrak{X}$ denote the matrix of three homogeneous coordinates of three points on a plane

$$\mathfrak{X} = \begin{bmatrix} \xi_{11} & \xi_{12} & \xi_{13} \\ \xi_{21} & \xi_{22} & \xi_{23} \\ \xi_{31} & \xi_{32} & \xi_{33} \end{bmatrix}. \tag{16.4}$$

Multiply matrices $\mathfrak{X}$ and $\mathfrak{A}$:

$$\mathfrak{X}\mathfrak{A} = \mathfrak{Y}. \tag{16.5}$$

The product $\mathfrak{Y}$ is a singular matrix if and only if $\mathfrak{X}$ is a singular matrix, i.e. if the points $A_1(\xi_{11}, \xi_{12}, \xi_{13})$, $A_2(\xi_{21}, \xi_{22}, \xi_{23})$, $A_3(\xi_{31}, \xi_{32}, \xi_{33})$ are collinear. The terms of matrix $\mathfrak{Y}$

$$\eta_{11} = a_{11}\xi_{11}+a_{21}\xi_{12}+a_{31}\xi_{13}, \qquad \eta_{12} = a_{12}\xi_{11}+a_{22}\xi_{12}+a_{32}\xi_{13},$$

$$\eta_{21} = a_{11}\xi_{21}+a_{21}\xi_{22}+a_{31}\xi_{23}, \qquad \eta_{22} = a_{12}\xi_{21}+a_{22}\xi_{22}+a_{32}\xi_{23},$$

$$\eta_{31} = a_{11}\xi_{31}+a_{21}\xi_{32}+a_{31}\xi_{33}, \qquad \eta_{32} = a_{12}\xi_{31}+a_{22}\xi_{32}+a_{32}\xi_{33},$$

$$\eta_{13} = a_{13}\xi_{11}+a_{23}\xi_{12}+a_{33}\xi_{13},$$

$$\eta_{23} = a_{13}\xi_{21}+a_{23}\xi_{22}+a_{33}\xi_{23},$$

$$\eta_{33} = a_{13}\xi_{31}+a_{23}\xi_{32}+a_{33}\xi_{33},$$

are homogeneous coordinates of the points $B_1(\eta_{11}, \eta_{12}, \eta_{13})$, $B_2(\eta_{21}, \eta_{22}, \eta_{23})$, $B_3(\eta_{31}, \eta_{32}, \eta_{33})$, which, as we know (Chapter I, §4), correspond to the points A_1, A_2, A_3 in the projective transformation defined by matrix $\mathfrak{A}$, i.e. by the equations

$$\begin{aligned} y_1 &= a_{11}x_1+a_{21}x_2+a_{31}x_3, \\ y_2 &= a_{12}x_1+a_{22}x_2+a_{32}x_3, \\ y_3 &= a_{13}x_1+a_{23}x_2+a_{33}x_3. \end{aligned} \tag{16.6}$$

Substituting in (16.5), instead of an arbitrary matrix $\mathfrak{X}$, a matrix consisting of the terms of determinant (16.1), we shall obtain matrix $\mathfrak{Y}$, which is singular if and only if $\mathfrak{X}$ is a singular matrix. Thus, instead of the equation $\mathfrak{X} = 0$, we shall have an equivalent equation, $\mathfrak{Y} = 0$.

The procedure described above is of fundamental importance for nomography; as can be seen from the examples given in the preceding sections, a nomogram obtained through the direct application of certain rules very often has a geometrical form which is unsuitable for practical use: for the accuracy at the various points of the drawing often differs widely, and in order to ensure the required accuracy we should have to enlarge the drawing so as to obtain suitable dimensions at the least accurate spot. The total dimensions of the drawing might then prove too large.

Through a suitable selection of matrix $\mathfrak{A}$ we shall try to transform the nomogram by projection in such a way as to increase the accuracy where it is too small and at the same time to reduce it where it is too great in the original drawing.

We shall explain the procedure by means of examples.

EXAMPLE 1. Construct a nomogram for the equation of the third degree

$$z^3+az+b = 0$$

where the coefficients a and b satisfy the inequalities $-1 \leqslant a \leqslant 0$, $1 \leqslant b \leqslant 10$.

a. Dividing both sides of the equation by z^3,

$$a/z^2+b/z^3+1 = 0$$

we can see that it is of the Cauchy form. Using transformation (16.1) we can write this equation in the form of a determinant,

$$\begin{vmatrix} 1 & 0 & -a \\ 0 & 1 & -b \\ 1/z^2 & 1/z^3 & 1 \end{vmatrix} = 0 \tag{16.7}$$

or

$$\begin{vmatrix} -1/a & 0 & 1 \\ 0 & -1/b & 1 \\ 1/z^2 & 1/z^3 & 1 \end{vmatrix} = 0.$$

The scales of the variables a, b and z are defined by the equations

$$x_1 = -1/a, \quad y_1 = 0,$$
$$x_2 = 0, \quad y_2 = -1/b,$$
$$x_3 = 1/z^2, \quad y_3 = 1/z^3.$$

Basing ourselves on these equations we outline a nomogram, marking on it the variability intervals of a, b, and z (Fig. 73).

b. This nomogram gives a too great accuracy for large values of b and has unlimited dimensions since the zero point of the a-scale is at infinity. It is thus necessary to extend the neighbourhood of the side DA and to reduce considerably the sides CD and AB by reducing the point at infinity B^∞ to an ordinary point.

In order to make the required deformations of the drawing, we shall transform the plane by projection in such a manner as to change the quadrilateral $ABCD$ (with one vertex at infinity) into a rectangle $A'B'C'D'$; then of course the opposite sides, i.e. the lines $A'B'$ and $C'D'$ on one hand and $B'C'$ and $A'D'$ on the other hand, would have to be pairwise parallel. This means that the point of intersection P of the sides AB^{∞} and CD would become a point at infinity and the point of intersection Q of the

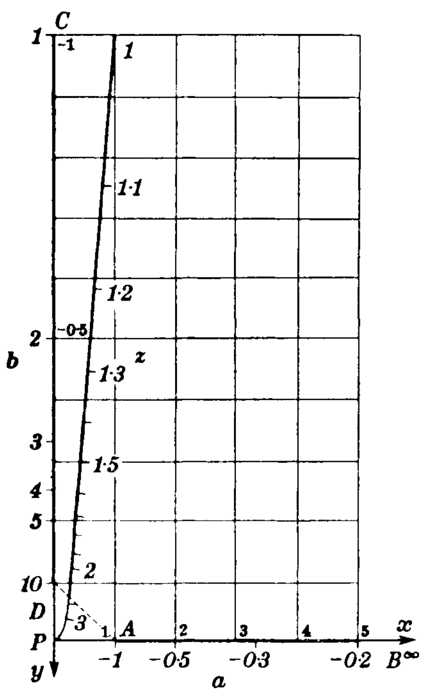

FIG. 73

opposite sides AD and CD would also become a point at infinity. Let those points be points on the axes of coordinates and let point A' be the origin of the system on the new plane.

The coordinates of point P are 0, 0. The coordinates of point Q will be obtained by solving the system of equations of the straight lines AD and $B^{\infty}C$,

$$x/1+y/(-0{\cdot}1) = 1, \qquad y = -1,$$

whence $x = -9$, $y = -1$.

We are thus to find a projective transformation which assigns

point $X'^{\infty}(1, 0, 0)$ to point $Q(-9, -1, 1)$,
point $Y'^{\infty}(0, 1, 0)$ to point $P(0, 0, 1)$,
point $O'(0, 0, 1)$ to point $A(0, -1, 1)$.

Let us write this in the form of a matrix:

$$\mathfrak{X} = \begin{bmatrix} -9 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \quad \text{is to correspond to} \quad \mathfrak{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus the unknown in our calculation is the matrix $\mathfrak{A} = [a_{ik}]$ satisfying the condition

$$\begin{bmatrix} -9 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since we have a unit matrix on the right side, we can, by using a well-known formula (see Chapter 1, § 4), represent the elements a_{ik} in the form

$$a_{ik} = \frac{X_{ki}}{X};$$

X_{ki} denotes here a minor of the determinant $|\mathfrak{X}|$ corresponding to the term which is found in place k, i (¹), and X denotes the numerical value of the determinant $|\mathfrak{X}|$.

Putting the common factor $1/X$ before the symbol of the matrix, we obtain the equalities

$$\mathfrak{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \frac{1}{X} \begin{bmatrix} \begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix} & -\begin{vmatrix} -1 & 1 \\ -1 & 1 \end{vmatrix} & \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} \\ -\begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} -9 & 1 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} -9 & 1 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} 0 & 0 \\ 0 & -1 \end{vmatrix} & -\begin{vmatrix} -9 & -1 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} -9 & -1 \\ 0 & 0 \end{vmatrix} \end{bmatrix} = \frac{1}{X} \begin{bmatrix} 1 & 0 & -1 \\ 0 & -9 & 9 \\ 0 & -9 & 0 \end{bmatrix}.$$

(¹) It should be noted that the order of the indices i, k in the terms a_{ik} and X_{ki} is reversed.

The common factor $1/X$ plays no part in our calculations since it appears in all the terms a_{ik} and on multiplying $\mathfrak{X}\mathfrak{A}$ it will pass into all the terms of the product, i.e. into all the triples of homogeneous coordinates. Disregarding this factor, let us write our projective transformation, which turns points P, Q and A into points P'^{∞}, Q'^{∞} and A':

$$\begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & -9 & 9 \\ 0 & -9 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} x_{11} & -9x_{12}-9x_{13} & -x_{11}+9x_{12} \\ x_{21} & -9x_{22}-9x_{23} & -x_{21}+9x_{22} \\ x_{31} & -9x_{32}-9x_{33} & -x_{31}+9x_{32} \end{bmatrix}.$$

Substituting for matrix $\mathfrak{X}$ a matrix formed of the terms of determinant (16.7) we finally obtain

$$\begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{bmatrix} = \begin{bmatrix} 1 & 9a & -1 \\ 0 & 9b-9 & 9 \\ z^{-2} & -9z^{-3}-9 & -z^{-2}+9z^{-3} \end{bmatrix}.$$

Returning to the non-homogeneous coordinates we obtain the following scale equations:

$$\xi_a = y_{11}/y_{13} = -1,$$

$$\eta_a = y_{12}/y_{13} = -9a,$$

$$\xi_b = y_{21}/y_{23} = 0,$$

$$\eta_b = y_{22}/y_{23} = b-1,$$

$$\xi_z = y_{31}/y_{33} = z/(9-z),$$

$$\eta_z = y_{32}/y_{33} = (9z^3+9)/(z-9).$$

We have obtained for the variables a and b regular scales on parallel lines, the end-points of our scales forming the four vertices of a rectangle (Fig. 74); the curvilinear scale z is contained between the scales a and b.

In order to make the drawing easier to read, let us mutliply the abscissas by -5 and the ordinates by $10/9$:

$$\xi'_a = 5, \qquad \eta'_a = -10a,$$
$$\xi'_b = 0, \qquad \eta'_b = 10(b-1)/9,$$
$$\xi'_z = 5z/(z-9), \qquad \eta'_z = 10(z^3+1)/(z-9).$$

The nomogram defined by these equations is shown in Fig. 74.

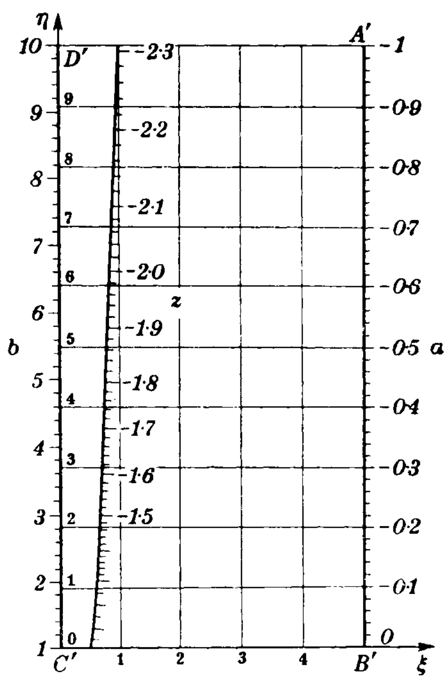

FIG. 74

EXAMPLE 2. Draw a nomogram for the relation

$$T^{\infty} = \frac{T_1^2}{2T_1 - T_2} \tag{16.8}$$

where T_1 and T_2 vary in the interval from 0° to 30°, and we always have $T_1 < T_2$.

These conditions do not define the variability limits of T^{∞}; let us assume that $10 \leqslant T^{\infty} \leqslant 100$.

We already considered this equation in § 12, Example 1 (Fig. 58), where we assumed reading the rise of temperatures T_1 and T_2 and finding (possibly by means of a slide-rule) the ratio T_2/T_1.

a. Writing equation (16.8) in the form

$$-T_2\frac{1}{2T_1}-\frac{1}{2T^{\infty}}T_1+1 = 0$$

we can see that it is of the Cauchy type. It can thus be expressed by means of a determinant in the following way:

$$\begin{vmatrix} 1 & 0 & T_2 \\ 0 & 1 & 1/2T^{\infty} \\ 1/2T_1 & T_1 & 1 \end{vmatrix} = 0. \qquad (16.9)$$

We thus have, on dividing by $T_2/2T^{\infty}$, the scale equations

$$\xi_1 = 1/T_2, \quad \eta_1 = 0,$$
$$\xi_2 = 0, \quad \eta_2 = 2T^{\infty},$$
$$\xi_3 = 1/2T_1, \quad \eta_3 = T_1.$$

Taking into consideration the given variability intervals of T_1 and T_2 we shall have a nomogram shown in outline in Fig. 75 (since the relevant points of the T_2-scale lie on the ξ-axis on the segment from 0·02 to 0·1 and the points of the T^{∞}-scale lie on the η-axis on the segment from 20 to 200, for the sake of clarity we have taken on the ξ-axis a unit that is 1000 times as large as the unit on the η-axis).

b. This nomogram is not suitable because it has unlimited dimensions; the T_2-scale has its zero point in infinity. Moreover, the units of the scales T_1 and T_2 have large variations in the interval from 0° to 30°.

Let us turn the quadrilateral $AB^{\infty}C^{\infty}D$ into a rectangle. To do this we must find the coordinates of the diagonal points of our quadrilateral. One of the diagonal points is the origin of the system as the intersection point of the sides AB^{∞} and $C^{\infty}D$. The second point is the intersection point of the sides AD and $B^{\infty}C^{\infty}$, i.e. the point at infinity of the straight line CD. Let us write the equation of this line: since point A has coordinates

1/30, 0 and point D has coordinates 0, 20, the equation of the line is

$$30x+y/20 = 1,$$

or in the homogeneous form

$$30x_1+x_2/20-x_3 = 0.$$

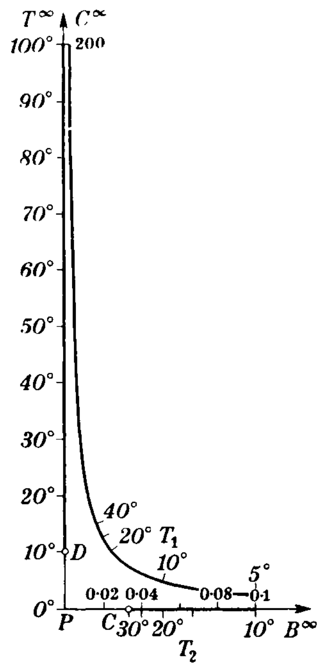

FIG. 75

Since we have $x_3 = 0$ for the point at infinity, let us assume

$$x_1 = 1, \quad x_2 = -600.$$

Therefore we look for a projective transformation that will assign

point (1, 0, 0) to point $Q(1, -600, 0)$

point (0, 1, 0) to point $P(0, 0, 1)$,

point (0, 0, 1) to point $C^\infty(0, 1, 0)$.

The matrix of our transformation should satisfy the equation

$$\begin{bmatrix} 1 & -600 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \mathfrak{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

As before, we have

$$\mathfrak{A} = \begin{bmatrix} \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} -600 & 0 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} -600 & 0 \\ 0 & 1 \end{vmatrix} \\ -\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & -600 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & -600 \\ 0 & 0 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -1 & 0 & -600 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}.$$

The equation of the transformed nomogram is of the form

$$\begin{bmatrix} 1 & 0 & T_2 \\ 0 & 2T^{\infty} & 1 \\ 1 & 2T_1^2 & 2T_1 \end{bmatrix} \begin{bmatrix} -1 & 0 & -600 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -T_2 & -600 \\ 0 & -1 & -2T^{\infty} \\ -1 & -2T_1 & -600-2T_1^2 \end{bmatrix}$$

and consequently the scale equations are

$$\xi_1' = 1/600, \qquad \eta_1' = T_2/600,$$
$$\xi_2' = 0, \qquad \eta_2' = 1/2T^{\infty},$$
$$\xi_3' = 1/(2T_1^2+600), \qquad \eta_3' = T_1/(T_1^2+300).$$

The first two scales lie on parallel lines and the third on an ellipse since from the equations with parameter T_1 we have successively

$$\eta = 2T_1\xi, \qquad T_1 = \frac{\eta}{2\xi}, \qquad \xi = \frac{1}{2(\eta/2\xi)^2+600} = \frac{2\xi^2}{\eta^2+1200\,\xi^2},$$

$$\eta^2+1200\xi^2 = 2\xi, \qquad 1200(\xi-1/1200)^2+\eta^2 = 1/1200.$$

We can see that the scales T_2 and T^{∞} lie on tangents to the ellipse. Les us divide the ordinates by $\sqrt{1200}$ in order to obtain a circle instead of an ellipse and then let us multiply all the coordinates by 600:

$$\xi_{\mathrm{I}} = 1, \qquad \eta_{\mathrm{I}} = T_2/20\sqrt{3},$$
$$\xi_{\mathrm{II}} = 0, \qquad \eta_{\mathrm{II}} = 5\sqrt{3}/T^{\infty},$$

$$\xi_{III} = 600/(2T_1^2+600), \qquad \eta_{III} = 10\sqrt{3}\,T_1/(T_1^2+300).$$

Fig. 76 shows the ultimate shape of the nomogram.

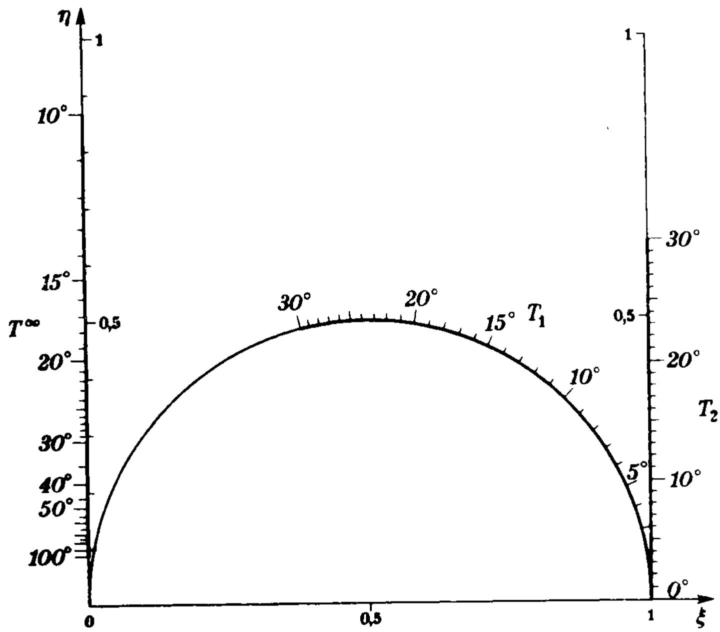

FIG. 76

16.3. The procedure described here does not comprise cases where the smallest convex multilateral containing the nomogram is a triangle. In those cases the ultimate form is obtained by writing the formulas of projective transformation of a plane in the form

$$\xi = \frac{a_1x+a_2y+a_3}{c_1x+c_2y+c_3}, \qquad \eta = \frac{b_1x+b_2y+b_3}{c_1x+c_2y+c_3},$$

where $c_1x+c_2y+c_3$ is the left side of the general equation of the straight line which is transferred to infinity, $a_1x+a_2y+a_3$ is the left side of the general equation of the straight line which is transformed into the straight line $\xi = 0$, and $b_1x+b_2y+b_3$ is the left side of the general equation of the straight line which is transferred upon the axis $\eta = 0$.

EXAMPLE 3. Draw a nomogram for the equation

$$w = vu^v$$

where u and v vary in the interval from 0 to 1.

By logging we obtain $\log w - \log v - v \log u = 0$. This is an equation of the Cauchy type which can be written by means of a determinant

$$\begin{vmatrix} 1 & 0 & -\log w \\ 0 & 1 & -\log u \\ 1 & -v & -\log v \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 0 & -\log w & 1 \\ 1 & -\log u & 1 \\ -v & -\log v & 1-v \end{vmatrix} = 0.$$

The latter form implies the following scale equations:

$$\begin{aligned} x_w &= 0, & x_u &= 1, & x_v &= v/(v-1), \\ y_w &= -\log w, & y_u &= -\log u, & y_v &= \log v/(v-1). \end{aligned} \tag{16.10}$$

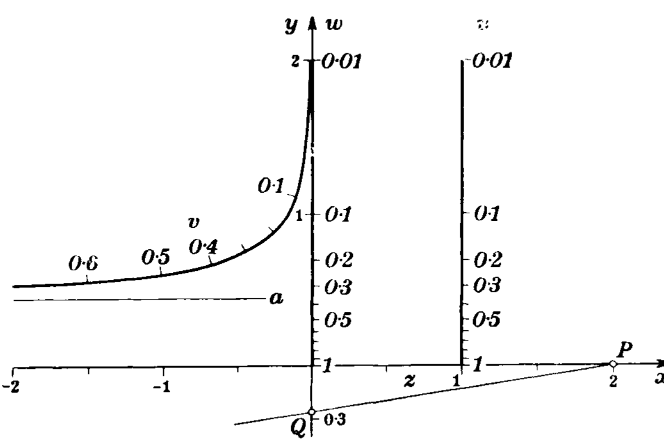

FIG. 77

As can be seen in Fig. 77, the nomogram lies in a triangle with an ordinary vertex (1, 0, 1) and two vertices at infinity (0, 1, 0) and (1, 0, 0).

The curve v has two asymptotes: the y-axis and a straight line a parallel to the x-axis; the equation of the line a is obtained by finding the limit of the expression y_v for $v \to 1$. By the L'Hospital rule we have

$$\lim_{v\to 1}\frac{\log_{10} v}{v-1} = \lim_{v\to 1}\frac{\log_e v}{v-1}\log_{10} e = 0{\cdot}4343 : \lim_{v\to 1}\frac{1/v}{1} = 0{\cdot}4343.$$

We shall now convert our triangle into a finite-dimensional triangle in such a way as to have 1. the origin of the w-scale, i.e. point 1_w, at the mid-point of the segment with end-points 1_u and 1_v and 2. point $0{\cdot}5_w$ at the mid-point of the transformed w-scale. The fulfilment of condition 2. will make the w-scale similar to a regular scale.

Condition 1. will be satisfied if point $P(2, 0, 1)$, forming a harmonic four with points $1_w, 1_u, 1_v$, turns into a point at infinity. Similarly, condition 2. will be satisfied if point $Q(0, \log 0{\cdot}5)$ on the y-axis, forming a harmonic four with points $0{\cdot}5_w$, 1_w and 0_w, turns into a point at infinity. It can thus be seen that the straight line joining points (2, 0) and (0, log 0·5) must be transferred to infinity. Replacing log 0·5 by number $-0{\cdot}3$ we obtain the equation of the straight line PQ:

$$3x-20y-6 = 0.$$

Leaving the w-scale on the y-axis and the line $1_u 1_v 1_w$ on the x-axis, we obtain a projective transformation in the form

$$\xi = \frac{x}{3x-20y-6}, \qquad \eta = \frac{y}{3x-20y-6}. \tag{16.11}$$

Substituting into the right sides of these equations expressions (16.10) defining the scales w, u and v in the original system of the axes x and y, we finally obtain the equations

of the w-scale: $\quad \xi_w = 0, \qquad \eta_w = \dfrac{-\log w}{20\log w-6},$

of the u-scale: $\quad \xi_u = \dfrac{1}{20\log u-3}, \qquad \eta_u = \dfrac{-\log u}{20\log u-3},$

of the v-scale: $\quad \xi_v = \dfrac{-v}{2v+20\log v-6}, \qquad \eta_v = \dfrac{-\log v}{3v+20\log v-6}.$

The nomogram defined by these equations is shown in Fig. 78. As we know, the asymptote a with the equation $y = \log e$ has turned into a tangent a' to the curve v; the equation of the tangent a' will be obtained by (16.11) in a parametric form:

$$\xi = x/(3x-20\log e-6), \qquad \eta = \log e/(3x-20\log e-6).$$

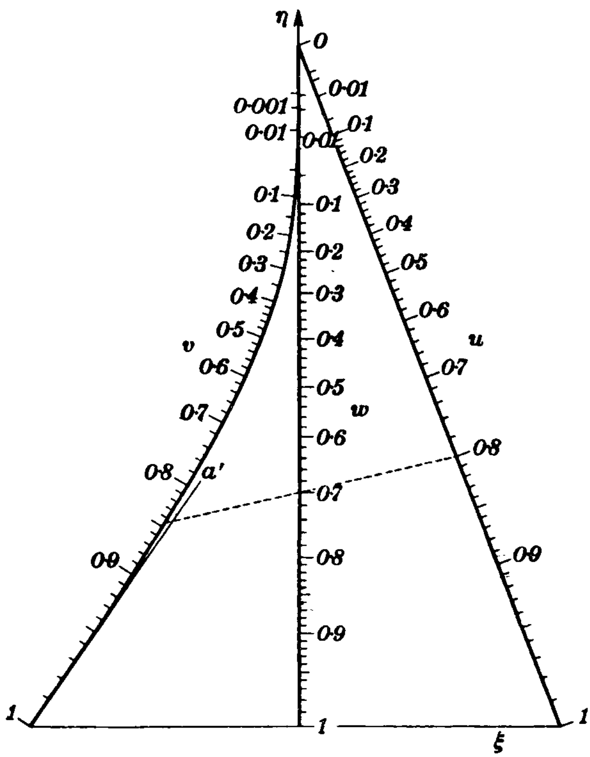

FIG. 78

By eliminating parameter x we obtain

$$3\log e \,.\, \xi-(20\log e+6)\,\eta-\log e = 0.$$

Exercises

1. Draw a nomogram for the equation

$$z^2+az+b = 0$$

for the intervals $0 \leqslant a \leqslant 1$, $1 \leqslant b \leqslant 10$.

2. Make the transformation described in Example 2 taking instead of the origin P of the system a point P_1 with coordinates 0, -20, 1.

3. Construct a nomogram for the equation

$$W = (H^4 - h^4)/6H,$$

where h varies in the interval from 10 to 25 and H varies in the interval from 20 to 30.

4. Construct a nomogram for the equation

$$l\sqrt{l_1+l_2} = l_1\sqrt{l_1},$$

where the two variables l_1 and l_2 run over the interval from 1 to 10.

5. Draw a nomogram for the equation

$$T\infty = T_1^2/(T_1 - T'_2),$$

where T'_2 is the rise of the temperature of a motor in the time from t_1 to t_2.

6. Draw a nomogram for the equation of the third degree

$$az^3 + bz^2 - 1 = 0$$

where the coefficients a and b assume all values greater than 10; only a positive value of z is required.

§ 17. The Clark equation

The equation

$$f_1(x)\,f_2(y)\,g_3(z) + [f_1(x) + f_2(y)]\,h_3(z) + 1 = 0$$

is called the *Clark equation.* For all values of x and y such that

$$f_1(x) \neq f_2(y) \tag{17.1}$$

this equation can be written in the form

$$\frac{1}{f_1(x) - f_2(y)} \begin{vmatrix} 1 & -f_1 & f_1^2 \\ 1 & -f_2 & f_2^2 \\ g_3 & h_3 & 1 \end{vmatrix} = 0. \tag{17.2}$$

The scale equations will be of the form

$$\xi_1 = 1/f_1^2(x), \quad \eta_1 = -1/f_1(x),$$

$$\xi_2 = 1/f_2^2(y), \quad \eta_2 = 1/f_2(y),$$

$$\xi_3 = g_3(z), \quad \eta_3 = h_3(z).$$

It can easily be seen that the first two scales are the same curve of the second degree $\xi = \eta^2$.

We shall thus obtain a nomogram (Fig. 79) consisting of three scales two of which lie on a parabola and the third on a curve with parametric equations defined by the functions $\xi = g_3(z)$, $\eta = h_3(z)$.

Each straight line joining two points x and y, a so called cord of the parabola, intersects the third scale at a point z which, together with the given x and y, satisfies the Clark equation. In the limiting case, where instead of a curve we take a tangent, i.e. where $f_1(x_0) = f_2(y_0)$, we shall obtain, if the functions occurring in the equation are continuous, a point z_0 which also satisfies the Clark equation.

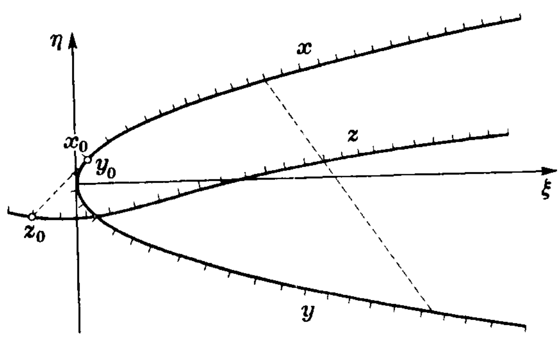

FIG. 79

EXAMPLE. Construct a nomogram for finding v from the equation

$$w = \frac{5u^2v-1}{5v-u^2v^2},$$

where $0 \leqslant w \leqslant 1$, $1 \leqslant u$.

This equation can be reduced to the Clark form:

$$-u^2v^2w+5(w-u^2)v+1 = 0.$$

We have here

$$f_1 = w, \quad f_2 = -u^2, \quad g_3 = v^2, \quad h_3 = 5v.$$

Writing this equation in the form of a determinant,

$$\begin{vmatrix} 1 & -w & w^2 \\ 1 & u^2 & u^4 \\ v^2 & 5v & 1 \end{vmatrix} = 0$$

(inequality (17.1) being always staisfied under our conditions because $w \neq -u^2$), we can see that

$$\xi_1 = 1/w^2, \quad \eta_1 = -1/w,$$
$$\xi_2 = 1/u^4, \quad \eta_2 = 1/u^2,$$
$$\xi_3 = v^2, \quad \eta_3 = 5v.$$

An outline of the nomogram defined by these equations is shown in Fig. 80. We can see that the u-scale is an arc of a pa-

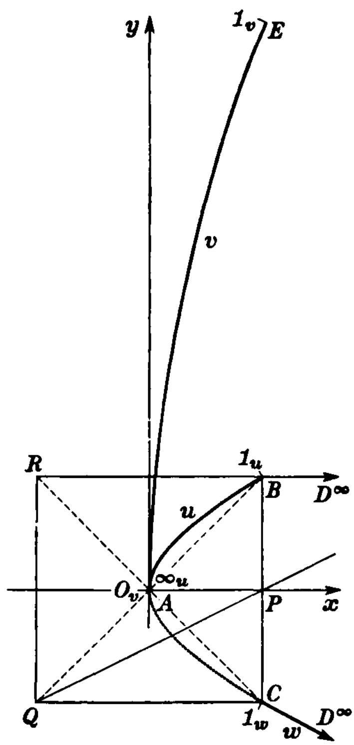

FIG. 80

rabola from point A to point B, the w-scale is an arc lying on a half-plane with negative ordinates from point C to infinity, and the v-scale is an arc of the parabola $\eta^2 = 25\xi$ from point A to point E with coordinates 1, 5. This arc cannnot be replaced by a smaller arc in this drawing because we also want to read values of v that correspond to numbers u and v close to unity.

Our nomogram is unsuitable because it has infinite dimensions; moreover, the reading of the v-scale involves considerable error since that scale lies outside scales u and w.

In order to give this nomogram a more convenient form we transform the plane by projection in such a manner as to turn the quadrilateral $A'B'C'D'$ into a rectangle and make the arc $A'E'$ of the v-scale lie inside that rectangle. In order to do this we must transfer two diagonal points of the quadrilateral to infinity. Since the (only) point at infinity of the parabola lies on the axis of symmetry of the quadrilateral, joining pairs of vertices by diagonals we shall obtain diagonal points P, Q and R. The drawing makes it obvious that the w-scale can only be carried over to the other side of the v-scale by intersecting the plane by the straight line PQ and by transferring it to infinity. Therefore, we must assign points at infinity to points P and Q, leaving point A, for example, at the origin of the system.

Since

point $(1, 0, 0)$ is to correspond to point $P(1, 0, 1)$,
point $(0, 1, 0)$ is to correspond to point $Q(-1, -1, 1)$,
point $(0, 0, 1)$ is to correspond to point $A(0, 0, 1)$

we obtain matrix $\mathfrak{A}$ from the matrix

$$\begin{bmatrix} 1 & 0 & 1 \\ -1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

by inversion; we then have

$$\mathfrak{A} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & -2 \\ 0 & 0 & -1 \end{bmatrix}.$$

Multiplying the matrices

$$\begin{bmatrix} 1 & -w & w^2 \\ 1 & u^2 & u^4 \\ v^2 & 5v & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & -2 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -w-1 & -w & 1+2w-w^2 \\ -1+u^2 & u^2 & 1-2u^2-u^4 \\ 5v-v^2 & 5v & v^2-10v-1 \end{bmatrix}$$

we obtain the scale equations

$$\xi_1 = (w+1)/(w^2-2w-1),$$
$$\eta_1 = w/(w^2-2w-1),$$
$$\xi_2 = (1-u^2)/(u^4+2u^2-1),$$
$$\eta_2 = -u^2/(u^4+2u^2-1),$$
$$\xi_3 = (5v-v^2)/(v^2-10v-1),$$
$$\eta_3 = 5v/(v^2-10v-1).$$

The scales u and w are still on the same curve of course; after the transformation it is a hyperbola, which is shown by the following:

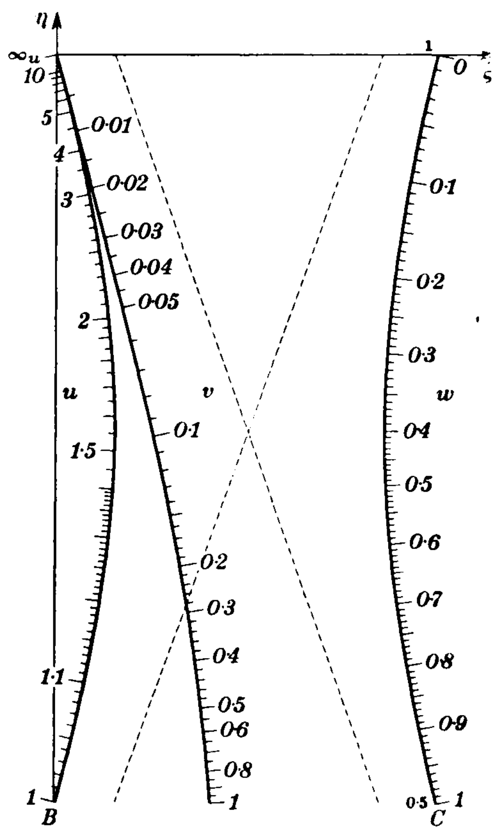

FIG. 81

$$\frac{\xi}{\eta} = \frac{w+1}{w} = 1+\frac{1}{w},$$

$$\eta = \frac{\eta(\xi-\eta)}{\eta^2-2\eta(\xi-\eta)-(\xi-\eta)^2} = \frac{\eta^2-\xi\eta}{\xi^2-2\eta^2},$$

$$\xi^2-2\eta^2 = -\xi+\eta,$$

$$(\xi+1/2)^2-2(\eta+1/4)^2 = 1/4-1/8,$$

$$8(\xi+1/2)^2-16(\eta+1/4)^2 = 1.$$

Substituting values from the interval (0, 1) for the variable w and values from 1 to infinity for the variable u we verify that the arcs corresponding to these intervals indeed have their end-points at the vertices of the rectangle $A'B'C'D'$.

Let us now find the v-scale. Using the parametric equations we must find the coordinates of points corresponding to values from the interval (0, 1). We shall obtain a scale lying on the arc $A'E'$, i.e. extending from the origin A of the system to the point with coordinates $\xi = 0{\cdot}4$ and $\eta = -0{\cdot}5$ (Fig. 81).

To give the nomogram a more convenient form we make the ordinates of all the points five times larger.

Exercises

1. Write the equation

$$u \sin w - \frac{u}{v}\cos^2 w + \frac{1}{v}\sin^2 w = 0$$

in the Clark form and construct a nomogram for the intervals $-2 \leqslant u \leqslant 0$, $1 \leqslant v$.

2. Regarding the equation

$$uvw+1 = 0$$

as a Clark equation ($h_3 = 0$) draw a nomogram for this equation making two scales lie on the same curve of the second degree and the third scale on a straight line.

Find in which cases this nomogram is better than a nomogram consisting of three rectilinear scales.

3. Given the equation of the second degree

$$wx^2+w^2x+2 = 0,$$

with the parameter w varying in the interval from 2 to 4, draw a nomogram from which it would be possible to read the value of the root x_1 of this equation if

$$-3{\cdot}9 \leqslant w \leqslant -1, \qquad -3{\cdot}9 \leqslant x_2 \leqslant -1.$$

4. Draw a nomogram for the equation

$$xy^2z^2+y^2z+xz = 3$$

where the parameters x and y vary in the interval from 1 to 3.

§ 18. The Soreau equation of the first kind

The equation

$$f_1(x) = \frac{f_2(y)+f_3(z)}{g_2(y)+g_3(z)} \tag{18.1}$$

is called the *Soreau equation of the first kind.*

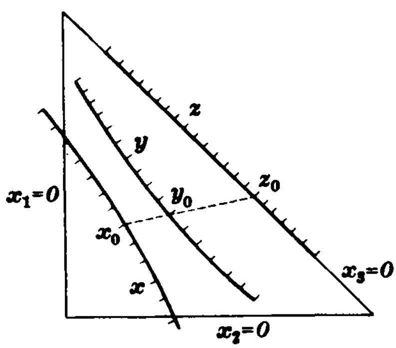

FIG. 82

An equation of this type can be written in the form

$$\begin{vmatrix} f_1 & 1 & 0 \\ f_2 & g_2 & -1 \\ f_3 & g_3 & 1 \end{vmatrix} = 0. \tag{18.2}$$

The homogeneous scale equations

$$\begin{aligned} x_1 &= f_1, & x_2 &= 1, & x_3 &= 0, \\ \bar{x}_1 &= f_2, & \bar{x}_2 &= g_2, & \bar{x}_3 &= -1, \\ \bar{\bar{x}}_1 &= f_3, & \bar{\bar{x}}_2 &= g_3, & \bar{\bar{x}}_3 &= 1, \end{aligned}$$

show that we are dealing with one rectilinear scale (on a straight line at infinity) and two curvilinear scales (Fig. 82).

Usually the scale on the straight line at infinity is transferred by means of a projective transformation upon an ordinary straight line.

EXAMPLE. Draw a nomogram for the equation

$$3\pi\varrho(R^2-r^2) = 4(R^3-r^3)$$

for the intervals $0 \leqslant r \leqslant 4$, $5 \leqslant R \leqslant 10$.

From this formula we can find the distance ϱ of the centre of gravity of a quarter of a circular ring with radii r and R from the geometrical centre of the ring.

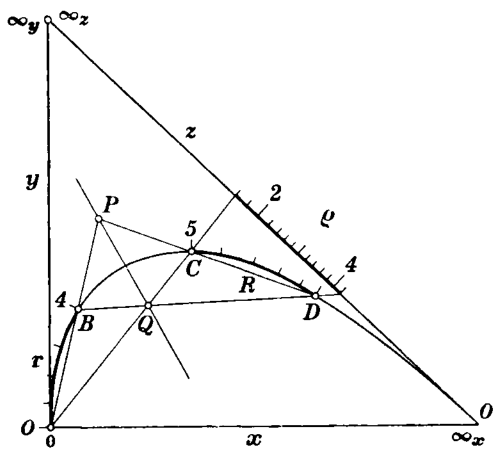

FIG. 83

Let us write this equation in the form

$$\frac{3\pi}{4}\varrho = \frac{R^3-r^3}{R^2-r^2}.$$

It obviously belongs to the Soreau type which we have been considering; using form (18.2) we obtain

$$|\mathfrak{X}| = \begin{vmatrix} \frac{3}{4}\pi\varrho & 1 & 0 \\ R^3 & R^2 & -1 \\ -r^3 & -r^2 & 1 \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} -\frac{3}{4}\pi\varrho & -1 & 0 \\ R^3 & R^2 & 1 \\ r^3 & r^2 & 1 \end{vmatrix} = 0.$$

The sketch in Fig. 83 shows us that the scale which gives the readings of the values of ϱ lies outside the scales r and R. In order to locate it between them we must make a transformation transferring the straight line dividing the scales r and R to infinity; then the ϱ-scale will appear between the scales r and R.

Let us do this in such a manner as to make the end-points of the scales r and R the vertices of a rectangle. The end-points of the r-scale are the points

$$O(0,0) \quad \text{for} \quad r=0, \qquad B(64,\ 16) \quad \text{for} \quad r=4;$$

the end-points of the R-scale are the points

$$C(125,\ 25) \quad \text{for} \quad R=5 \quad \text{and} \quad D(1000, 100) \quad \text{for} \quad R=10.$$

In order to find the coordinates of the diagonal points P and Q, let us write the equations of the lines OB and CD; we obtain

$$y = x/4$$

and

$$\begin{vmatrix} x & y & 1 \\ 125 & 25 & 1 \\ 1\,000 & 100 & 1 \end{vmatrix} = 0, \quad \text{i.e.} \quad -75x+875y-12\,500 = 0.$$

Solving these equations we obtain as the coordinates of point P numbers

$$x_P = 2\,000/23, \qquad y_P = 500/23.$$

Similarly, to find the coordinates of point Q we write the equations of the lines OC and BD:

$$y = x/5$$

and

$$\begin{vmatrix} x & y & 1 \\ 64 & 16 & 1 \\ 1000 & 100 & 1 \end{vmatrix} = 0, \quad \text{i.e.} \quad -84x+936y-9\,600 = 0.$$

These equations give us

$$x_Q = 4\,000/43, \qquad y_Q = 800/43.$$

Leaving point O at the origin of the system we shall require that

point (1, 0, 0) should correspond to point $P(2000,\ 500,\ 23)$,
point (0, 1, 0) should correspond to point $Q(4000,\ 800,\ 43)$,
point (0, 0, 1) should correspond to point $O(0,\ 0,\ 1)$.

The inverse of the matrix

$$\begin{bmatrix} 2000 & 500 & 23 \\ 4000 & 800 & 43 \\ 0 & 0 & 1 \end{bmatrix}$$

is the matrix

$$\mathfrak{A} = \begin{bmatrix} 800 & -500 & 3\,100 \\ -4000 & 2000 & 6\,000 \\ 0 & 0 & -400\,000 \end{bmatrix}.$$

Let us take instead of $\mathfrak{A}$ a matrix with terms multiplied by 0·01:

$$\mathfrak{A} = \begin{bmatrix} 8 & -5 & 31 \\ -40 & 20 & 60 \\ 0 & 0 & -4000 \end{bmatrix}.$$

Multiplying matrices $\mathfrak{X}$ and $\mathfrak{A}$

$$\begin{bmatrix} -\frac{3}{4}\pi\varrho & -1 & 0 \\ R^3 & R^2 & 1 \\ r^3 & r^2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 8 & -5 & 31 \\ -40 & 20 & 60 \\ 0 & 0 & 4000 \end{bmatrix}$$

$$= \begin{bmatrix} -6\pi\varrho+40 & \frac{15}{4}\pi\varrho-20 & -\frac{93}{4}\pi\varrho-60 \\ 8R^3-40R^2 & -5R^3+20R^2 & 31R^3+60R^2-4000 \\ 8r^3-40r^2 & -5r^3+20r^2 & 31r^3+60r^2-4000 \end{bmatrix},$$

we finally obtain the scale equations

$$\xi_1 = \frac{24\pi\varrho-160}{93\pi\varrho+240}, \qquad \eta_1 = \frac{80-15\pi\varrho}{93\pi\varrho+240},$$

$$\xi_2 = \frac{8R^3-40R^2}{31R^3+60R^2-4000}, \qquad \eta_2 = \frac{-5R^3+20R^2}{31R^3+60R^2-4000},$$

$$\xi_3 = \frac{8r^2-40r^2}{31r^3+60r^2-4000}, \qquad \eta_3 = \frac{-5r^3+20r^2}{31r^3+60r^2-4000}.$$

By interchanging ξ with η and *vice versa*, by changing the signs and by a suitable choice of limits we shall obtain the nomograms shown in Fig. 84.

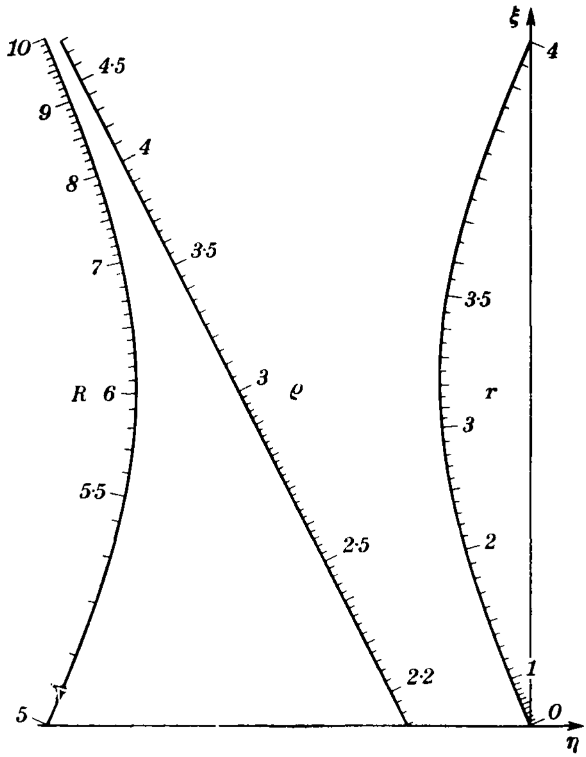

FIG. 84

Exercises

1. Construct a nomogram for the equation

$$V = \frac{\pi}{3} \cdot \frac{R^3 - r^3}{R - r}$$

for $0 \leqslant r \leqslant 10$ and $20 \leqslant R \leqslant 30$.

2. Construct a nomogram for the equation

$$u^2 - uw + v(1 - w) = 0$$

giving the readings of the values of w when u and v vary in the intervals $0 \leqslant u \leqslant 4$ and $1 \leqslant v \leqslant 8$.

3. Draw a nomogram for the equation

$$uw = vw + u^3 + v^4$$

where u varies in the interval from 1 to 2·5 and w varies in the interval from 1 to 1·5.

§ 19. The Soreau equation of the second kind

The equation

$$\frac{f_1(x)+f_2(y)}{g_1(x)+g_2(y)} = \frac{f_1(x)+f_3(z)}{g_1(x)+g_3(z)}$$

is called the *Soreau equation of the second kind.*

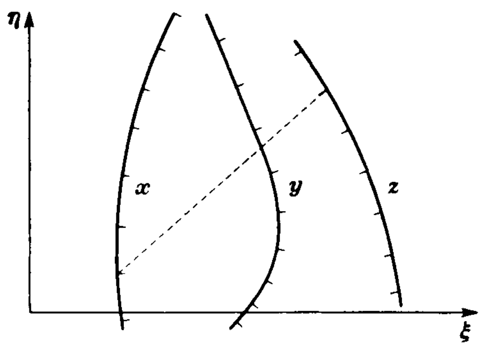

Fig. 85

An equation of this kind can be written in the form

$$\begin{vmatrix} f_1 & g_1 & 1 \\ f_2 & g_2 & -1 \\ f_3 & g_3 & -1 \end{vmatrix} = 0. \qquad (19.1)$$

As can be seen from equation (19.1), in the general case each of the three scales lies on a curve (Fig. 85). The scale equations are:

$$\xi_1 = f_1(x), \qquad \eta_1 = g_1(x),$$
$$\xi_2 = -f_2(y), \qquad \eta_2 = -g_2(y),$$
$$\xi_3 = -f_3(z), \qquad \eta_3 = -g_3(z).$$

Example. Draw a nomogram for the equation

$$3xz^2+2x^2y+6y^2z-2xy^2-6yz^2-3x^2z = 0$$

where each variable runs over the interval (0, 1).

This equation can be written in the form of the Soreau equation

$$\frac{x-2y}{x^2-2y^2} = \frac{x-3z}{x^2-3z^2},$$

and thus also in form (19.1):

$$\begin{vmatrix} x & x^2 & 1 \\ -2y & -2y^2 & -1 \\ -3z & -3z^2 & -1 \end{vmatrix} = 0.$$

We obtain hence the following scale equations:

$$\xi_1 = x, \quad \eta_1 = x^2,$$
$$\xi_2 = 2y, \quad \eta_2 = 2y^2,$$
$$\xi_3 = 3z, \quad \eta_3 = 3z^2.$$

The nomogram defined by these formulas is shown in Fig. 86.

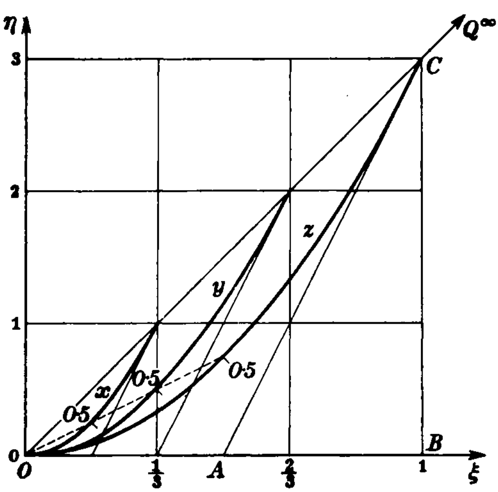

FIG. 86

This nomogram does not give great accuracy because the straight lines joining the points of the u-scale with the points of the w-scale that are close to unity intersect the v-scale at acute angles. In order to give the nomogram a better shape we should have to enlarge considerably the distance of point $0{\cdot}5_w$ from the diagonal line $x = y$ leaving unchanged the points on that diagonal.

This can be achieved in a number of ways, e.g. by transforming the plane so as to turn

$$\begin{aligned}&\text{point } Q(1,\ 1,\ 0) \text{ into point } X'^{\infty}(1,\ 0,\ 0),\\ &\text{point } A(0{\cdot}5,\ 0,\ 1) \text{ into point } Y'^{\infty}(0,\ 1,\ 0),\\ &\text{point } O(0,\ 0,\ 1) \text{ into point } O(0,\ 0,\ 1).\end{aligned} \qquad (*)$$

Since point A is a pole of the straight line $y = x$ with respect to the parabola on which the w-scale lies, the arc $0_w 1_w$ of the parabola, as we know, turns into a half of an ellipse or of a circle (the straight line AQ^{∞} does not intersect any of the three parabolas in question).

This does not seem the best way, however, because the neighbourhood of point $0{\cdot}5_w$ would be enlarged much more than, for instance, the neighbourhood of point 0, which is not absolutely necessary here. Let us choose another method. Take an affine transformation of the triangle OBC into another triangle, in which the ratio of the sides OC to the remaining sides will be considerably less. We shall perform this in a purely geometrical manner without writing the equations.

We draw an arbitrary triangle $O'C'B'$ transferring in an affine manner (i.e. by retaining the ratios of parallel segments) the network of lines $x = a$ and $y = b$ as in Fig. 86. We obtain an oblique system of coordinates, in which we draw the scales x, y and z, using the original equations (Fig. 87).

Since the tangents at points $0{\cdot}5_x$, $0{\cdot}5_y$ and $0{\cdot}5_z$ to the corresponding parabolas are parallel to the straight line OC, in the new nomogram the tangents at the corresponding points will be parallel to the straight line $O'C'$.

Exercises

1. Make the transformation $(*)$ of the nomogram shown in Fig. 86 for example.

2. Write the equation of the nomogram shown in Fig. 87 where OQ^{∞} and OA are axes of coordinates.

3. Draw a nomogram for the equation

$$uv^2+u^2w^2+wv-v^3w^2-u^2v-uw = 0$$

where u varies from $0{\cdot}6$ to $0{\cdot}7$ and w from $0{\cdot}5$ to $0{\cdot}6$.

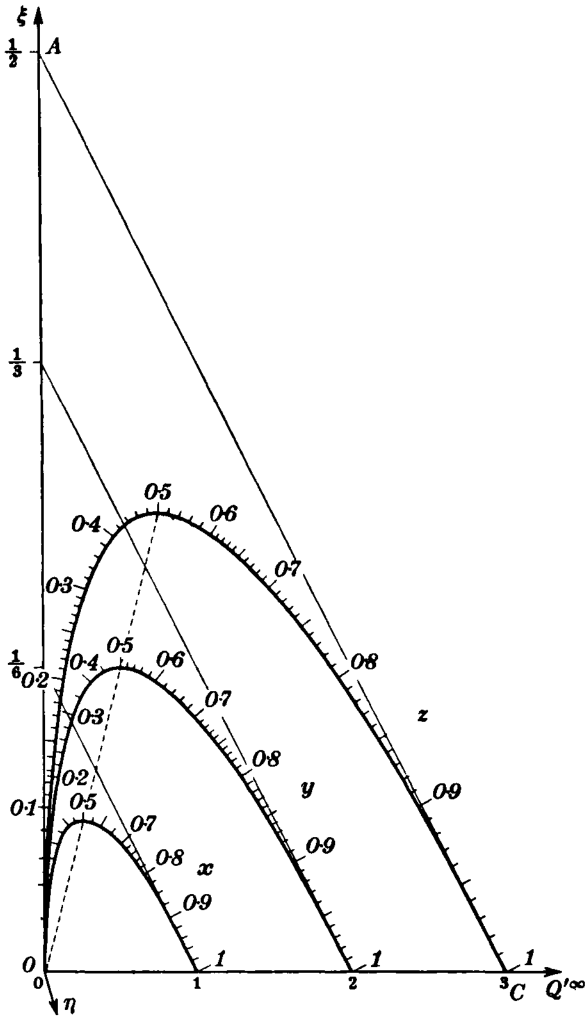

FIG. 87

§ 20. An arbitrary equation with three variables. Nomograms consisting of two scales and a family of envelopes

Let the equation

$$F(u, v, w) = 0 \tag{20.1}$$

be an arbitrary equation with variables u, v and w.

Consider arbitrary functional scales

$$x = \varphi_1(u), \qquad y = \psi_1(u), \tag{20.2}$$

$$x = \varphi_2(v), \qquad y = \psi_2(v). \tag{20.3}$$

We assume that the function $F(u, v, w)$ has partial continuous derivatives and that the functions φ_i and ψ_i have continuous derivatives.

Let us choose a certain value w_0 and a family of straight lines w_0 joining such pairs of points u and v on selected scales that the following equation is satisfied:

$$F(u, v, w_0) = 0. \tag{20.4}$$

The straight lines of the family $\boldsymbol{w}_0$ have an envelope w_0, which is a curve or a point (Fig. 88).

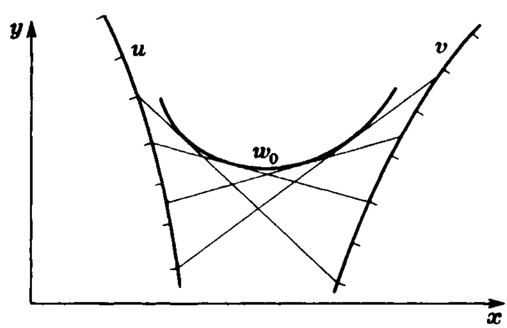

FIG. 88

The equation of the envelope w_0 is obtained in the following manner:

We write the equation of the straight line joining the point u with coordinates $\varphi_1(u)$, $\psi_1(u)$ with the point with coordinates $\varphi_2(v)$, $\psi_2(v)$ in the form

$$y-\psi_1(u) = \frac{\psi_2(v)-\psi_1(u)}{\varphi_2(v)-\psi_1(u)}\big(x-\varphi_1(u)\big).$$

Substituting for v the function

$$v = g(u, w_0)$$

found from the relation $F(u, v, w_0) = 0$, we transform the equation into the equation of a family $\boldsymbol{w}_0$ of straight lines

$$y-\psi_1(u) = f(u)\left(x-\varphi_1(u)\right) \tag{20.5}$$

depending on one parameter u.

The equation of the envelope w_0 is obtained, as we know, by differentiating (20.5) with respect to u:

$$-\psi_1'(u) = f'(u)\left(x-u_1(u)\right)-f(u)\,.\,\varphi_1'(u) \tag{20.6}$$

and eliminating u from equations (20.5) and (20.6).

Thus, for different values of the variable w we have obtained, in general, different curves w (Fig. 89).

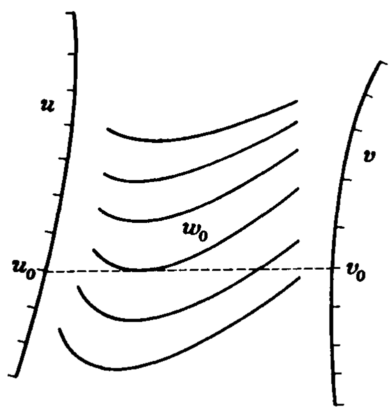

FIG. 89

The manner of using a nomogram consisting of two scales and one family of envelopes is obvious: three numbers u_0, v_0 and w_0 satisfy equation (20.1) if and only if the straight line joining point u_0 of the u-scale with point v_0 of the v-scale is tangent to the curve w_0.

The calculations necessary to determine the envelope equations are generally cumbersome. In many cases we can simplify them considerably by choosing the scales u and v on straight lines.

For instance, if the scales u and v are regular and lie on parallel lines, i.e., if

$$x = 0, \quad y = u,$$

$$x = 1, \quad y = bv.$$

and the given equation is of the form $v = g(u, w)$, then the envelope is defined by the pair of equations

$$y-u = \big(bg(u, v)-u\big)x,$$
$$-1 = \left(b\frac{\partial g}{\partial u}-1\right)x,$$

from which the parameter u must be eliminated.

EXAMPLE. Construct a nomogram for the equation

$$w\big(u+v-\sqrt{u^2+v^2}\big) = uv$$

for

$$0 \leqslant u \leqslant 10, \quad 0 \leqslant v \leqslant 5.$$

Let us select for the variable u a regular scale on the y-axis,

$$x = 0, \quad y = u,$$

and for the variable v a regular scale on the x-axis,

$$x = v, \quad y = 0.$$

The straight line joining points u and v has the equation

$$\frac{x}{v}+\frac{y}{u} = 1.$$

Finding from the given equation

$$v = 2w\frac{u-w}{u-2w}$$

and substituting, we obtain the equation of the family of straight lines w dependent on parameter u

$$(u^2-2uw)x+2uwy-2w^2y = 2u^2w-2uw^2.$$

Differentiating with respect to u,

$$2(u-w)x+2wy = 4uw-2w^2,$$

and substituting into the preceding equation, we obtain—after a certain amount of calculation—the equation

$$(2w-x)\,[w^2-2w(x+y)+x^2+y^2] = 0,$$

which contains the envelope

$$w^2-2w(x+y)+x^2+y^2 = 0.$$

This is a circle tangent to the axes of the system,

$$(x-w)^2+(y-w)^2 = w^2.$$

Thus our nomogram consists of the u-scale on the y-axis, the v-scale on the x-axis and a family of circles tangent to both axes of the system (Fig. 90).

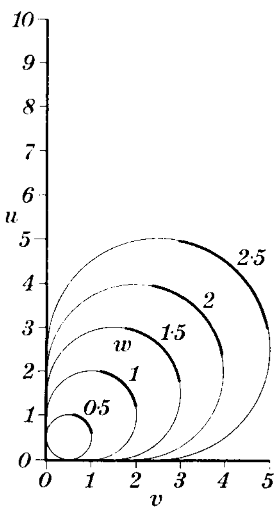

FIG. 90

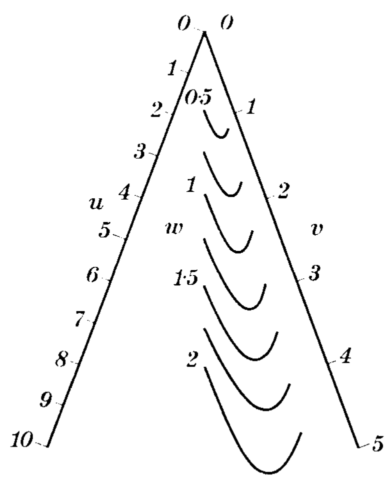

FIG. 91

By means of an affine transformation we can obtain a form as in Fig. 91. The family of circles w has been turned into a family of ellipses.

As can be seen from the example given (where the function has been of a very simple form) the calculations necessary to write the equations of a family of envelopes are usually very cumbersome. That is why, in more complex cases, we follow a different procedure in drawing a nomogram.

Finding for a certain value of w_0 several pairs of values of u_k and v_k we draw lines joining the points u_k and v_k of the chosen scales u and v, and we draw the envelope w_0 on the grounds of knowing several tangents. Similarly, we draw another curve w.

This remark refers particularly to cases where instead of the equation $F(u, v, w) = 0$ we have a table of values, of the variable w, for instance, in relation to the variables u and w, which is the case in numerous experiments in technological research.

Exercises

1. Construct a nomogram for the equation

$$(u-v)^2 - 8w\,(u+v) - 16w^2 = 0$$

for $0 \leqslant u \leqslant 10$ and $0 \leqslant v \leqslant 10$, choosing for u and v two regular scales on parallel lines.

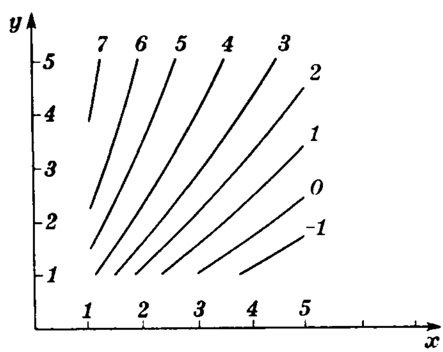

FIG. 92

2. Construct a nomogram for the equation

$$w\sqrt{u^2+v^2} = uv - 10u + 12v$$

for

$$0 \leqslant u \leqslant 5, \quad 0 \leqslant v \leqslant 4,$$

choosing for u and v two regular scales on intersecting straight lines.

3. Construct a nomogram consisting of two scales and one envelope family for the relation between x, y and z shown in Fig. 92.

II. LATTICE NOMOGRAMS

§ 21. General form of lattice nomograms

Let $f(x, y, w) = 0$ be any function of three variables satisfying the following condition:

There exists on the plane (x, y) a domain D such that, if the point (x_0, y_0) belongs to D, then there is at least one value of w_0 such that

$$f(x_0, y_0, w_0) = 0, \quad f_x'^2(x_0, y_0, w_0)+f_y'^2(x_0, y_0, w_0) > 0.$$

This condition implies the existence of a curve passing through the point (x_0, y_0) each point of which satisfies the equation

$$f(x, y, w_0) = 0.$$

The set of all points satisfying the equation

$$f(x, y, w_0) = 0$$

forms one or more curves. This set is called the *w_0-line*. The *x_0-line* is the straight line $x = x_0$; the *y_0-line* is the straight line $y = y_0$. We thus have three families of lines (Fig. 93):

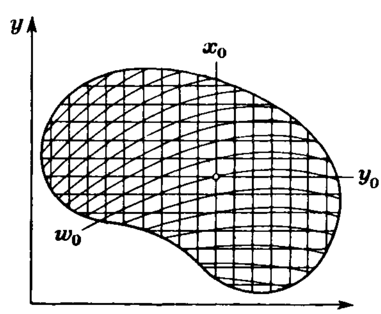

FIG. 93

1. the family of straight lines parallel to the y-axis,
2. the family of straight lines parallel to the x-axis,
3. the family of curves $f(x, y, w_0) = 0$.

As follows from the definition of lines x_0, y_0 and w_0, they have a point in common if and only if numbers x_0, y_0 and w_0 satisfy the equation

$$f(x_0, y_0, w_0) = 0.$$

We can make another drawing. Let us take three families of lines (Fig. 94) on a plane (ξ, η)

$$U(\xi, \eta, x) = 0, \qquad V(\xi, \eta, y) = 0, \qquad W(\xi, \eta, w) = 0,$$

selected so that the curve $U(\xi, \eta, x_0) = 0$, the curve $V(\xi, \eta, y_0) = 0$ and the curve $W(\xi, \eta, w_0) = 0$ have a point in common if and only if the following equation is satisfied:

$$f(x_0, y_0, w_0) = 0.$$

Our assumptions regarding Fig. 94 recall an analogous property of collineation nomograms, where numbers u, v and w were

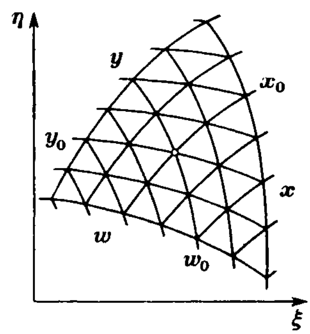

Fig. 94

represented by points and the fulfilment of a given equation by numbers u_0, v_0 and w_0 was equivalent to the collinearity of the points representing those numbers. Now we have lines instead of points, and instead of the collinearity of three points we have the possession of a point in common by three lines. Drawings such as Figs. 93 and 94 play a similar role to that of collineation nomograms. We call them *lattice nomograms.*

In §§ 10–19 we considered only certain types of equations and showed the methods of drawing nomograms for them. Now we can draw a lattice nomogram for practically every function of three variables. Moreover, as can easily be observed, lattice nomograms can be subjected to any continuous and one-to-one transformations on a plane or in space, because any three lines having a point in common will then turn into three lines having a point in common. This is a very important property since, in case where the domain D, which contains our nomogram, has different degrees of accuracy in its different parts, we can alter this by enlarging less accurate parts and reducing those which are too accurate.

For example, let us take the equation

$$w = uv$$

in the intervals $0 \leqslant u \leqslant 4$ and $0 \leqslant v \leqslant 4$.

The equation $x = u$ and the equation $y = v$ represent two families of straight lines parallel to the axes of coordinates, and the equations $xy = w$ represent hyperbolas whose asymptotes are the axes of coordinates (Fig. 95).

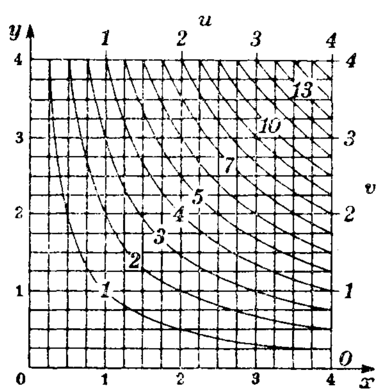

FIG. 95

In order to find, for instance, the product 3·5 . 1·5 we make the lines $u = 3{\cdot}5$ and $v = 1{\cdot}5$ intersect and read what number, approximately, is represented by the hyperbola which passes through that point: the figure shows that the number is 5·25.

Let us now make another drawing for the same equation, substituting $u = 2\sqrt{x}$ and $v = 2\sqrt{y}$. (This is not a projective transformation of course.)

Lines parallel to the axes of coordinates turn again into lines parallel to the axes of coordinates; but the regular scales on the axes have been changed to scales of the second powers in order to turn the hyperbolas corresponding to the values $w = 1, 2, \ldots,$

which in the former nomogram ran more and more closely together with the growth of w, into curves more evenly spaced.

With our substitution we have

$$w = uv = 2\sqrt{x} \,.\, 2\sqrt{y} \quad \text{or} \quad xy = w^2/16;$$

we have again obtained hyperbolas, but now they intersect the straight line $y = x$ at points of the regular scale.

The nomogram of Fig. 95 can be used when a relative accuracy of reading on the w-line is needed, and that of Fig. 96— when we require absolute accuracy on the w-line. In the latter nomogram the accuracy of reading the products uv is rather small when the ratio of those numbers is great.

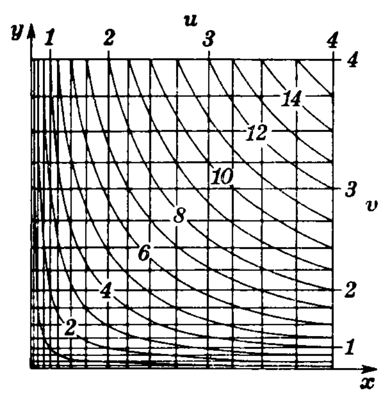

FIG. 96

That accuracy can be improved without enlarging the dimensions of the drawing. It is easy to see that this can be done by changing the lines u and v, which in both nomograms were parallel to the axes of coordinates, into curves deviating from the axes more and more as their distance from the origin of the system increases. Let us simply join by straight lines the points of intersection of the u-line and the line $y = 4$ in the first drawing with the points of intersection of the u-line and the x-axis in the second drawing; let us do the same with the v-lines (Fig. 97). We obtain a lattice of lines which, in point of density, has properties intermediate between the first and the second drawing.

How can we now draw lines corresponding to the different values of the variable w? The simplest way would be—without entering into calculations, which are unnecessary here—to draw the lines by means of this very system of coordinates u and v by using the equation $uv = w$. That is how the third form of our nomogram has been executed.

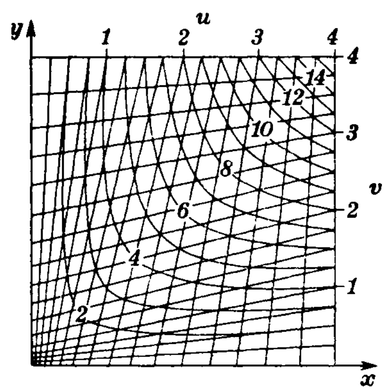

FIG. 97

Obviously, lattice nomograms admit a much greater variety of forms than collineation nomograms. We make our choice, drawing in an entirely arbitrary way two families of lines corresponding to two values of the variables, and then, assuming a certain value of w_0 we find, as in every system of coordinates, those pairs of numbers u_0 and v_0 which satisfy the equation $f(u_0, v_0, w_0) = 0$.

Now, having drawn the line w_0, we follow the same procedure in drawing other w-lines.

Comparing lattice and collineation nomograms we can observe that each of the two types has its advantages and disadvantages:

1. Lattice nomograms can be used to represent any relation of three variables; collineation nomograms can represent only those relations which can be written in the form

$$\begin{vmatrix} \varphi_1(x) & \psi_1(x) & 1 \\ \varphi_2(y) & \psi_2(y) & 1 \\ \varphi_3(z) & \psi_3(z) & 1 \end{vmatrix} = 0.$$

(As we know, reducing a given equation to this form is a complex problem even for relatively simple functions.)

2. The drawing and calculation labour involved in making a lattice nomogram is much greater than that necessary for making a collineation nomogram. The former necessitates the drawing of several curves each of which must be determined by a large number of points.

3. The ease and precision of reading is, as shown by experience, much greater in nomograms where a number is represented by a point and not by a line or curve, i.e. in collineation nomograms. This is due to the fact that the lines and curves drawn in a lattice nomogram cannot run so densely as the points marked on functional scales; to a certain extent, what makes it more difficult to use lattice nomograms is the fact that the mark denoting the numbers assigned to a line is removed far from the place of reading the line, while in functional scales the marks are close to the points. Finally the drawing itself is in the case of a lattice nomogram densely covered with lines, which hampers the user and increases the chances of making mistakes. Collineation nomograms, on the other hand, are clear and easy to use, excluding the possibility of mistakes altogether.

4. The wear of a lattice nomogram is much greater with frequent use—and that is what nomograms are for—than that of a collineation nomogram, for in order carry out an interpolation in a lattice nomogram we draw the missing lines in pencil and then rub them out damaging the drawing proper. With collineation nomograms we can put over the drawing a thin piece of cellophane with a straight line drawn on its reverse.

The features we have mentioned of the two types of nomograms show that we should always endeavour to represent an equation by means of a collineation nomogram. Only when the execution of such a nomogram is impossible or very difficult we have to content ourselves with a lattice nomogram.

The drawing labour involved in making a lattice nomogram, which has been mentioned in 2., is much simpler if all the three families of lines u, v and w consist of straight lines. Lattice nomograms of this kind are called *rectilinear nomograms.*

We shall prove the following theorem:

If for the equation:

$$f(u, v, w) = 0 \tag{21.1}$$

there exists a collineation nomogram, then there also exists a rectilinear lattice nomogram for that equation, and conversely.

Let N be a collineation nomogram for equation (21.1). Consider a correlation on a plane (Chapter I, § 5) which assigns to every point $X(x_1, x_2, x_3)$ a straight line $p(u_1, u_2, u_3)$. As we know, every point X_u lying on the u-scale has a corresponding number u; let us assign to that number a straight line p_u which corresponds in the correlation to our point X_u; we have thus obtained a family of lines p_u assigned to the variable u. Similarly, we define two other families of straight lines, p_v and p_w, which correspond to the variables v and w. Now let X_u^0, X_v^0 and X_w^0 be points of the scales u, v, w, respectively, which lie on a straight line l_0. Since the straight line l_0 has in the correlation a corresponding point L_0, through which pass the lines p_u^0, p_v^0 and p_w^0 corresponding to the points X_u^0, X_v^0 and X_w^0, the condition that three lines representing three values u_0, v_0 and w_0 such that $f(u_0, v_0, w_0) = 0$ should pass through one point is seen to be satisfied. The drawing which corresponds in the correlation to the collineation nomogram is thus a lattice nomogram consisting of three families of straight lines.

Exercises

1. Draw a lattice nomogram for the equation

$$v = \pi r^2 h/3$$

for the intervals $1 \leqslant r \leqslant 6$ and $3 \leqslant h \leqslant 12$.

2. Draw a lattice nomogram for the equation

$$\gamma = 0{\cdot}0017\, H \frac{273}{273+t}$$

for the intervals $0 \leqslant t \leqslant 40$ and $600 \leqslant H \leqslant 800$.

3. Draw a lattice nomogram for the equation

$$w = uv$$

taking concentric circles as the u-lines and parallel lines as the v-lines.

4. Construct a lattice nomogram for finding the roots of the equation of the third degree

$$aw^3+2aw+b = 0$$

in relation to numbers a and b assuming that a runs over the interval from 2 to 5 and b—from -5 to $-0{\cdot}1$. Does there exist a collineation nomogram for this equation? Does there exist a lattice nomogram consisting of three pencils of straight lines?

5. Draw a lattice nomogram for the equation

$$w^u = uv+1$$

for the intervals $1{\cdot}05 \leqslant v \leqslant 1{\cdot}45$ and $-0{\cdot}5 \leqslant u \leqslant -0{\cdot}45$.

§ 22. Rectilinear lattice nomograms

As follows from the considerations of § 20, all types of equations which have been considered in §§ 10–19 can be presented by rectilinear lattice nomograms. Although we already know the methods of constructing collineation nomograms for those equations, yet on account of the so called combining of nomograms for functions of many variables it is necessary to discuss the construction of lattice nomograms for those equations.

a. The equation

$$w = u+v \tag{a}$$

can be represented by means of a lattice nomogram consisting of three families of straight lines: the family of lines $x = u$, the family of lines $y = v$, and the family of lines $x+y = w$. They are families of parallel lines (Fig. 98); by analogy with scales they could be called *regular families* because the distances between pairs of lines of the same family are proportional to the differences of numbers corresponding to those lines. Nomograms based on such three families of parallel lines are called the *Lalanne nomograms*.

An affine transformation of a plane retains the parallelism of lines belonging to the same family but alters the angles and the ratios of the units of distance of the line families. For example, transforming the triangle ABC (Fig. 98) into the triangle $A'B'C'$ (Fig. 99) we shall obtain a nomogram in which the units and the angles will be different.

If we make a projective transformation of the plane, then each of the three families of parallel lines will turn into a family of lines of a certain pencil; since in Fig. 98 the vertices of the pencils are points at infinity, we shall now obtain three pencils

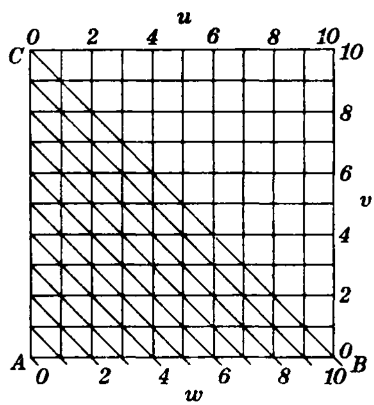

FIG. 98

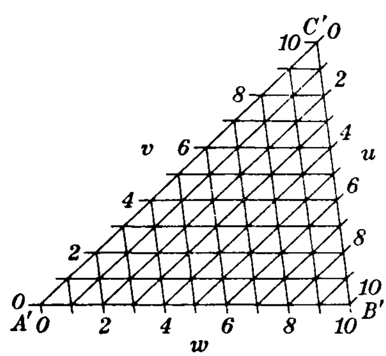

FIG. 99

whose vertices U, V and W will lie on a straight line (Fig. 100). The manner of assigning, for instance, numbers u to the elements of the pencil U is obvious: on an arbitrary straight line l parallel to the line UVW we draw a regular scale and then assign the

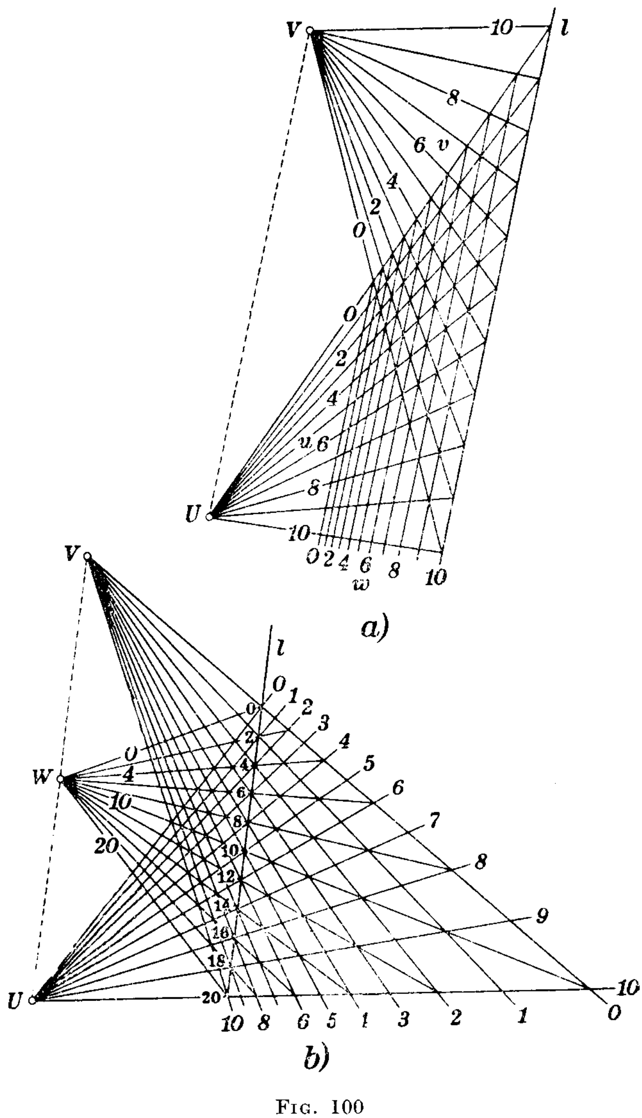
V
10
l
8
6 v
4
2
0
0
2
4
u 6
8
U
10
0 2 4 6 8 10
w
a)
V
l
0
0 1 2 3 4 5 6 7 8 9
W
0
4
10
20
2
4
6
8
10
12
14
16
18
20
U
10
10 8 6 5 4 3 2 1 0
b)

Fig. 100

value u to that line of the pencil U which passes through point u of the regular scale; for any straight line drawn on the plane will intersect the lines of the pencil U in the projective scale, but only those lines which are parallel to UVW retain the regularity of the scale, since it is the point at infinity of the scale that corresponds to the value ∞.

EXAMPLE 1. Draw a Lalanne nomogram for the equation

$$\Delta = 3160\, G^{1\cdot 85}/d^{4\cdot 97}$$

for the intervals $40 \leqslant d \leqslant 350$ and $1000 \leqslant G \leqslant 10000$.

This equation is equivalent to the equation

$$\log \Delta - \log 3160 = 1\cdot 85 \log G - 4\cdot 97 \log d.$$

Assuming $w = \log \Delta - \log 3160$, $u = 1\cdot 85 \log G$, $v = -4\cdot 97 \log d$, we have

$$5\cdot 55 \leqslant u \leqslant 7\cdot 4, \qquad -12\cdot 6 \leqslant v \leqslant -8\cdot 5,$$

i.e. the equation

$$w = u+v.$$

Figure 98 shows that the nomogram is contained in a rectangle with the horizontal side equal to $7\cdot 4 - 5\cdot 55 = 1\cdot 85$ in length and the vertical side $-8\cdot 5 + 12\cdot 6 = 4\cdot 1$ in length; the lines corresponding to the constant values of parameter w will then be inclined with respect to the axis $x(u)$ at an angle of 135°. Changing the units on the axes x and y we shall obtain a different angle between w and the x-axis. The required nomogram will be obtained by replacing the uniform scales on u and v by logarithmic scales on G and d according to the substitutions (Fig. 101).

Since all the scales appearing here are logarithmic, we can make the construction, without basing it on the system of axes u, v, as follows.

G. From points of the logarithmic scale from 1000 to 10 000 we draw parallel lines, marking them with numbers from 1000 to 10000, the unit being chosen arbitrarily.

d. From points of the logarithmic scale from 40 to 350 we draw parallel lines (but not parallel to the G-lines), marking

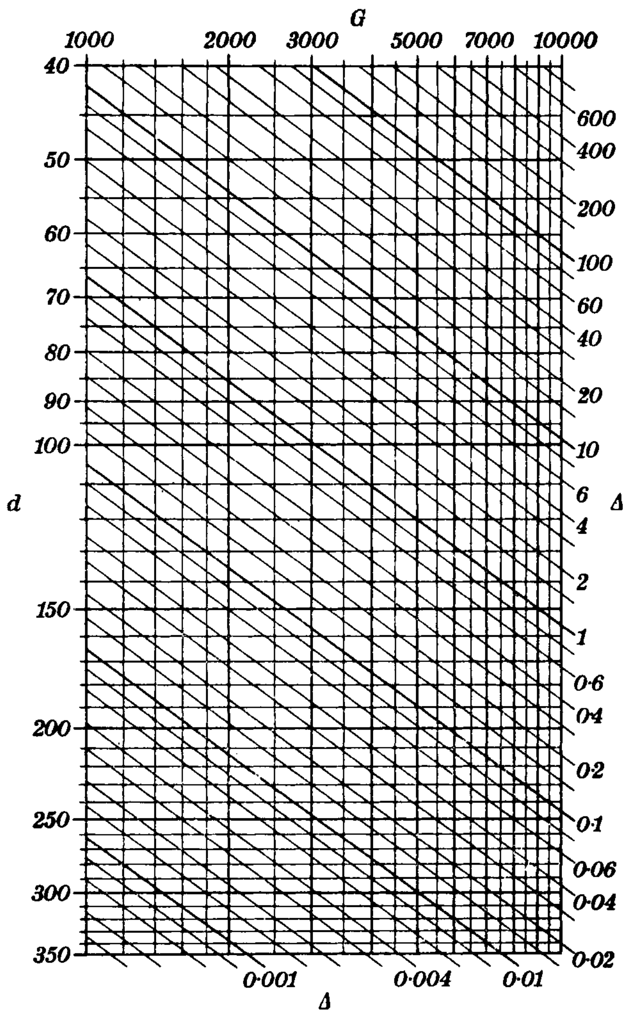

Fig. 101

them with numbers from 40 to 350, the unit being chosen arbitrarily.

Δ. We draw one line Δ—in Fig. 101 $\Delta = 1$ has been chosen by joining two points, (G_0, d_0) and (G_1, d_1), which satisfy the equation

$$1 = 3160\, G_0^{1\cdot 85}/d_0^{4\cdot 97} = 3160\, G_1^{1\cdot 85}/d_1^{4\cdot 97}$$

or

$$1{\cdot}85 \log G_0 - 4{\cdot}97 \log d_0 = 1{\cdot}85 \log G_1 - 4{\cdot}97 \log d_1 = -\log 3160.$$

Substituting $G_0 = 1000$ we obtain $d_0 = 66{\cdot}2$, and substituting $G_1 = 10000$ we obtain $d_1 = 156{\cdot}3$.

We then select the point (G_2, d_2) so as to satisfy the equatio n

$$100 = 3160\, G_2^{1\cdot 85}/d_2^{4\cdot 97}.$$

Substituting $G_2 = 10000$ we obtain $d_2 = 61{\cdot}7$.

We draw through the point (G_2, d_2) a straight line parallel to the one drawn before. We denote the first by number $\Delta = 1$ and the second by number $\Delta = 100$. We obtain the family of lines Δ by adding the missing lines and using the logarithmic scale as before. It will be observed that the procedure described here is dual to the well-known method of drawing a nomogram consisting of three parallel scales.

EXAMPLE 2. Draw a lattice nomogram for the equation

$$1/R = 1/r_1 + 1/r_2$$

for the intervals $0{\cdot}01 \leqslant r_1 \leqslant 1$ and $0{\cdot}01 \leqslant r_2 \leqslant 1$.

Substituting $u = 1/r_1$, $v = 1/r_2$ and $w = 1/R$ we can see that u and v vary in the interval from 1 to 100.

Using the remarks given above we draw a lattice nomogram for the equation $w = u+v$ by means of a regular scale y and of projecting it from two points, U and V, lying on a line parallel to this scale. The vertex W of the w-scale, which, as we know, must be on the straight line UV, is found by drawing a straight line joining point 50 of the auxiliary scale with the point at which the 0 line of family u intersects the 100-line of family v or by using the symmetry of the drawing through cutting the segment UV in two (Fig. 102). The line passing through point y of the auxiliary regular scale receives the notation $2y$.

Proceeding to the construction of our nomogram for the given equation, we find in the lattice thus obtained lines of the family u corresponding to numbers $r_1 = 0{\cdot}01,\ 0{\cdot}02, \ldots,\ 1$, lines of the family v corresponding to numbers $r_2 = 0{\cdot}01,\ 0{\cdot}02, \ldots, 1$ and lines of the family w corresponding to numbers $R = 0,\ 0{\cdot}005,\ 0{\cdot}01,\ 0{\cdot}02,\ \ldots,\ 0{\cdot}5$.

The method of drawing this nomogram can be simplified considerably by using the following properties of the projective scale.

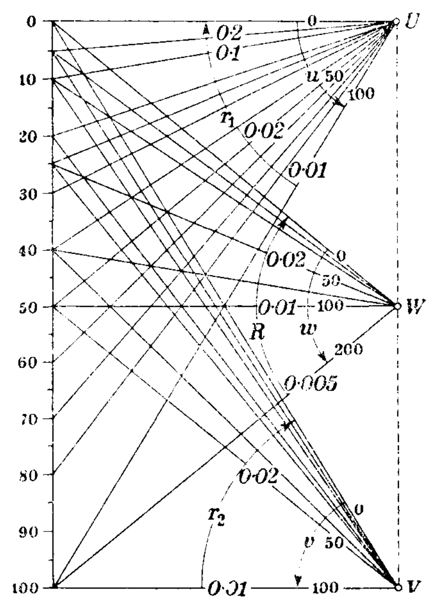

FIG. 102

Suppose we are given two axes x and y, situated so that the zero point O_x does not lie on the y-axis and the zero point O_y does not lie on the x-axis; let the point P be the intersection point of a straight line PO_x parallel to the y-axis and a straight line PO_y parallel to the x-axis. We shall show that the projection of a regular scale on one axis is a scale of the inverses on the other axis.

Indeed, similarity of triangles (Fig. 103) implies that:

$$y{:}b = a{:}x;$$

thus, if we draw a regular scale on the straight line y, we shall obtain on x a projective scale by projecting from point P:

$$y = ab/x.$$

Thus, instead of drawing an auxiliary y-scale in Fig. 102 and then replacing it by a projective scale $1/r$, we can use a regular scale r on a straight line parallel to UO_u and take O_v on the line UV.

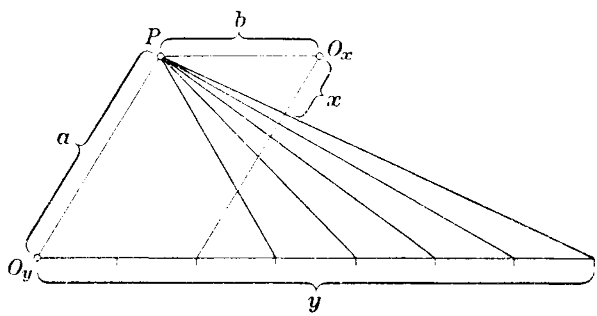

FIG. 103

The manner of making the drawing in Fig. 104 has been as follows:

1. The family of straight lines r_1 has been drawn by projecting the regular scale r from point U.

2. The family of straight lines r_2 has been drawn by projecting the regular scale r_2 from point V.

In order to obtain a nomogram of a slightly different shape than Fig. 102, we have changed the sense of the y_2-axis (and at the same time of r_2), which in Fig. 102 was the same as the sense of the axis $y_1 = y$. Therefore:

3. The family of straight lines has been drawn by projecting from point W^∞ (and not the mid-point of UV) the points at which lines of family r_1 intersect the straight line Vr_2^∞ and assigning to them numbers R equal to the corresponding numbers r_1.

b. Equations of the type

$$w = uv \tag{b}$$

were dealt with in § 14: we drew for them lattice nomograms with two families of straight lines and one family of curves. It is not difficult, however, to give a rectilinear nomogram for this

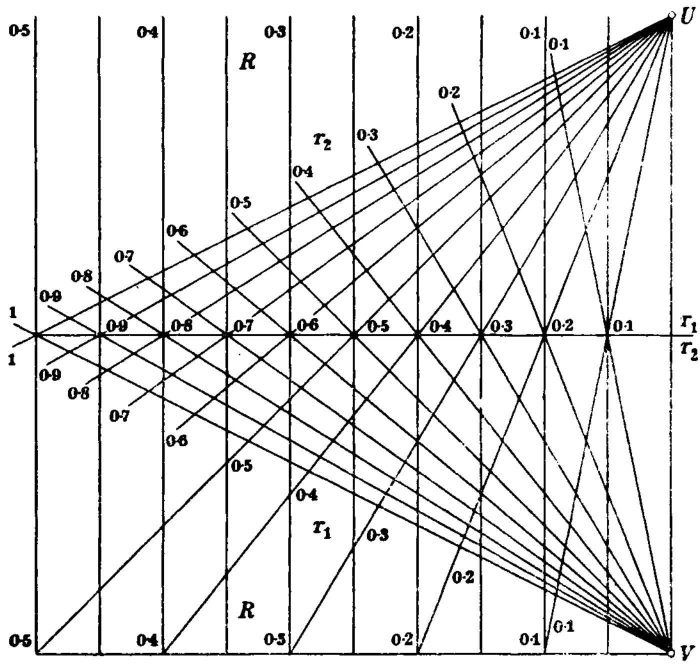

FIG. 104

equation; it is sufficient to substitute $u = x$ and $w = y$, since then $v = w/u = y/z$ will be the value of the slope of the straight line $y = vx$.

The new nomogram for this equation consists therefore of the family of straight lines $x = u$, the family of straight lines $y = w$ and the family of straight lines $y = vx$ (Fig. 105).

Because of its simple construction this nomogram is often used in practice; it is called the *Crepin nomogram*. A special variety of the Crepin lattices are those drawings in which the scale on the x-axis, the scale on the y-axis or both scales are replaced by logarithmic scales.

From the point of view of projective geometry we have here three pencils of lines whose vertices do not lie on the same line, i.e. they are not equivalent by projection to a Lalanne lattice. By means of a projective transformation of the plane a Crepin nomogram can be changed into a lattice whose three pencils all have ordinary vertices.

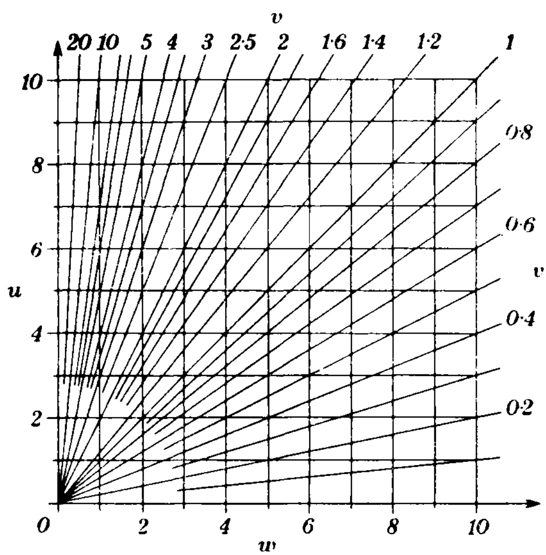

FIG. 105

A transformation of a plane which turns a lattice nomogram containing a family of curves (e.g. a family of hyperbolas for the equation $w = uv$) into a nomogram constructed from straight lines only is called an *anamorphosis*.

EXAMPLE 3. Draw a Crepin nomogram for the function

$$z = y^x.$$

Reducing the equation to the form

$$\log z = x \log y$$

we assume $u = \log y$, $v = x$ and $w = \log z$.

Number y is assigned to the straight line $\xi = \log y$, number z to the line $\eta = \log z$, and number x to a line with the slope x, i.e. to the line with the equation $\eta = x\xi$.

This nomogram (Fig. 106) is often used in practice owing to its simplicity and the facility of executing it with precision.

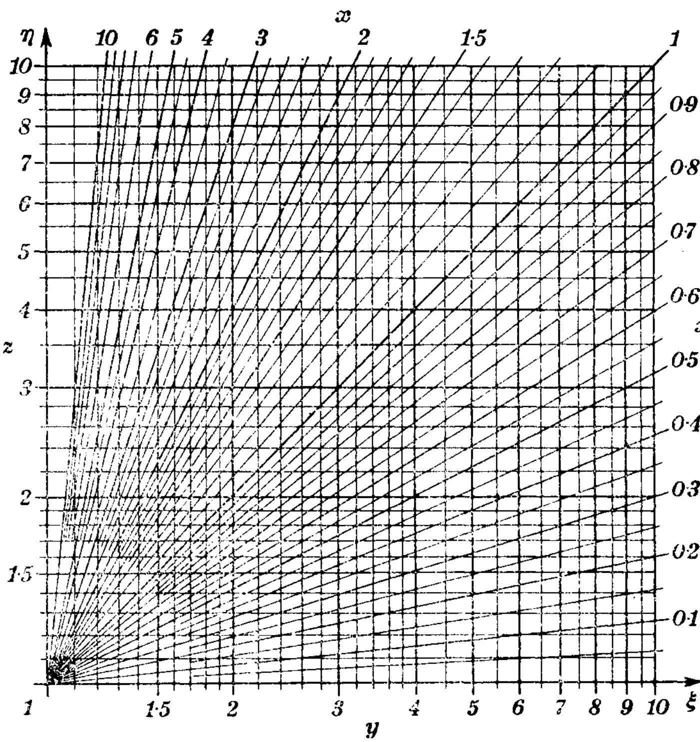

FIG. 106

c. The Cauchy equation

$$f_1(u)\, g_3(w)+f_2(v)\, h_3(w)+1 = 0 \tag{c}$$

can also be represented by a rectilinear nomogram; for, if we assume

$$x = f_1(u), \tag{I}$$

$$y = f_2(v), \tag{II}$$

we have on substituting these in equation (c)

$$g_3(w)x+h_3(w)y+1 = 0, \qquad \text{(III)}$$

i.e. we have a straight line for every value of w; equations (I) and (II) also represent straight lines of course. A schematic image and the method of drawing are shown in Fig. 107:

1. From points of the scale of function (I) drawn on the x-axis we draw the straight lines of family (I), assigning to them numbers contained in the given interval $(\underline{u}, \overline{u})$.

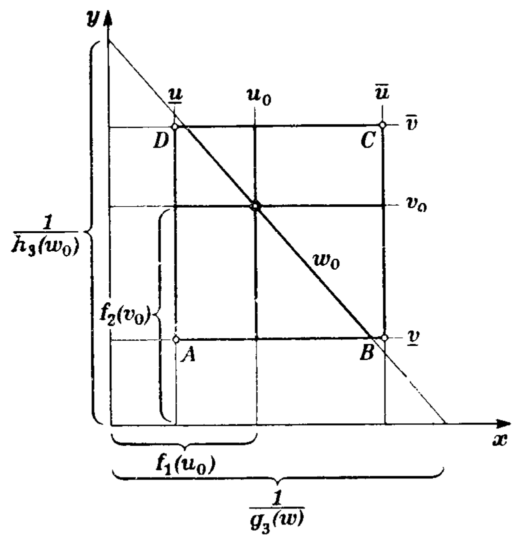

FIG. 107

2. From points of the scale of function (II) drawn on the y-axis we draw the straight lines of family (II) assigning to them numbers contained in the given interval $(\underline{v}, \overline{v})$.

3. Using equation (III) we find the intersection points of the straight line corresponding to the selected value of w_0 with the sides of the rectangle $\underline{u}$, $\overline{u}$, $\underline{v}$, $\overline{v}$. According to the inclination of the line w_0, we take either the intersection with the vertical sides $\underline{u}$, $\overline{u}$ or the intersection with the horizontal sides $\underline{v}$, $\overline{v}$. This means that we draw the functional scales which family (III)

determines on the sides of the rectangle; e.g. on the side AD (Fig. 108), by substituting in equation (III) $x = f_1(u)$, we obtain

$$y = -\frac{g_3(w)\,f_1(\underline{u})+1}{h_3(w)}.$$

Similarly on the side BC we have the functional scale

$$y = -\frac{g_3(w)\,f_1(\bar{u})+1}{h_3(w)},$$

which is obtained by substituting number $f_1(\bar{u})$ for x.

This method of constructing a nomogram, which can also be applied to equations (a) and (b), requires certain modifications in cases where the density of the lattice in different parts of the drawing does not answer the assumptions. If we want to retain the rectilinearity of the nomogram, only projective transformations are admissible.

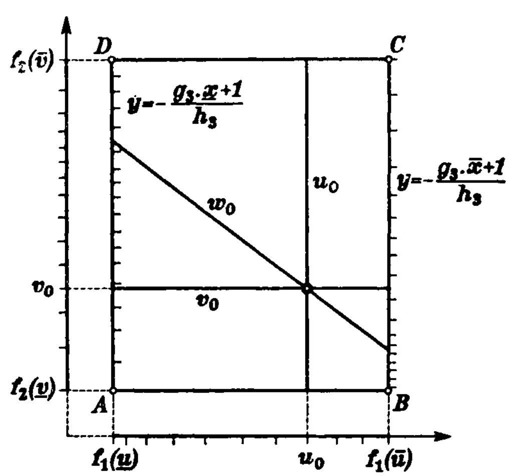

FIG. 108

For example, suppose that the line families u and v are both too dense in the neighbourhood of point C (Fig. 109a) and that we want to move a certain point N on the side DC to the mid-point of that side and a certain point M of the side BC to the mid-point of that side.

It is admissible to begin by moving the system of coordinates

$$x = x'+c_1, \quad y = y'+c_2$$

so as to make point $C(c_1, c_2)$ the origin of the system.

We now determine the coordinates of the points $P(0, p, 1)$ and $Q(q, 0, 1)$, which form with points M or N harmonic pairs separating the vertices B, C and D, C of the rectangle, i.e. such pairs that

$$(BCMP) = -1 \quad \text{and} \quad (DCNQ) = -1.$$

Finally, let us make such a projective transformation of the plane (x, y) as will turn points P and Q into points at infinity

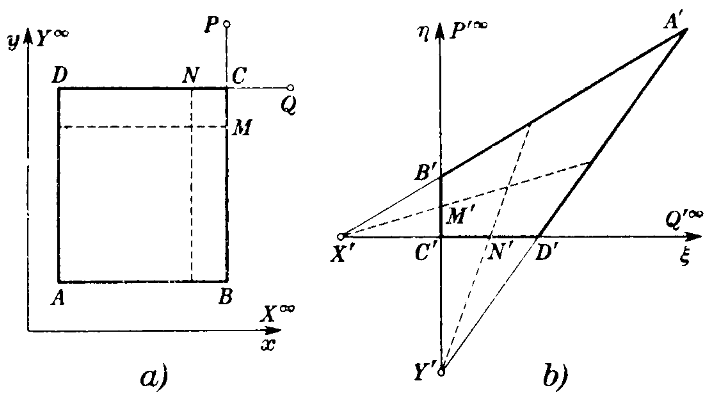

Fig. 109

on the axes of coordinates and leave point C in place. As we know (Chapter I, § 4), this is done by multiplying the three homogeneous coordinates x_1, x_2, x_3 by an inverse of the matrix

$$\begin{bmatrix} q & 0 & 1 \\ 0 & p & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

i.e., by the matrix

$$\begin{bmatrix} p & 0 & -p \\ 0 & q & -q \\ 0 & 0 & pq \end{bmatrix}.$$

The two families of parallel lines, u and v, will be turned in this transformation into pencils of lines. E.g., let us find the coordinates of the vertex X' of pencil v (Fig. 109b). It is a point corresponding of course to the point $X(1, 0, 0)$; we thus have

$$[1 \;\; 0 \;\; 0]\begin{bmatrix} p & 0 & -p \\ 0 & q & -q \\ 0 & 0 & pq \end{bmatrix} = [p \;\; 0 \;\; -p],$$

i.e., point X' has non-homogeneous coordinates -1, 0.

Similarly, we find that point Q has non-homogeneous coordinates 0, -1.

In order to draw the pencils u and v it is now sufficient to find the equations of the scales which those pencils determine on the axes of coordinates. Family u, which determined the scale $x = f_1(u)$ on the x-axis and the scale

$$x' = x - c_1 = f_1(u) - c_1$$

on the x'-axis, will determine after the projective transformation a scale with the equation

$$[x' \;\; 0 \;\; 1]\begin{bmatrix} p & 0 & -p \\ 0 & q & -q \\ 0 & 0 & pq \end{bmatrix} = [px' \;\; 0 \;\; -px' + pq],$$

i.e., the scale

$$\xi_u = \frac{px'}{-px' + pq} = \frac{f_1(u) - c_1}{-f_1(u) + c_1 + q}, \qquad \eta_u = 0.$$

Similarly, we find the v-scale on the η-axis:

$$y' = y - c_2 = f_2(v) - c_2,$$

$$[0 \;\; y' \;\; 1]\begin{bmatrix} p & 0 & -p \\ 0 & q & -q \\ 0 & 0 & pq \end{bmatrix} = [0 \;\; qy' \;\; -qy' + pq],$$

$$\xi_v = 0, \qquad \eta_v = \frac{qy'}{-qy' + pq} = \frac{f_2(v) - c_2}{-f_2(v) + c_2 + p}.$$

The equation of family w is obtained after the transformation also by multiplying the parametric equations of that family by the matrix of the mapping. Family w had in the original system of coordinates the equation

$$g_3(w)\,x+h_3(w)\,y+1=0.$$

We must represent it in a parametric form since that is the only form in which we can make a projective transformation by multiplying matrices. This can be done by taking

$$x=t \quad \text{and} \quad y=-\frac{g_3(w)t+1}{h_3(w)}.$$

Passing to homogeneous coordinates we have

$$x_1=h_3(w)\,t, \quad x_2=-g_3(w)\,t-1, \quad x_3=h_3(w).$$

It is only now that we multiply by the matrix of the transformation

$$[h_3t \quad -g_3t-1 \quad -h_3]\begin{bmatrix} p & 0 & -p \\ 0 & q & -q \\ 0 & 0 & pq \end{bmatrix}$$

$$=[ph_3t \quad -qg_3t-q \quad -ph_3t+qg_3t+q+pqh_3]$$

and obtain the parametric equations of family (III′):

$$\xi_{\text{III}}=\frac{ph_3t}{-ph_3t+qg_3t+q+pqh_3}, \qquad \eta_{\text{III}}=\frac{-qg_3t-q}{-ph_3t+qg_3t+q+pqh_3}.$$

It will be observed that the procedure here described can also be applied to the case where we want to make a projective transformation of a given quadrangle drawn on a lattice nomogram into a rectangle. What is essential in this method is that for each family of lines (I), (II) and (III) we must write parametric equations and then multiply the one-row matrix whose terms are the right sides of the parametric equations by the matrix of the transformation.

d. For the Clark equation

$$f_1(u)\,f_2(v)\,g_3(w)+[f_1(u)+f_2(v)]\,h_3(w)+1=0, \tag{d}$$

the construction of a lattice nomogram is not so direct as it is for the equations discussed in points a, b and c, where either three or two families of lines were pencils of straight lines. Here the geometrical aspect is more complex—we are confronted with three families of straight lines tangent to certain curves.

Analytically, however, we have the same kind of difficulties to deal with as before, such as occur with all the types considered in the chapter on collineation nomograms.

As we know, the Clark equation can be written in the form

$$\begin{vmatrix} 1 & -f_1 & f_1^2 \\ 1 & -f_2 & f_2^2 \\ g_3 & h_3 & 1 \end{vmatrix} = 0.$$

In § 12 we assumed the terms of the first, second and third row of this determinant to be the coordinates of three points, u, v and w. We now regard these numbers as the coordinates of a straight line and assume

$$\begin{aligned} u_1 &= 1, & u_2 &= -f_1(u), & u_3 &= f_1^2(u), \\ u_1' &= 1, & u_2' &= -f_2(v), & u_3' &= f_2^2(v), \\ u_1'' &= g_3(w), & u_2'' &= h_3(w), & u_3'' &= 1. \end{aligned}$$

We obtain the following three families of straight lines, whose equations on dividing by x_3 can be represented in a non-homogeneous form:

$$x - f_1(u)y + f_1^2(u) = 0 \quad \text{(family } u\text{)},$$

$$x - f_2(v)y + f_2^2(v) = 0 \quad \text{(family } v\text{)},$$

$$g_3(w)x + h_3(w)y + 1 = 0 \quad \text{(family } w\text{)}.$$

Of course, if the domain which interests us in a given problem requires certain deformations, we can do it by means of the same method as before, consisting in multiplication by a suitably chosen matrix.

For example suppose that the variables u and w vary in the intervals $\underline{u} \leqslant u \leqslant \overline{u}$, $\underline{w} \leqslant w \leqslant \overline{w}$ (Fig. 110). Suppose we want to turn a quadrilateral with sides $\underline{u}$, $\overline{u}$, $\underline{w}$, $\overline{w}$ into a rectangle with sides parallel to the axes of coordinates. Accordingly, we write the matrix of the coordinates of the diagonal lines p, q and r:

$$\mathfrak{A} = \begin{bmatrix} u_{1p} & u_{2p} & u_{3p} \\ u_{1q} & u_{2q} & u_{3q} \\ u_{1r} & u_{2r} & u_{3r} \end{bmatrix}$$

and we find the inverse matrix $\mathfrak{A}^{-1}$.

Reasoning as in § 4, where we transformed a plane by writing formulas for the coordinates of points, we can see that the

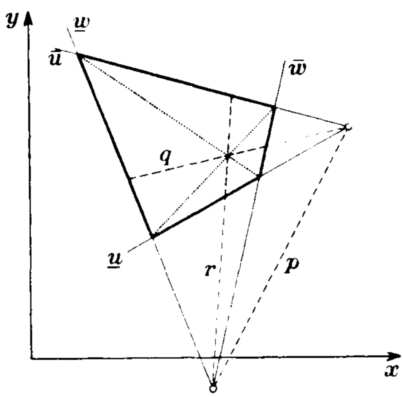

FIG. 110

transformation of the coordinates of straight lines is defined by the product of matrices

$$\begin{bmatrix} 1 & -f_1 & f_1^2 \\ 1 & -f_2 & f_2^2 \\ g_3 & h_3 & 1 \end{bmatrix} \mathfrak{A} .$$

e. The Soreau equation of the first kind

$$f(u) = \frac{f_2(v)+f_3(w)}{g_2(v)+g_3(w)}, \tag{e}$$

can be written, as we know (§ 18), in the form

$$\begin{bmatrix} f_1 & 1 & 0 \\ f_2 & g_2 & -1 \\ f_3 & g_3 & 1 \end{bmatrix} = 0.$$

Regarding the terms of each row as threes of coordinates of straight lines, we have, just as in the case of the Clark equation,

$$f_1(u)x+y+0 = 0 \quad \text{(family } u\text{)},$$
$$f_2(v)x+g_2(v)\,y-1 = 0 \quad \text{(family } v\text{)},$$
$$f_3(w)x+g_3(w)\,y+1 = 0 \quad \text{(family } w\text{)}.$$

The lines of the first family form a pencil with the vertex at the origin of the system; the lines of each of the other two families are tangents to certain curves C_v, or C_w (Fig. 111).

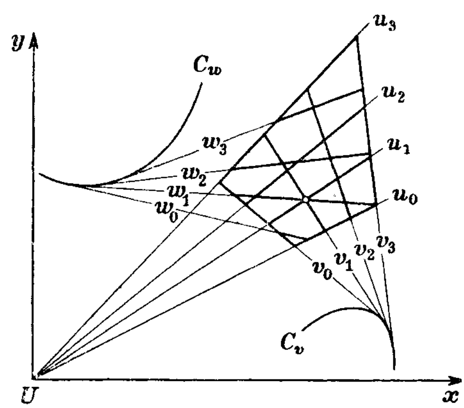

FIG. 111

In cases where the given range for the variables involves a deformation of the drawing, and we want to retain its rectilinearity, we make a transformation of the plane as described in point d.

f. The Soreau equation of the second kind

$$\frac{f_1(u)+f_2(v)}{g_1(u)+g_2(v)} = \frac{f_1(u)+f_3(w)}{g_1(u)+g_3(w)} \tag{f}$$

can be written, as has been seen in § 13, in the form

$$\begin{bmatrix} f_1 & g_1 & 1 \\ f_2 & g_2 & 1 \\ f_3 & g_3 & -1 \end{bmatrix} = 0.$$

Again, let the numbers figuring in the rows denote the coordinates of straight lines. Their equations are obtained by writing

$$f_1(u)x+g_1(u)y+1 = 0 \qquad \text{(family } u\text{)},$$
$$f_2(v)x+g_2(v)y-1 = 0 \qquad \text{(family } v\text{)},$$
$$f_3(w)x+g_3(w)y-1 = 0 \qquad \text{(family } w\text{)}.$$

If the problem required a deformation of the domain in which the nomogram is contained, the procedure would be similar to that shown in d.

Exercises

1. Draw a rectilinear lattice nomogram for the equation

$$3Q = 10(B-0{\cdot}24)H^{3/2}$$

for the intervals $0 \leqslant H \leqslant 5$, $0 \leqslant B \leqslant 5$, $0 \leqslant Q \leqslant 150$.

2. Draw a rectilinear lattice nomogram for the equation

$$z^3+z^2x-3z^2y-3xyz+2 = 0,$$

in which the parameters x and y run over the intervals $1{\cdot}5 \leqslant x \leqslant 2{\cdot}5$, $1 \leqslant y \leqslant 2{\cdot}3$.

3. Draw a rectilinear lattice nomogram for the equation

$$w = 0{\cdot}95\sqrt{(u+v)/uv},$$

where u and v run over the interval from 3 to 80.

4. Draw a rectilinear lattice interval for the equation

$$\frac{2\sqrt{2}}{3}\pi\varrho = \frac{R^2+Rr+r^2}{R+r}$$

for the intervals $0 \leqslant r \leqslant 3$, $3 \leqslant R \leqslant 10$.

5. Draw a lattice nomogram for the equation

$$\varrho = v_0t+\tfrac{1}{2}gt^2$$

for $0 \leqslant v_0 \leqslant 10$ and $0 \leqslant t \leqslant 60$.

CHAPTER IV

EQUATIONS WITH MANY VARIABLES

§ 23. Collineation nomograms of many variables

In Chapter III we dealt with the method of constructing collineation nomograms for certain types of equations containing three variables and with the ways of drawing lattice nomograms for any equation with three variables. Passing to equations with four or more variables we shall observe that certain types can be solved by the reduction to two or more nomograms for functions of three variables; however, there exists no method for constructing a nomogram on a plane for every function of four (or more) variables.

We shall list here those equations of more than three variables which can be represented in a simple manner by nomograms similarly constructed to those discussed in Chapter III.

23.1. Suppose we are given the equation of four variables

$$f(u, v, w, t) = 0,$$

which can be reduced to the form

$$\varphi(u, v) = \psi(w, t), \tag{23.1}$$

i.e. in which we can separate two pairs of variables. In this case we introduce a new (fifth) variable s, writing instead of one equation (23.1) two equations,

$$\varphi(u, v) = s \quad \text{and} \quad \psi(w, t) = s. \tag{23.2}$$

Let us now draw a lattice nomogram for the equation $s = \varphi(u, v)$ adopting a system of coordinates in which u (or v) is represented by a family of straight lines $x = u$, and s by a family of straight lines $y = s$ (Fig. 112); the lines of family v (or u) will, on the whole, be curves.

In addition, let us draw a lattice nomogram for the equation $s = \psi(v, t)$, adopting a system of coordinates in which v (or t)

is represented by lines $x = w$, and s by lines $y = s$; in this system to the values of the variable t corresponds a family of curves with the equation $y = \psi(x, t)$.

The manner of using a nomogram prepared in this way is obvious; e.g. given the numbers u_0, v_0 and t_0, we find the point P of intersection of the straight line u_0 with the curve v_0, we draw through it a parallel to the x-axis, intersect the curve t_0 at point Q and read the value w_0 assigned to the straight line of family w which passes through Q.

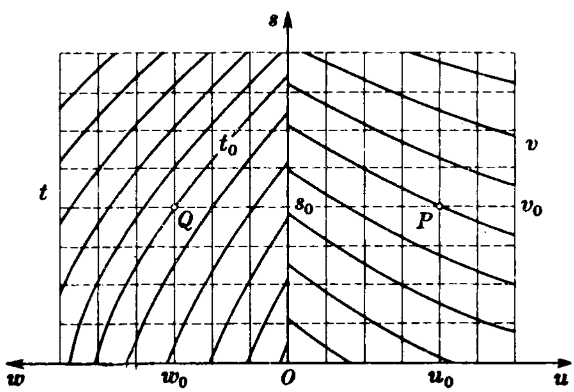

FIG. 112

This suggests a few remarks concerning the nomogram in Fig. 112. It consists of two lattice nomograms for the two equations (23.2) having a common family of lines s. The identity of family s for both parts of the nomogram is its only essential feature. We can thus change the scale on the x-axis, assuming $x' = g(x)$ for the first equation of (23.2) and $x' = g_1(x)$ for the second equation of (23.2), where g and g_1 are arbitrary continuous and monotone functions, and change the scale on the ordinate assuming $y' = h(y)$; the manner of reading a nomogram constructed in this way in the system of coordinates x', y' will be the same. Finally, we can subject the drawing to any transformation, e.g. a projection transformation.

In cases where the functions $\varphi(u, v)$ and $\psi(w, t)$ have simple shapes, we can construct for equation (23.2) a more exact nomogram, consisting of functional scales. They are the subject of the next section.

23.2. A special class of equations of four variables is formed by those equations which can be written in the form of a determinant,

$$\begin{vmatrix} \varphi_1(u) & \psi_1(u) & 1 \\ \varphi_2(v) & \psi_2(v) & 1 \\ \varphi_{34}(w,t) & \psi_{34}(w,t) & 1 \end{vmatrix} = 0, \tag{23.3}$$

which contains functions of the variable u in the first row, functions of the variable v in the second row and functions of the variables w and t in the third row. Assume that the terms of the third row satisfy the following condition:

There exist two intervals,

$$\underline{w} \leqslant w \leqslant \overline{w}, \quad \underline{t} \leqslant t \leqslant \overline{t}, \tag{23.4}$$

in which the functions $\varphi_{34}(w, t)$ and $\psi_{34}(w, t)$ have continuous partial derivatives of the first order and we have

$$\begin{vmatrix} \dfrac{\partial\varphi_{34}}{\partial w} & \dfrac{\partial\psi_{34}}{\partial w} \\ \dfrac{\partial\varphi_{34}}{\partial t} & \dfrac{\partial\psi_{34}}{\partial t} \end{vmatrix} \neq 0. \tag{*}$$

As we know, in this case equations

$$x = \varphi_{34}(w, t), \quad y = \psi_{34}(w, t) \tag{23.5}$$

define a correspondence between points of the rectangle (23.4) and points of a certain domain D (Fig. 113) on the plane (x, y) such that for every point (w_0, t_0) of the rectangle there exists a neighbourhood which has on the plane (x, y) a certain small domain D_0 corresponding to it in a one-to-one manner.

Drawing nomograms for equations (23.3) we shall always divide the rectangle (23.4) into small rectangles in which the correspondence (23.5) is one-to-one.

Let us take a certain number w_0 of the interval $(\underline{w}, \overline{w})$ and draw the curve

$$x = \varphi_{34}(w_0, t), \quad y = \psi_{34}(w_0, t).$$

We shall call it the w_0-*line*. It will be observed that the assumption of the bi-uniqueness of transformation (23.5) implies that the lines w_0 and w_0' corresponding to different numbers w_0 and w_0' have no point in common in our domain.

Similarly, for every number t_0 of the interval $(\underline{t}, \bar{t})$ we have a t_0-*line* defined by the equations

$$x = \varphi_{34}(w, t_0), \qquad y = \psi_{34}(w, t_0), \tag{23.6}$$

and the lines corresponding to different numbers have no point in common in the domain under consideration.

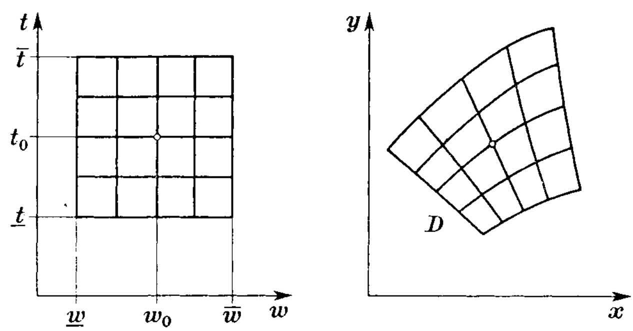

FIG. 113

Proceeding to the construction of a nomogram for equation (23.3) we draw

1. A functional scale with equations containing the parameter u

$$x_1 = \varphi_1(u), \qquad y_1 = \psi_1(u),$$

2. A functional scale with equations containing the parameter v

$$x_2 = \varphi_2(v), \qquad y_2 = \psi_2(v),$$

3. Two families of lines: lines (23.5) of the w-family, each of them corresponding to a certain value of the variable w, and lines (23.6) of the t-family, each of them corresponding to a certain value of the variable t.

Let us take three points: A_u of the u-scale, A_v of the v-scale and A_{wt}, the intersection point of the w-line and the t-line

(Fig. 114). Since the coordinates of point A_{wt} are the terms of the third line of determinant (23.3), the points A_u, A_v and A_{wt} are seen to lie on a straight line if and only if the corresponding numbers u, v, w and t satisfy equation (23.3).

Figure 114 is thus a collineation nomogram. It differs from the collineation nomograms for equations with three variables in having a lattice composed of two families of lines instead of one functional scale.

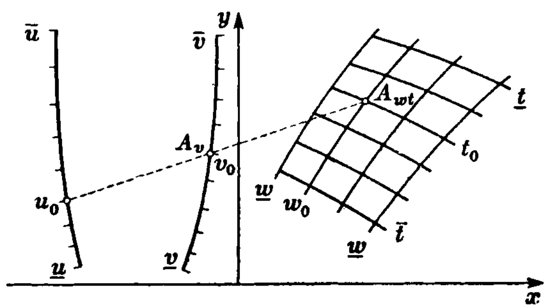

FIG. 114

Assume that condition (*) is not satisfied at any point of a certain domain D, i.e., that

$$\begin{vmatrix} \dfrac{\partial\varphi_{34}}{\partial w} & \dfrac{\partial\psi_{34}}{\partial w} \\ \dfrac{\partial\varphi_{34}}{\partial t} & \dfrac{\partial\psi_{34}}{\partial t} \end{vmatrix} = 0$$

in domain D.

Then, as we know, there exists a function of one variable $f(x)$ such that the equation

$$\psi_{34}(w, t) = f\big(\varphi_{34}(w, t)\big)$$

is satisfied for all points of domain D.

In this case every pair of values (w, t) has a corresponding point lying on the curve

$$y = f(x).$$

Consider the following two figures:

1. A collineation nomogram of the equation

$$\begin{vmatrix} \varphi_1(u) & \psi_1(u) & 1 \\ \varphi_2(v) & \psi_2(v) & 1 \\ r & f(r) & 1 \end{vmatrix} = 0,$$

i.e., the figure shown in Fig. 115.

2. A lattice nomogram of the equation

$$y = \psi_{34}(w, t),$$

in which the values of y are represented by straight lines parallel to the x-axis, the values of w (or t) by lines parallel to the y-axis, and the values of t (or v) by curves $y' = \psi_{34}(w, t)$, the value of t (or w) being constant. This nomogram is shown in Fig. 116.

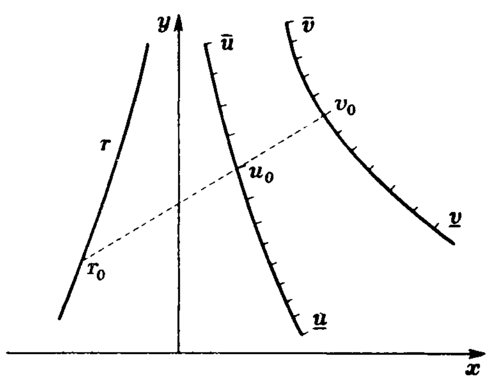

FIG. 115

Let us join the two nomograms. We shall obtain a nomogram shown in Fig. 117, which is a combined nomogram of equation (23.3) in the case of the interdependence of functions φ_{34} and ψ_{34}.

We use this nomogram in the following manner. Choosing, for example, arbitrary values for v_0, w_0 and t_0 we find the lines w_0 and t_0, and then we draw from their intersection point a line parallel to the x-axis as far as the intersection with the curve r at the point R. Joining the points R and v_0 by a straight line we find at the intersection with the u-scale the required value of u_0. If we were given the values u_0, v_0 and, for instance, t_0, we should first determine point R by the intersection of the straight line

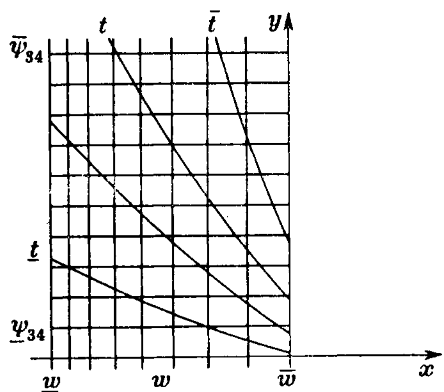

FIG. 116

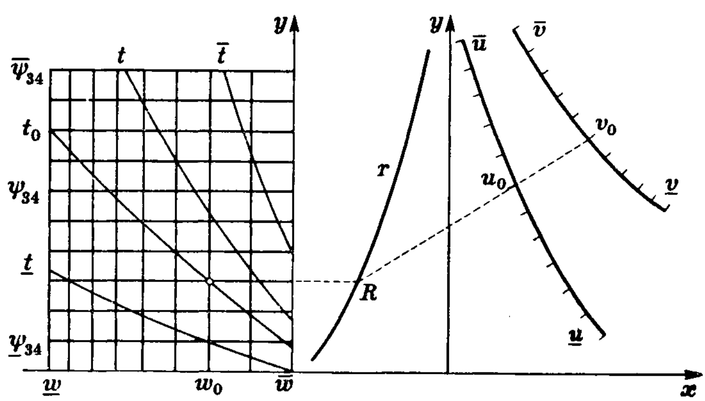

FIG. 117

joining u_0 and v_0 and then, drawing from R a parallel to the x-axis and intersecting the t_0-line we should find the value w_0.

Remark. Since in Fig. 112 from the intersection points of the lines w and t parallel lines to the x-axis are drawn, no deformation of the lattice $[w, t, y]$ that turns lines parallel to y

into lines parallel to y will alter the significance of the nomogram. This means that the scale of the variable w on the x-axis can be chosen arbitrarily.

23.3. We shall now deal with the question what equations can be reduced to form (23.3).

On the basis of the considerations in Chapter III it can easily be observed that if in the equations

$$f_1g_3+f_2h_3+1 = 0 \qquad \text{(the Cauchy equation)},$$

$$f_1f_2g_3+(f_1+f_2)h_3+1 = 0 \qquad \text{(the Clark equation)},$$

$$f_1 = \frac{f_2+f_3}{g_2+g_3} \qquad \text{(the Soreau equation I)},$$

$$\frac{f_1+f_2}{g_1+g_2} = \frac{f_1+f_3}{g_1+g_3} \qquad \text{(the Soreau equation II)},$$

the functions $g_3(w)$, $h_3(w)$ (or $f_3(w)$) were replaced by functions of two variables, then, using the determinant forms of these equations, we could obtain equations of type (23.3).

It is thus obvious that the following equations can be represented by collineation nomograms with two scales and one lattice of lines:

$$f_1(u)g_{34}(w,t)+f_2(w)h_{34}(w,t)+1 = 0,$$

$$f_1(u)f_2(v)g_{34}(w,t)+[f_1(u)+f_2(v)]h_{34}(w,t)+1 = 0,$$

$$f_1(u) = \frac{f_2(v)+f_{34}(w,t)}{g_2(v)+g_{34}(w,t)},$$

$$\frac{f_1(u)+f_2(v)}{g_1(u)+g_2(v)} = \frac{f_1(u)+f_{34}(w,t)}{g_1(u)+g_{34}(w,t)}.$$

EXAMPLE. Construct a nomogram for the equation

$$\Theta = R^2(m_p/2+m_0) \tag{23.7}$$

where R varies in the interval from 30 to 160, while m_p and m_0 vary in the interval from 0·0005 to 0·01.

Equation (23.7) can be written in the form

$$-\Theta\frac{1}{R^2m_0}+\frac{m_p}{2}\cdot\frac{1}{m_0}+1 = 0.$$

This is a Cauchy equation in which the factors containing two variables, R and m_0, are the functions $1/R^2m_0$ and $1/m_0$. Therefore we can use the transformation shown in § 16 and write the equation in the form of a determinant:

$$\begin{vmatrix} 1 & 0 & \Theta \\ 0 & 1 & -m_p/2 \\ 1/R^2m_0 & 1/m_0 & 1 \end{vmatrix} = 0.$$

The nomogram is composed of

1. A rectilinear scale with equations

$$x_1 = 1/\Theta, \quad y_1 = 0, \tag{23.8}$$

2. A rectilinear scale with equations

$$x_2 = 0, \quad y_2 = -2/m_p, \tag{23.9}$$

3. A lattice of two families of lines with parametric equations

$$x_{34} = 1/R^2m_0, \quad y_{34} = 1/m_0. \tag{23.10}$$

The Θ-scale will be a part of the x-axis. The m_p-scale will be a part of the y-axis from $y = -2/0{\cdot}01 = -200$ to $y = -2/0{\cdot}0005 = -4000$ (Fig. 118).

The partial derivatives of the functions x_{34} and y_{34} satisfy condition (*) because

$$\begin{vmatrix} -\dfrac{1}{R^2m_0^2} & -\dfrac{1}{m_0^2} \\ \dfrac{-2}{R^3m_0} & 0 \end{vmatrix} = -\frac{2}{R^3m_0^3} \neq 0.$$

The lines $R = R_0$ are straight lines $y = R^2x$; they pass through the origin of the system and have slopes contained between numbers $30° = 900$ and $160° = 25600$.

The lines corresponding to the constant values of m_0 are horizontal straight lines $y = 1/m_0$ contained between the lines

$$\underline{y} = 1/0{\cdot}01 = 100 \quad \text{and} \quad \bar{y} = 1/0{\cdot}0005 = 2000.$$

The nomogram is contained in a quadrilateral with vertices

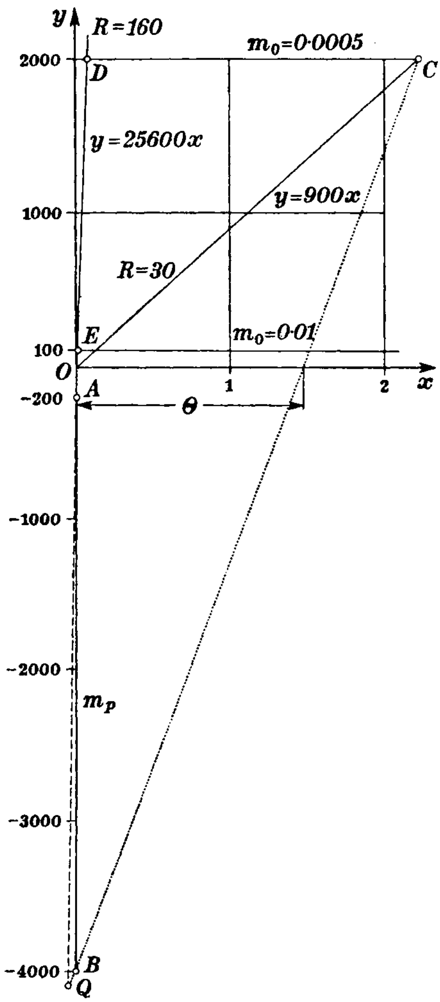

FIG. 118

$B(0, -4000)$, $C(20/9, 2000)$, $D(5/64, 2000)$ and $O(0,\ 0)$. Figure 118 shows that it should be transformed so as to make the scales m_0 and Θ regular and to give to the pencil of lines R, which is very dense in the vicinity of the value $R = 160$, a form approaching the regular.

The first objective will be reached by transferring point O to infinity; it will then be observed that every line R will intersect the pencil of lines m_0 in a regular scale. The second objective, i.e., increasing the accuracy in the pencil R near $R = 160$, will be gained by transferring to infinity a certain line of that pencil contained between the y-axis and the line $R = 160$. We shall choose that line so as to make the end-points of scales m_p and Θ the vertices of a rectangle. Accordingly, we must transfer to infinity point Q, which is the intersection point of the lines BC and AE, E lying at the intersection point of the lines $m_0 = 0{\cdot}01$ and $R = 160$.

Point E has coordinates $y_E = 100$ and $x_E = 100/25600 = 1/256$.

The straight line AE has the equation

$$y+200 = \frac{300}{1/256}x \quad \text{or} \quad y = 76800x-200.$$

The straight line BC has the equation

$$y+4000 = \frac{6000}{20/9}x \quad \text{or} \quad y = 2700x-4000.$$

These equations give us the coordinates of point Q:

$$x_Q = -2/39, \quad y_Q = -50200/13,$$

and the equation of the straight line OQ, which we have decided to transfer to infinity:

$$y = 75300x.$$

Moreover, let us transform the y-axis into an η-axis and the line at infinity into an ξ-axis. Passing to homogeneous coordinates let us write explicitly that:

the straight line $75300x_1-x_2 = 0$ is transformed into the line at infinity,

the line at infinity $x_3 = 0$ is transformed into the line $\eta = 0$,

the y-axis, i.e. the line $x_1 = 0$ is transformed into the line $\xi = 0$.

From the considerations of § 4, p. 39, we derive the formulas of transformation

$$y_1 = x_1, \quad y_2 = x_2, \quad y_3 = 75300x_1 - x_2.$$

Assuming $x_1/x_3 = x$, $x_2/x_3 = y$, $y_1/y_3 = \xi$ and $y_2/y_3 = \eta$, i.e., passing to non-homogeneous coordinates, we finally obtain

$$\xi = x/(75300x - y), \quad \eta = 1/(75300x - y). \tag{23.11}$$

We obtain the equations of the Θ-scale by substituting the right sides of equations (23.8) in equations (23.11):

$$\xi_\Theta = 1/75300, \quad \eta_\Theta = \Theta/75300.$$

We obtain the equation of the m_p-scale by substituting the right sides of equations (22.9) in equations (22.11):

$$\xi_p = 0, \quad \eta_p = m_p/2.$$

The equations of the lattice of two line families are obtained by substituting the right sides of equations (23.10) in equations (23.11):

$$\xi = 1(75300 - R^2), \quad \eta = R^2 m_p/(75300 - R^2).$$

Taking a constant for R, we can see that lines m_p form a pencil of lines with its vertex at the origin of the system,

$$\eta = 75300\, m_p \xi,$$

and taking a constant for m_p, we can see that the lines R form a pencil of lines parallel to the η-axis,

$$\xi = 1/(75300 - R^2).$$

By a suitable choice of units on the axes of the system we obtain a nomogram represented in Fig. 119.

23.4. Assume that we have an equation with five variables

$$f(u, v, w, t, s) = 0$$

which can be reduced to the form

$$\begin{vmatrix} \varphi_1(u) & \psi_1(u) & 1 \\ \varphi_{23}(v, w) & \psi_{23}(v, w) & 1 \\ \varphi_{45}(t, s) & \psi_{45}(t, s) & 1 \end{vmatrix} = 0.$$

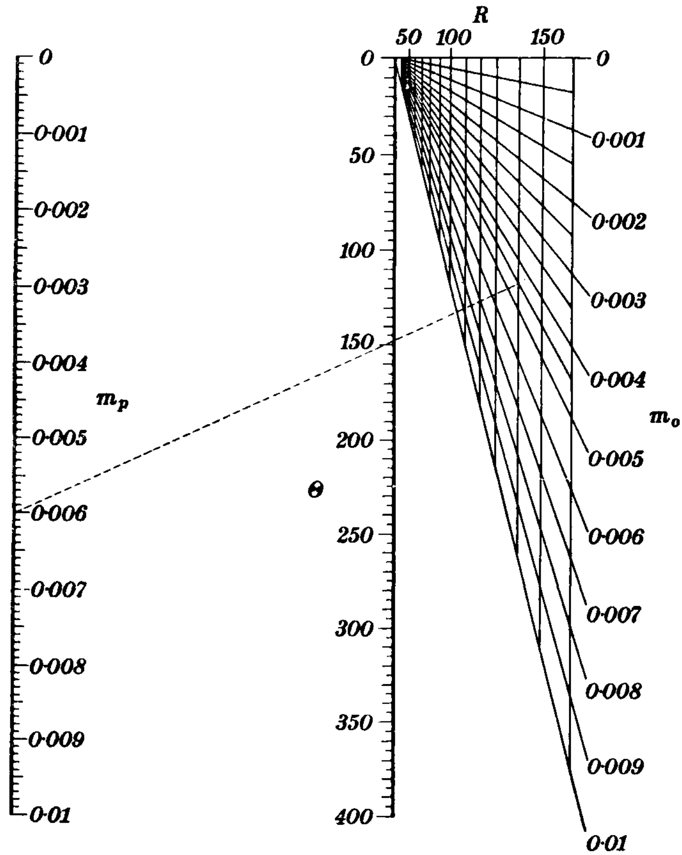

FIG. 119

Equations

$$x_1 = \varphi_1(u), \quad y_1 = \psi_1(u) \tag{23.12}$$

represent, in general, a curvilinear scale Z_1.

Let us take equations

$$x_2 = \varphi_{23}(v, w), \quad y_2 = \psi_{23}(v, w). \tag{23.13}$$

If the variable v runs over an interval (a, b) and the variable w over an interval (c, d), then on a plane where the axes are marked with the letters v and w pairs of numbers v, w denote points contained in a rectangular domain. Equations (23.13) define the transformation of this rectangle into a certain set Z_{23} of the plane (Fig. 120), where x and y are orthogonal coordinates. Now let us assume that the set Z_{23} is a domain and that the equations in question define a one-to-one correspondence between the points of this set and the points of the rectangle on the plane (v, w).

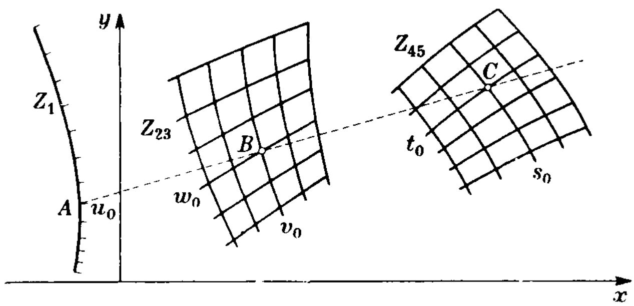

FIG. 120

Let us mark on the set Z_{23} the lines corresponding to constant values of v_0 and the lines corresponding to constant values of w_0. The curve corresponding to the value of v_0 has of course an equation with parameter w,

$$x = \varphi_{23}(v_0, w), \qquad y = \psi_{23}(v_0, w),$$

and the curve corresponding to the value of w_0 has an equation with parameter v,

$$x = \varphi_{23}(v, w_0), \qquad y = \psi_{23}(v, w_0).$$

In addition, let us assume that the equations

$$x_{45} = \varphi_{45}(t, s), \qquad y_{45} = \psi_{45}(t, s) \tag{23.14}$$

also transform a certain rectangle of the plane (t, s) defined by the inequalities

$$e \leqslant t \leqslant f, \qquad g \leqslant s \leqslant h$$

into a plane domain Z_{45} in a one-to-one manner. Moreover, let us mark on the set Z_{45}, as on set Z_{23}, lines corresponding to constant values of the variable t and lines corresponding to constant values of the variable s.

We have obtained a drawing consisting of a scale Z_1, on which points assigned to the values of the variable u are marked, of a set Z_{23}, which is a lattice of lines assigned to the values of the variables v and w, and of a set Z_{45}, which is a lattice of lines assigned to the values of the variables t and s. It will be observed that the five numbers

$$u_0, \; v_0, \; w_0, \; t_0, \; s_0 \tag{23.15}$$

have three points in the drawing assigned to them: number u_0 has a corresponding point A of the set Z_1, the pair of numbers v_0, w_0 have a corresponding point B of the set Z_{23} and the pair of numbers t_0, s_0 have a corresponding point C of the set Z_{45}. Obviously, points A, B and C lie on a straight line if and only if numbers (23.15) satisfy equation (23.7); Fig. 120 is thus a collineation nomogram for that equation.

Replacing in the Cauchy, Clark, Soreau I and Soreau II equations two functions of the first variable by a pair of functions dependent on two variables and two functions of the second variable by a new pair of functions depending on the new variables, we shall obviously obtain types of equations with five variables which can be represented by collineation nomograms with two lattices of lines and one scale.

23.5. Collineation nomograms can also be constructed for certain equations containing six variables, u_1, u_2, u_3, u_4, u_5 and u_6. This holds when the equation is of the form

$$\begin{vmatrix} \varphi_{12} & \psi_{12} & 1 \\ \varphi_{34} & \psi_{34} & 1 \\ \varphi_{56} & \psi_{56} & 1 \end{vmatrix} = 0\,, \tag{23.16}$$

where φ_{ik} and ψ_{ik} are functions of the variables u_i, u_k satisfying the condition of bi-uniqueness of the correspondence

$$x = \varphi_{ik}(u_i, u_k), \qquad y = \psi_{ik}(u_i, u_k).$$

The drawing then consists of three lattices: Z_{12} with lines u_1 and u_2, Z_{34} with lines u_3 and u_4 and Z_{56} with lines u_5 and u_6 (Fig. 121).

Points A_{12}, A_{34} and A_{56} lie on a straight line if and only if the pairs of lines u passing through them,

$$u_1,\ u_2, \quad u_3,\ u_4, \quad u_5,\ u_6,$$

satisfy equation (23.16).

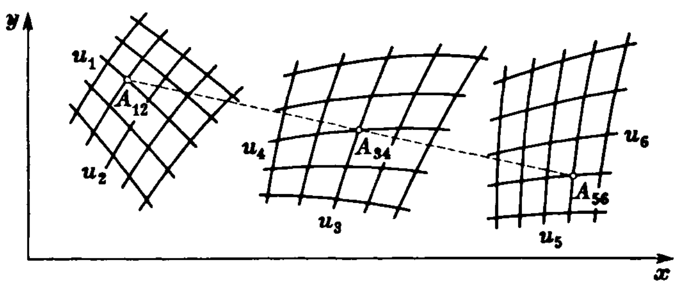

FIG. 121

§ 24. Elementary geometrical methods of joining nomograms

In this section we shall present a few very special types of equations with four and more variables, for which nomograms composed of collineation and lattice nomograms can be constructed by simple geometrical methods. They are equations frequently encountered in practice, and that is why it is particularly important to know the methods of working them out.

24.1. Consider the equation

$$f_1(u)+f_2(v) = f_3(w)+f_4(t). \tag{a}$$

In this case we draw two nomograms:

1. A nomogram with three scales on parallel lines for the equation

$$f_1(u)+f_2(v) = a,$$

2. A nomogram with three scales on parallel lines for the equation

$$a = f_3(w)+f_4(t).$$

We then join the two by superposing identical scales on one another.

This method can be generalized to a greater number of variables. Given the equation

$$f_1(u)+f_2(v) = f_3(w)+f_4(t)+f_5(s),$$

we construct three nomograms for the equations

$$f_1(u)+f_2(u) = \alpha, \quad \alpha = f_3(w)+\beta, \quad \beta = f_4(t)+f_5(s)$$

and join them together by superposing the α-scale of the second nomogram upon the (identical) α-scale of the first nomogram, and then superposing the β-scale of the third nomogram on the (identical) β-scale of the second nomogram.

EXAMPLE 1. Construct a nomogram for the equation

$$\tau = 5\varphi M/d^3$$

for the intervals $30 \leqslant d \leqslant 100$, $1{\cdot}5 \leqslant \varphi \leqslant 4$, $5 \leqslant M \leqslant 100$, $0{\cdot}001 \leqslant \tau \leqslant 0{\cdot}1$.

Write the equation in the form

$$\log \tau + 3 \log d = \log 5 + \log \varphi + \log M$$

and assume

$$\log \tau = u, \quad 3 \log d = v, \quad \log \varphi + \log 5 = w, \quad \log M = t.$$

To begin with, let us draw a nomogram for the equation

$$u+v = w+t;$$

as follows from the intervals of the variables φ, M and d, we have

$$-3 \leqslant u \leqslant -1, \quad 4{\cdot}5 \leqslant v \leqslant 6, \quad 0{\cdot}8 \leqslant w \leqslant 1{\cdot}3, \quad 0{\cdot}7 \leqslant t \leqslant 2.$$

Accordingly, let us construct two nomograms for two equations

$$u+v = \alpha \quad \text{and} \quad w+t = \alpha.$$

1. We draw a nomogram (Fig. 122) for the first equation selecting scales u and v and marking only the construction of points 2_α and 3_α; similarly, we draw a nomogram for the second equation selecting 2_α and 3_α, and also 1_t and 2_t, and finding 1_w and 2_w by construction.

2. It is now easy to join the two nomograms so as to make points 2_a of the two drawings coincide and points 3_a of the two drawings coincide.

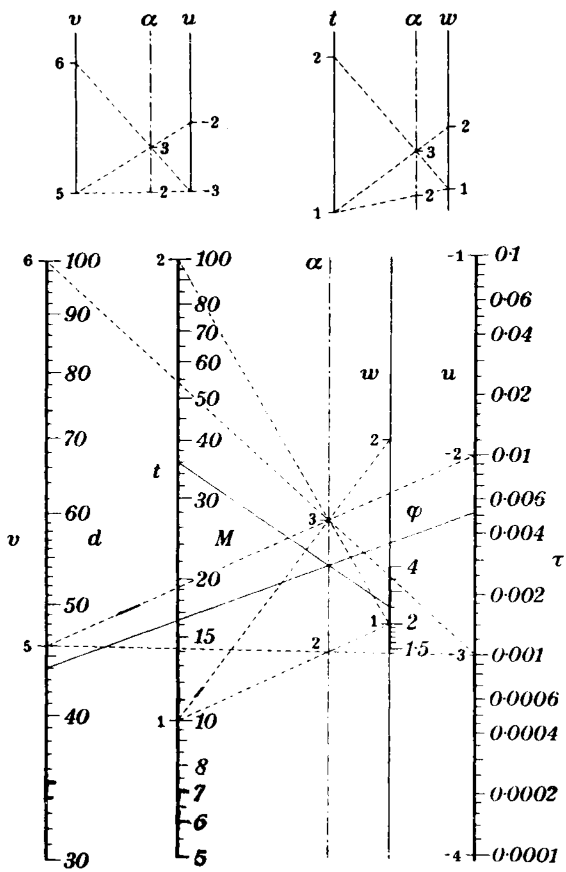

FIG. 122

We have obtained a nomogram in which we replace the scales u, v, w and t by logarithmic scales τ, d, φ and M.

24.2. Equation

$$\frac{1}{f_1(u)}+\frac{1}{f_2(v)}=\frac{1}{f_3(w)}+\frac{1}{f_4(t)} \tag{b}$$

can be reduced to the preceding form by assuming $1/f_i = \varphi_i$, since then

$$\varphi_1(u)+\varphi_2(v) = \varphi_3(w)+\varphi_4(t);$$

it is now possible to draw a nomogram composed of four parallel scales.

In certain cases, however, particularly when the scales of functions φ_i are unlimited, it is more useful to draw two such nomograms, as shown in § 11.

We then assume $u' = f_1(u)$, $v' = f_2(v)$, $w' = f_3(w)$ and $t' = f_4(t)$ and draw two nomograms for the equations $1/u+1/v = 1/a$ and $1/a = 1/w'+1/t'$, with four regular scales and a common zero point.

By joining the two nomograms so as to make the a-scales coincide, we obtain a combined nomogram for the equation $1/u'+1/v' = 1/w'+1/t'$. Replacing scales u', v', w' and t' in it by the corresponding functional scales, we finally obtain the required nomogram.

R e m a r k. Let us construct for the equations $u+v = a$ and $w+t = a$ nomograms in which the a-scale will be a line at infinity. The senses of the scales u and v will then, of course, be opposite and the units equal; similar senses of the scales w and t will be the same and their units equal (Fig. 123a). Joining the two nomograms means identifying all the points of the straight line of the a-scale. We must therefore choose the units and the distances of the scales u and v, and also w and t, in the two drawings so that points 2_a^∞ will coincide, i.e., that the invariants 2_a^∞ on the two drawings will be equal; and, similarly, that the points 3_a^∞ and 4_a^∞ will coincide. This means that the rectangles $-2_u{-}3_u 6_t 5_v$ and $2_w 1_w 2_t 1_t$ must be similar and similarly placed.

The construction of the nomogram in Fig. 123 does not essentially differ from the method shown in Figs. 43 and 122; after determining equal and identically directed scales u and v and a parallel scale t the point 1 of the scale has been found by intersecting the straight line $2_t 1_w$ parallel to $-2_u 5_v$ and the straight line $1_t 1_w$ parallel to $-3_u 5_v$.

The reading of nomogram 123 is performed by superimposing a transparent sheet on which a family of parallel lines has been drawn (Fig. 123a): when one line of the family passes through the points u_0 and v_0, then another line, passing through t_0, determines on the scale the required value of w_0.

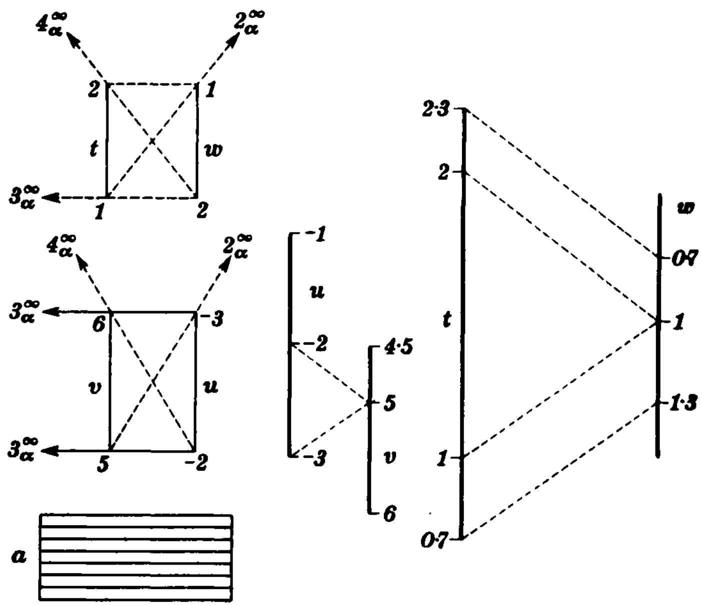

FIG. 123

EXAMPLE 2. Construct a nomogram for the equation

$$u = \sqrt[3]{\frac{3x\sqrt{y}\,z^2}{x\sqrt{y}-z^2\sqrt{y}+2xz^2}}$$

where each variable runs over the interval (0, 1).

Write the equation in the form

$$\frac{3}{u^3}+\frac{1}{x}=\frac{1}{z^2}+\frac{2}{\sqrt{y}}$$

and assume $v' = u^3$, $z' = z^2$, $y' = \sqrt{y}$.

We shall now draw two nomograms for the equations

$$\frac{3}{u'}+\frac{1}{x}=\frac{1}{a}, \tag{I}$$

$$\frac{1}{a}=\frac{1}{z'}+\frac{2}{y'}. \tag{II}$$

I. We select arbitrary points $1_{u'}$, $0_{u'} = 0_{x'}$ and $1_{x'}$ and find the point $0{\cdot}5_a$ by drawing a line joining $0{\cdot}5_x$ with $\infty_{u'}$ and a line joining $1{\cdot}5_{u'}$ with ∞_x (for we have $3/\infty+1/0{\cdot}5 = 1/0{\cdot}5$ and $3/1{\cdot}5+1/\infty = 1/0{\cdot}5$) (Fig. 124).

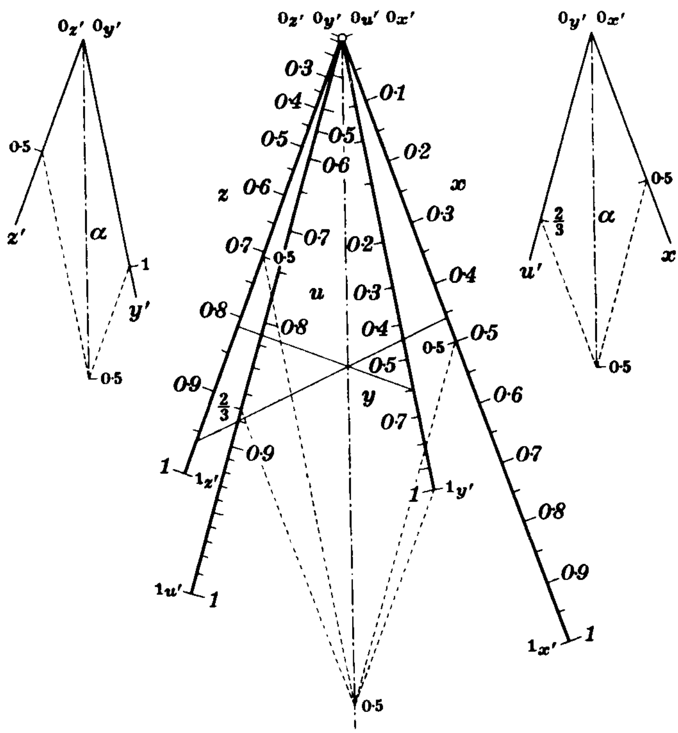

FIG. 124

II. We select arbitrary points $1_{z'}$, $0_{z'} = 0_{y'}$, $1_{y'}$ and find the point $0{\cdot}5_a$ by drawing a line joining $1_{y'}$ with $\infty_{z'}$ and a line joining $0{\cdot}5_{z'}$ with $\infty_{y'}$ (since we have $1/0{\cdot}5 = 1/\infty + 2/1$ and $1/0{\cdot}5 = 1/0{\cdot}5 + 2/\infty$).

We now make a joint drawing in such a way as to make the α-scale common, i.e., to have a common point 0_α and a common point $0{\cdot}5_\alpha$.

Finally, we replace the regular scales u', y', and z' by scales $u' = u^3$, $z' = z^2$, and $y' = \sqrt{y}$.

24.3. Consider the equation

$$f_1(u) + f_2(v) = f_3(w)\, f_4(t). \tag{c}$$

An equation of type

$$u' + v' = w't'$$

can also be replaced by a nomogram composed of two nomograms: it is sufficient to write

$$u' + v' = \alpha, \tag{I}$$

$$\alpha = w't' \tag{II}$$

and join together a nomogram with regular scales u', v' and α and an N-shaped nomogram for equation (II), in which α is a regular scale. Then of course only one variable, w' or t', will be given by a regular scale.

Example 3. Draw a nomogram for the equation

$$\varphi = \frac{R}{r + 0{\cdot}1/Y}$$

for the intervals $0{\cdot}1 \leqslant r \leqslant 0{\cdot}2$, $0{\cdot}7 \leqslant R \leqslant 3$, $1 \leqslant Y \leqslant 10$, $2{\cdot}5 \leqslant \varphi \leqslant 10$.

Write the equation in the form

$$r + 0{\cdot}1/Y = R/\varphi$$

and assume $0{\cdot}1/Y = u$, and then

$$r + u = \alpha, \tag{I}$$

$$\alpha = R/\varphi. \tag{II}$$

I. We draw a nomogram for equation (I), in which u varies from 0·01 to 0·1, marking the points 0·1 and 0·2 of the a-scale (Fig. 125).

II. We draw an N-shaped nomogram for the equation $R/a = \varphi$ having regular scales for a and R with a given a-scale.

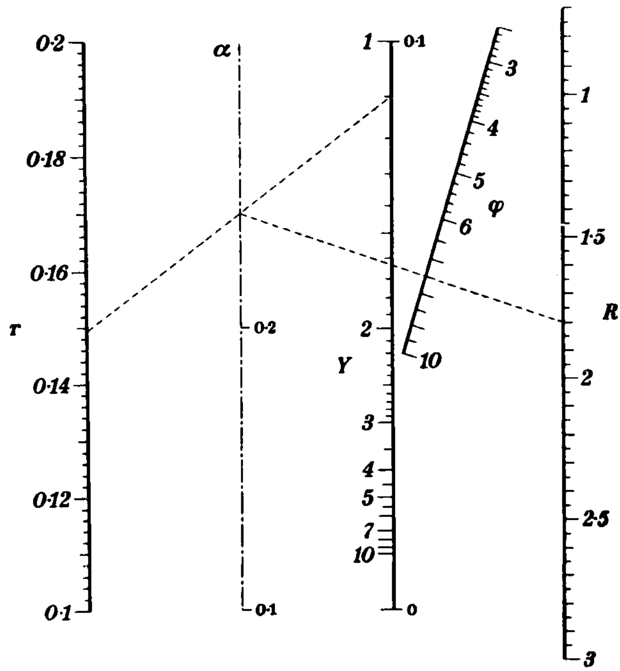

Fig. 125

Joining the two nomograms by making the a-scales coincide, we obtain a drawing which, on replacing the u-scale by a functional scale $u = 0{\cdot}1/Y$, becomes a nomogram for the given equation.

In cases where functions $f_1(u)$ and $f_2(v)$, in the given intervals, have very large values or tend to infinity, we can assume

$$f_1(u) = 1/\varphi_1(u), \quad f_2(v) = 1/\varphi_2(v)$$

and write the equation in the form

$$\frac{1}{\varphi_1(u)} + \frac{1}{\varphi_2(v)} = f_3(w)\, f_4(t);$$

we then construct nomograms for the equations

$$\frac{1}{\varphi_1(u)}+\frac{1}{\varphi_2(v)}=\frac{1}{a}, \tag{I}$$

$$f_3(w)\,f_4(t)=\frac{1}{a}. \tag{II}$$

For example, let us take the equation

$$1/u+1/v = w/t$$

and the intervals $0 \leqslant u \leqslant 1$, $-1 \leqslant v \leqslant 0$, $0{\cdot}5 \leqslant w \leqslant 4$, $1 \leqslant t \leqslant 2$.

I. We draw a nomogram for the equation

$$1/u+1/v = 1/a$$

taking arbitrary points 1_u, $0_u = 0_v$, 1_v and finding 1_a so as to satisfy the equation (Fig. 126a).

II. We construct a nomogram for the equation

$$1/a = w/t$$

with a regular a-scale, selecting the a-scale and a regular t-scale on parallel lines (Fig. 126b).

By joining the two nomograms so as to make points 0_a and 1_a of the two drawings coincide, we obtain the required nomogram (Fig. 126c).

24.4. Considering the equation

$$f_1(u)\,f_2(v) = f_3(w)\,f_4(t)$$

we proceed in the same way as in the preceding cases: we use the substitutions $u' = f_1$, $v' = f_2$, $w' = f_3$ and $t' = f_4$, we draw nomograms for the equations

$$u'v' = a \quad \text{and} \quad a = w't',$$

we join them together, and then introduce functional scales instead of the regular scales u', v', w' and t'.

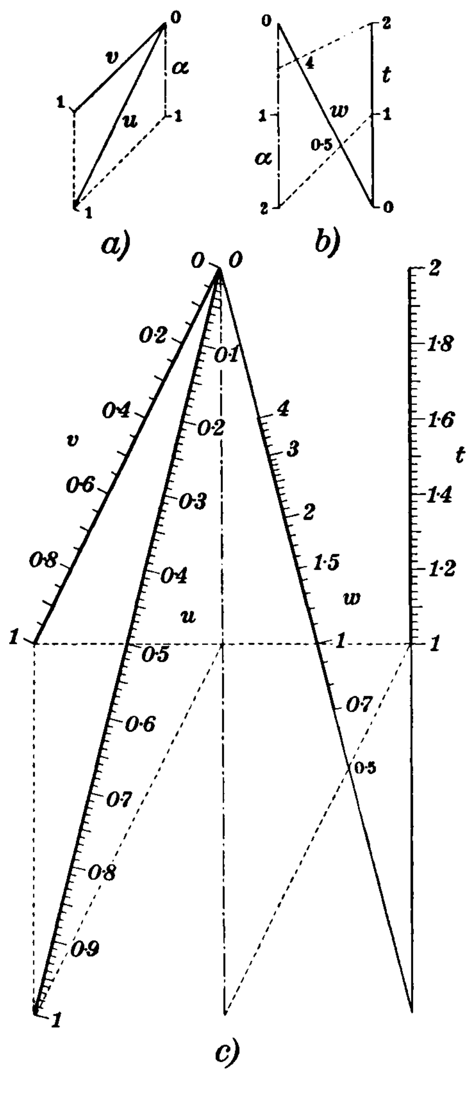

FIG. 126

EXAMPLE 4. Draw a nomogram for the equation

$$u = \sqrt[v]{w^t}$$

where v and t vary in the interval from 1 to 2, and w varies in the interval from 1 to 4.

We write

$$v \log u = t \log w$$

and substitute $vu' = a$ and $a = tw'$, where $u' = \log u$, $w' = \log w$.

I. A nomogram for the equation $u' = a/v$ with regular scales a and v is shown in Fig. 127a.

II. A nomogram for the equation $w' = a/t$ with regular scales a and t is shown in Fig. 127b.

Joining the two nomograms and replacing numbers u' and w' on the projective scales u' and w' by the values of u and w which are assigned to them by equations $u' = \log u$ and $w' = \log w$, we obtain the required nomogram (Fig. 127c).

Scales u and w can also be drawn by projecting an ordinary logarithmic scale drawn of the straight line a.

The problem can also be solved in a different way.

Assume

$$v/w' = t/u'$$

and construct an N-shaped nomogram for equations

$$v/w' = a, \tag{I'}$$

$$t/u' = a. \tag{II'}$$

The scales v and w', t and u' will now be regular and the scale on a will be projective; we mark on the a-scale three points, 0_a, 2_a and ∞_a, both on the nomogram for equation (I′) (Fig. 128a) and on that for equation (II′) (Fig. 128b). On the grounds of the theorem on the unique determination of a projective scale by giving three points of that scale, we can see that the nomograms should be joined in such a manner as to make the points 0_a, 2_a and ∞_a of one figure coincide with the points 0_a, 2_a and ∞_a of the other.

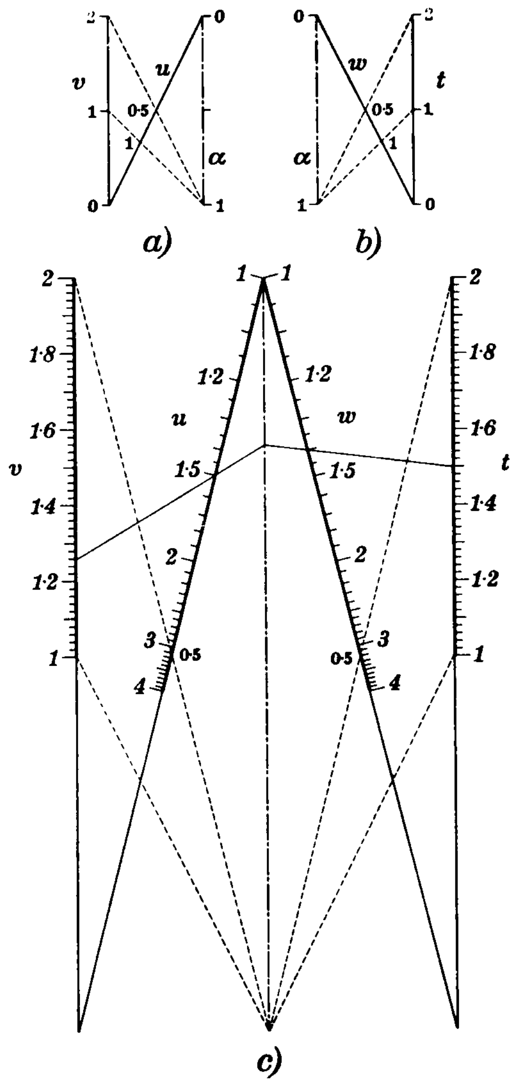

FIG. 127

The nomogram obtained is shown in Fig. 128c; replacing the regular scales w' and u' by logarithmic scales, we shall obtain the ultimate drawing.

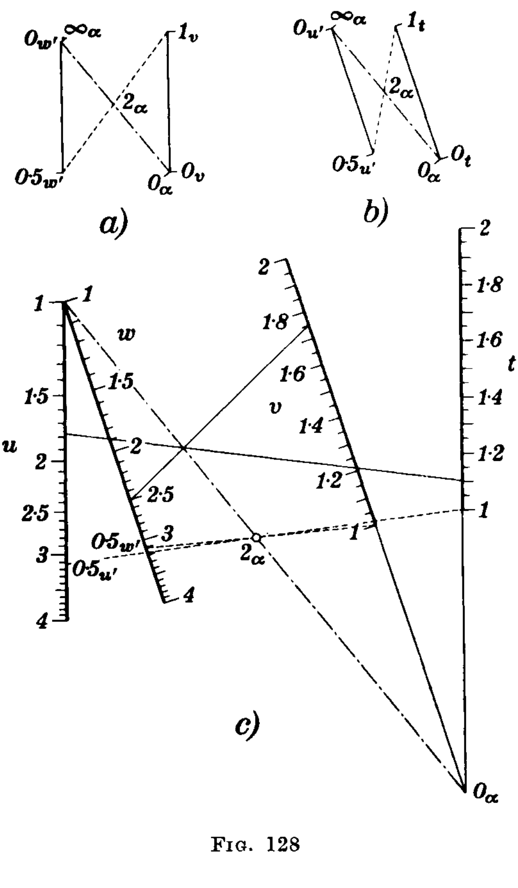

FIG. 128

24.5. Consider the equation

$$f_1(u)+f_2(v)=f_{34}(w,t). \tag{e}$$

If the function $f_{34}(w, t)$ is neither a sum nor a difference of two functions each of which depends on one variable, then we replace equation (e) by two equations,

$$f_1(u)+f_2(v) = \alpha, \tag{I}$$

$$\alpha = f_{34}(w, t); \tag{II}$$

for equation (II), however, we must draw here a family of curves in the Cartesian system α, t or α, w. E.g., if we have chosen the axes α and t (Fig. 129), the values of α will be represented by

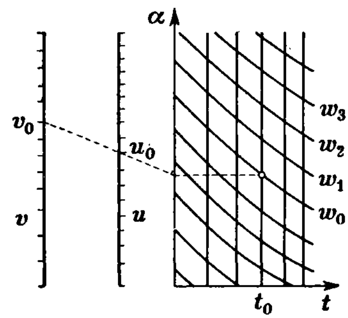

FIG. 129

a regular family of lines $y = \alpha$ or by a regular α-scale (we can have an arbitrary scale on the t-axis). Joining with this drawing the nomogram of equation (I), composed of three scales, of which the α-scale is identical with the previous scale on α, we obtain the ultimate form of the nomogram for equation (e).

It can easily be seen that equation (e) is a particular case of equation (23.3) on p. 204.

Namely, let us assume that functions $\varphi_3(w, t)$ and $\psi_3(w, t)$ are linearly dependent and that, for example,

$$\psi_3 = a\varphi_3 + b.$$

Multiplying the terms of the first column of determinant (23.3) by $-a$ and the terms of the third column of that determinant by $-b$ and adding them to the terms of the second column, we obtain

$$\begin{vmatrix} \varphi_1 & \psi_1 - a\varphi_1 - b & 1 \\ \varphi_2 & \psi_2 - a\varphi_2 - b & 1 \\ \varphi_{34} & 0 & 1 \end{vmatrix} = 0.$$

If, also, $\psi_1 - a\varphi_1 - b = -1$ and $\psi_2 - a\varphi_2 - b = 1$, then obviously

$$\begin{vmatrix} \varphi_1 & -1 & 1 \\ \varphi_2 & 1 & 1 \\ \varphi_{34} & 1 & 1 \end{vmatrix} = \varphi_1(u) + \varphi_2(v) - 2\varphi_{34}(w, t) = 0,$$

i.e., form (e).

24.6. For the equation

$$f_1(u)\, f_2(v) = f_{34}(w, t) \tag{f}$$

the procedure is the same; the difference lies in the fact that, instead of a nomogram for equation (I), we have an N-shaped nomogram for the equation

$$f_1(u)\, f_2(v) = \alpha,$$

in which the regular α-scale coincides with a regular scale of a lattice nomogram for the relation $\alpha = f_{34}(w, t)$ (Fig. 130).

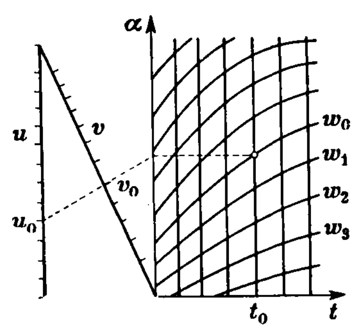

FIG. 130

Equation (f) is also a particular case of equation (23.3) on p. 204. In order to obtain equation (f) from equation (23.3) it suffices, as before, to assume

$$\psi_3 = a\varphi_3 + b$$

and $\psi_1 - a\varphi_1 - b = 1$ and $\varphi_2 = 0$. We obtain the equation

$$\varphi_1(u)\, \psi_2(v) - \big(1 - \psi_2(v)\big)\, \varphi_{34}(w, t) = 0$$

or

$$\varphi_1(u)\,\frac{\psi_2(v)}{1-\psi_2(v)} = \varphi_{34}(w, t),$$

i.e., an equation of type (f).

24.7. Considering the equation

$$f_1(u)\,f_2(v) = f_3(w)\,f_4(t)+g_4(t) \qquad \text{(g)}$$

we substitute

$$f_1(u)\,f_2(v) = \alpha, \quad f_3(w)\,f_4(t)+g_4(t) = \alpha.$$

The nomogram for the first equation is N-shaped; the second is a Cauchy equation since it can be written in the form

$$f_3(w)\,\frac{f_4(t)}{g_4(t)} - \alpha\,\frac{1}{g_4(t)} + 1 = 0;$$

as we know, the nomogram for this equation has a regular or a projective scale on α. In the case of a projective scale we must draw a projective scale on α also in the first nomogram. We shall obtain Fig. 131 or Fig. 132.

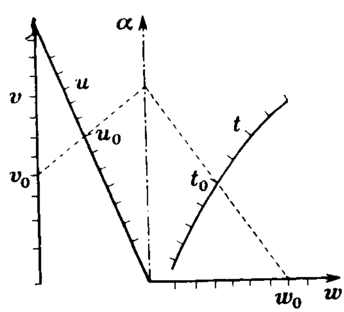

FIG. 131

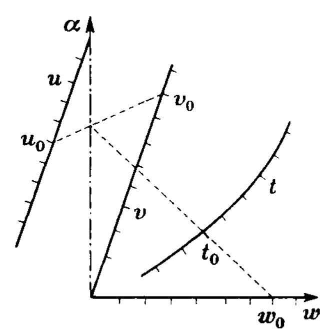

FIG. 132

24.8. Equation

$$f_1(u)+f_2(v)\,g_1(u) = f_3(w)+f_4(t)\,g_3(w) \qquad \text{(h)}$$

can be decomposed into two Cauchy equations,

$$f_1+f_2 g_1 = \alpha, \quad f_3+f_4 g_3 = \alpha,$$

or

$$f_2 \frac{g_1}{f_1} - a \frac{1}{f_1} + 1 = 0, \quad f_4 \frac{g_3}{f_3} - \frac{1}{f_3} + 1 = 0.$$

The nomogram will be composed of two nomograms with a common regular or projective scale a (Fig. 133).

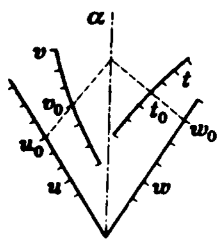

FIG. 133

24.9. Consider the equation

$$\frac{f_1(u)}{f_2(v)} = \frac{f_3(w)+f_4(t)}{1+f_3(w)\,f_4(t)}. \tag{i}$$

Assuming

$$\frac{f_1}{f_2} = a, \quad a = \frac{f_3+f_4}{1+f_3 f_4}$$

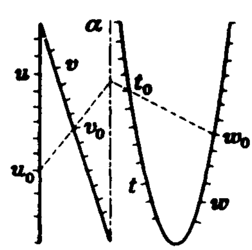

FIG. 134

we have an N-shaped nomogram for the first equation, and for the second a collineation nomogram consisting of a rectilinear a-scale and two scales, w and t, on a curve of the second degree, since it is a Clark equation,

$$f_3 f_4 - (f_3 + f_4)\frac{1}{\alpha} + 1 = 0,$$

where α occurs only in one component. The combined nomogram will be of the form shown in Fig. 134.

24.10. For the equation

$$\frac{f_1(u)+f_2(v)}{g_1(u)+g_2(v)} = \frac{f_3(w)+f_4(t)}{g_3(w)+g_4(t)} \tag{j}$$

we can also construct a nomogram composed of two nomograms with a common α-axis. Assuming

$$\frac{f_1+f_2}{g_1+g_2} = \alpha \tag{I}$$

and

$$\alpha = \frac{f_3+f_4}{g_3+g_4}, \tag{II}$$

we can see that both equation (I) and equation (II) are of the Soreau I type and can be represented by nomograms with a regular α-scale and two curvilinear scales. It is then sufficient to put them together, superimposing the α-scales on one another, in order to obtain a nomogram for equation (j) (Fig. 135).

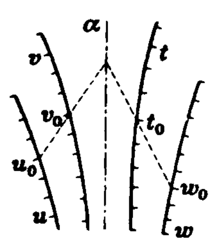

FIG. 135

Exercises

Construct combined nomograms for the following relations:

1. $t = u/vw$ for the intervals $18 \leqslant u \leqslant 500$, $3{\cdot}5 \leqslant v \leqslant 8{\cdot}5$, $6 \leqslant w \leqslant 50$.

2. $d = \pi\sqrt{l/m\gamma}$ for the intervals $1 \leqslant l \leqslant 25$, $1{\cdot}3 \leqslant \gamma \leqslant 3$, $3 \leqslant m \leqslant 8{\cdot}7$.

3. $p_0' = p_\alpha \varepsilon^n$ for the intervals $0{\cdot}75 \leqslant p_a \leqslant 0{\cdot}95$, $12 \leqslant \varepsilon \leqslant 20$, $1{\cdot}2 \leqslant n \leqslant 1{\cdot}5$.

4. $p = K(M+L)$ for the intervals $5 \leqslant K \leqslant 50$, $1 \leqslant M \leqslant 5{\cdot}8$, $3{\cdot}1 \leqslant L \leqslant 7{\cdot}2$.

5. $Q = D^3 \sqrt[3]{w/r}$ for the intervals $3 \leqslant D \leqslant 4$, $11 \leqslant w \leqslant 38$, $15 \leqslant r \leqslant 170$.

6. $\dfrac{T_1^2}{T_2^2} = \dfrac{r_1^3}{r_2^3}$ (third law of Kepler) for the intervals $0{\cdot}24 \leqslant T_1 \leqslant T_2 \leqslant 248{\cdot}6$, $58 \leqslant r_1 < r_2 \leqslant 5917$.

7. $Q_c = Q_0 + R^2(m_p/2+m_0)$ for the intervals $0{\cdot}02 \leqslant Q_0 \leqslant 11$, $30 \leqslant R \leqslant 160$, $0{\cdot}0003 \leqslant m_0 \leqslant 0{\cdot}008$, $0{\cdot}0005 \leqslant m_p \leqslant 0{\cdot}01$.

8. $x^4+y^4 = z^4+u^4$ where each variable runs over the interval from 1 to 10 under the assumption that it is desirable to have increasing accuracy as the number draws nearer to unity.

9. $\mu = 5\lambda/\omega + 3\varphi\psi$ for the intervals $7 \leqslant \lambda \leqslant 19$, $50 \leqslant \omega \leqslant 500$, $4 \leqslant \varphi \leqslant 8$, $7 \leqslant \psi \leqslant 14$.

10. $w = 9{\cdot}5 \sqrt{(u+v)/uvt}$ for the intervals $5 \leqslant u \leqslant 200$, $5 \leqslant v \leqslant 200$, $70 \leqslant t \leqslant 5000$.

11. $u+v-w-t = uvw+uvt-uwt-vwt$ where each interval runs over the interval from 5 to 8.

12. $u+v = (w+t)/(1+wt)$ for the intervals $3 \leqslant v \leqslant 8$, $2 \leqslant w \leqslant 5$, $7 \leqslant t \leqslant 11$.

13. $uv = (w+t^2)/(t+w^2)$ for the intervals $1 \leqslant w \leqslant 20$, $2 \leqslant t \leqslant 8$, $4 \leqslant v \leqslant 11$.

14. $(w+t)/(1+wt) = f(u, v)$ where f is a function of the variables u and v which assumes values between 5 and 7, and w and t vary in the intervals $3 \leqslant w \leqslant 8$, $10 \leqslant t \leqslant 15$.

15. $\dfrac{u-v}{u+v} = \dfrac{w+t}{1+wt}$ for the intervals $2 \leqslant v \leqslant 8$, $7 \leqslant w \leqslant 8$, $4 \leqslant t \leqslant 11$.

16. Construct a nomogram for the equation

$$\varphi = \frac{10YR}{10Yr+1}$$

in which the a-scale from Fig. 120 will be replaced by a line at infinity.

§ 25. Systems of equations. Nomograms consisting of two parts to be superimposed on each other

In § 24.2 we have shown a nomogram (Fig. 123) read by means of superimposing on it a transparent sheet, on which a family of parallel lines has been drawn. That nomogram is thus composed of two parts: an "immovable" part, containing four functional scales, and a "movable" part, also called a *transparent*, containing a family of parallel lines. The device of joining two drawings by superimposing them on each other can be extended to functions with more variables and to systems of equations.

25.1. Let us take a system of equations with six variables,

$$\begin{aligned} f_{12}(u, v)-f_5(z) &= f_{34}(w, s)-f_6(t), \\ g_{12}(u, v)-g_5(z) &= g_{34}(w, s)-g_6(t). \end{aligned} \tag{25.1}$$

We shall make two drawings:

I. On the plane $[x, y]$ we shall consider the following two pairs of families of curves (Fig. 136a):

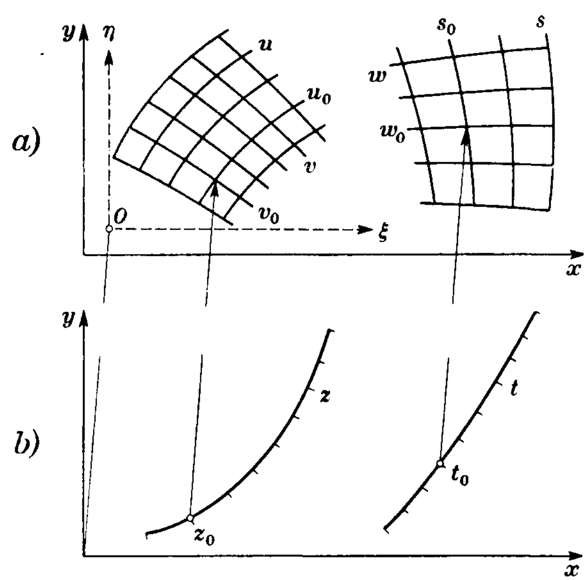

FIG. 136

$$\begin{aligned} x &= f_{12}(u, v), \\ y &= g_{12}(u, v), \end{aligned} \quad \text{and} \quad \begin{aligned} x &= f_{34}(w, s), \\ y &= g_{34}(w, s). \end{aligned} \tag{25.2}$$

II. On the plane $[\xi, \eta]$ we shall consider two functional scales (Fig. 136b):

$$\begin{aligned} \xi &= f_5(z), \\ \eta &= g_5(z), \end{aligned} \quad \text{and} \quad \begin{aligned} \xi &= f_6(t), \\ \eta &= g_6(t). \end{aligned} \tag{25.3}$$

Let us now imagine that drawing b) has been superimposed on drawing a) so that the axes ξ and x are parallel and identically directed and the origin O of the system coincides with O'. If the point z_0 of the z-scale coincides with the point (u_0, v_0) of the lattice $[u, v]$ and if the point t_0 of the t-scale coincides with the point (w_0, s_0) of the lattice $[w, s]$, then of course the vector with initial point z_0 and end-point t_0 will be equal to the vector with initial point (u_0, v_0) and end-point (w_0, s_0). The projections of those vectors on the axes of abscissas are equal, i.e.,

$$f_6(t_0)-f_5(z_0) = f_{34}(w_0, s_0)-f_{12}(u_0, v_0),$$

and, similarly, the projections on the axes of ordinates are equal:

$$g_6(t_0)-g_5(z_0) = g_{34}(w_0, s_0)-g_{12}(u_0, v_0);$$

the system of equations (25.1) is therefore satisfied.

A nomogram composed of parts a) and b) thus makes it possible to solve the system of equations (25.1) for given values of u_0, v_0, z_0 and t_0 or u_0, v_0, z_0 and, say, s_0; it is more troublesome to find z and t when the values of u_0, v_0, w_0 and s_0 are given. It is essential here to keep the axes of the systems parallel. To make this easier, we draw both on the immovable part a) and on the movable part b) a series of lines parallel to the axes of abscissas, or we use well-known systems of joint-connected rods admitting only translations.

As follows from the method of using nomogram 136, the scales on the axes x and ξ must be regular and have equal units; the scales on the axes y and η must also be regular and have equal units, but the units on the axes of abscissas may be different from the units on the axes of ordinates. However, we can move the lattice of curves $[u, v]$ away from the lattices of curves $[w, s]$ as far as we like, of course moving the t-scale away from the z-scale by the same vector, since

$$f_{12}-f_5 = f_{34}+a-(f_6+a),$$

$$g_{12}-g_5 = g_{34}+b-(g_6+b)$$

for arbitrary a and b.

EXAMPLE. Draw a nomogram for the system of equations

$$u\cos v - z = \sqrt{s-w} - t,$$
$$u\sin v - z = w - t^2$$

for the intervals

$$0 \leqslant u \leqslant 2{\cdot}5, \quad 0 \leqslant v \leqslant 90^\circ, \quad 0 \leqslant z \leqslant 2{\cdot}5,$$
$$0 \leqslant t \leqslant 1{\cdot}6, \quad 0 \leqslant w \leqslant 3, \quad -1 \leqslant s \leqslant 1.$$

In order to isolate in the drawing the lattice of w, s from the lattice of u, v we add and substract on the right sides number 4·5. We obtain

$$u\cos v - z = \left(\sqrt{s-w} + 4{\cdot}5\right) - (t + 4{\cdot}5),$$
$$u\sin v - z = w - t^2.$$

The equations of the $[u, v]$ lattice will have the form

$$x = u\cos v, \quad y = u\sin v,$$

and the equations of the $[w, s]$ lattice will have the form

$$x = \sqrt{s-w} + 4{\cdot}5, \quad y = w.$$

Consequently, the curves $u = c$ are circles with centres at the origin of the system, the lines $v = c$ will be lines of a pencil with vertex at the origin of the system, the lines $w = c$ will be lines parallel to the x-axis and the curves x will be parabolas with equations $y = -x^2 + s$ (Fig. 137a).

The functional scales have equations

$$\xi = z, \quad \eta = z,$$

and

$$\xi = t + 4{\cdot}5, \quad \eta = t^2.$$

25.2. The movable part of a nomogram of the type under consideration will have a particularly simple structure if $g_5(z) = 0$ and $g_6(t) = 0$.

In that case certain transformations of the nomogram are permissible. Namely, if

$$f_{12} - f_5 = f_{34} - f_6,$$
$$g_{12} = g_{34},$$

then, choosing two arbitrary continuous and monotonic functions

$\varphi(x)$ and $\psi(x)$, we shall obtain an equivalent system of the form

$$f_{12}+\varphi(g_{12})-f_5 = f_{34}+\varphi(g_{34})-f_6,$$
$$\psi(g_{12}) = \psi(g_{34}).$$

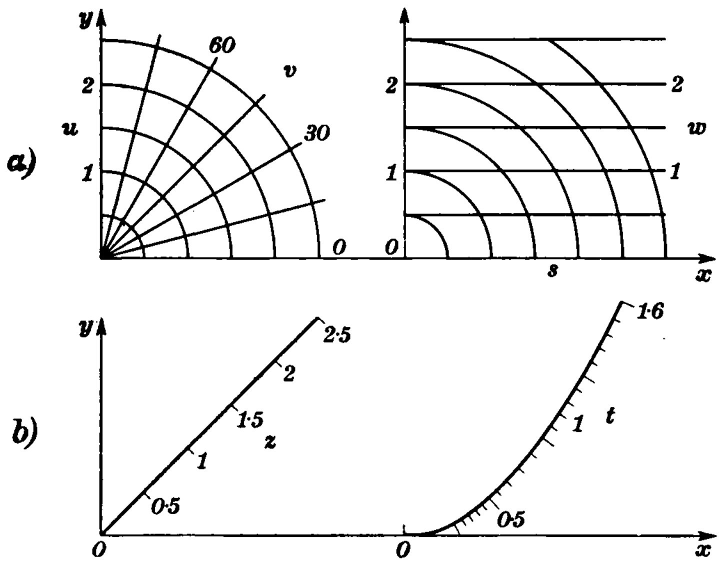

FIG. 137

Our nomogram will consist of two parts, the first being the same as in the general case and the second being a combination of two scales

$$\xi = f_5(z) \quad \text{and} \quad \xi = f_6(t)$$

on one straight line $\eta = 0$ (Fig. 138).

25.3. Nomograms composed of two parts can also be applied to a single equation with five variables. Let us take the equation

$$f_{12}-f_4 = f_{13}-f_5, \tag{25.4}$$

where the first variable appears in both function f_{12} and function f_{13}. Let us take two arbitrary monotonic continuous functions $\varphi(u)$ and $\psi(u)$ and write two equations,

$$f_{12}+\varphi(u)-f_4 = f_{13}+\varphi(u)-f_5,$$
$$\psi(u)-0 = \psi(u)-0.$$

This system leads to a nomogram which consists of two families of lines,

$$x = f_{12}(u, v)+\varphi(u), \qquad y = \psi(u),$$
$$x = f_{13}(u, w)+\varphi(u), \qquad y = \psi(u), \tag{25.5}$$

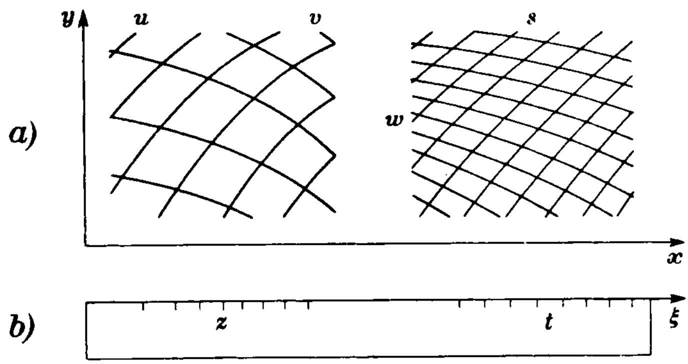

FIG. 138

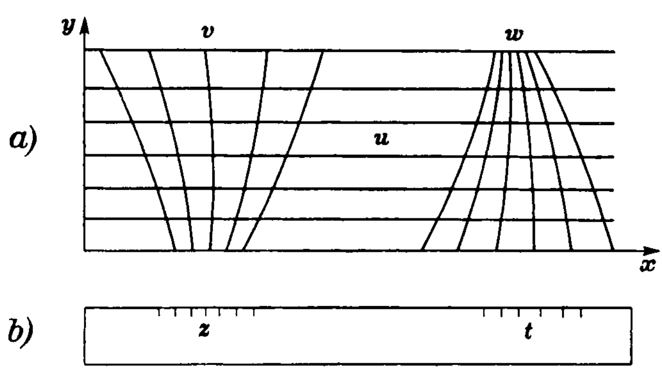

FIG. 139

and two scales,

$$\xi = f_4(z) \quad \text{and} \quad \xi = f_5(t),$$

on one straight line $\eta = 0$.

This nomogram is shown in Fig. 139. It can be seen from equations (25.5) that the first variable, u, is represented by means of a family of straight lines parallel to the x-axis.

Exercises

1. Draw a nomogram of the equation

$$w = xy^2 \sqrt{z}\, e^t\, (s-2x)$$

where the following relation holds between the variables:

$$x^4 = z(s+x)$$

in the intervals

$$0 \leqslant x \leqslant 2, \quad 0 \leqslant y \leqslant 3, \quad 1 \leqslant z \leqslant 4,$$
$$0 \leqslant t \leqslant 0{\cdot}3, \quad 4 \leqslant s \leqslant 8, \quad 0 \leqslant w \leqslant 10.$$

2. Construct a nomogram composed of two parts for the equation

$$(u+v)^w = (u+z)^t$$

for $10 \leqslant u \leqslant 20$, $0 \leqslant v \leqslant 10$, $0 \leqslant z \leqslant 5$, $1 \leqslant w \leqslant 2$, $1 \leqslant t \leqslant 2$.

CHAPTER V

PROBLEMS OF THEORETICAL NOMOGRAPHY

§ 26. The Massau method of transforming nomograms

Formulas occurring in practice usually have the forms listed in §§ 10–23. However, there exist equations, particularly in technological problems, whose structure is complicated and difficult to identify with any of the known types. In such cases we should, if possible, make suitable transformations, leading to the known types of equations.

In this section we shall deal with the problem of transforming equations.

26.1. Suppose we are given an equation containing three variables

$$f(u, v, w) = 0. \tag{26.1}$$

If the left side of the equation has the form of a third order determinant in which each row contains only one variable and one of the columns consists of unities only, i.e., if (26.1) is an equation of the form

$$\begin{vmatrix} \varphi_1(u) & \psi_1(u) & 1 \\ \varphi_2(v) & \psi_2(v) & 1 \\ \varphi_3(w) & \psi_3(w) & 1 \end{vmatrix} = 0, \tag{S}$$

we shall call it the *Soreau form*, or briefly the (S) *form*.

In view of the considerations of § 15 it is obvious that relation (26.1) can be represented by a collineation nomogram if, and only if, the equation is of the (S) form. Consequently, the question whether there exists a collineation nomogram for a given equation is equivalent to the question whether an (S) form exists for that equation.

In § 28 we shall prove that it is not possible to bring every equation to an equation of the (S) form. In cases where the (S) form does exist for an equation (26.1), it can be obtained by the following method, introduced by Massau and called the *Massau method*.

We introduce new variables x and y by means of the equalities

$$x = G(u, v, w) \quad \text{and} \quad y = H(u, v, w) \tag{26.2}$$

and then eliminate the variables v and w from the given equation (26.1) and the substitutions (26.2); suppose that the result of that elimination is an equation linear with respect to x and y:

$$\varphi_1(u)\,x+\psi_1(u)\,y+\chi_1(u) = 0. \tag{u}$$

Similarly, assume that by eliminating the variables u and w from equations (26.1) and (26.2) we also obtain a relation linear with respect to x and y:

$$\varphi_2(v)\,x+\psi_2(v)\,y+\chi_2(v) = 0; \tag{v}$$

and finally, assume that by eliminating u and v from (26.1) and (26.2), we obtain an equation linear with respect to x and y:

$$\varphi_3(w)\,x+\psi_3(w)\,y+\chi_3(w) = 0. \tag{w}$$

As we know, a system of three linear equations, (u), (v) and (w), with two variables x and y has a solution if and only if its determinant is equal to zero:

$$\begin{vmatrix} \varphi_1(u) & \psi_1(u) & \chi_1(u) \\ \varphi_2(v) & \psi_2(v) & \chi_2(v) \\ \varphi_3(w) & \psi_3(w) & \chi_3(w) \end{vmatrix} = 0. \tag{S$'$}$$

This equation is satisfied by any three numbers u, v, w satisfying equation (26.1). We shall show by examples that equation (S′) may be solved by other threes of numbers, such as do not satisfy equation (26.1).

It can easily be seen that (S′) can always be reduced by division to the (S) form. If the product $\chi_1(u)\,\chi_2(v)\,\chi_3(w)$ is not identically equal to zero for threes of numbers u, v, w satisfying equation (26.1), we divide both sides of equation (S′) by that product, obtaining the (S) form:

$$\begin{vmatrix} \varphi_1/\chi_1 & \psi_1/\chi_1 & 1 \\ \varphi_2/\chi_2 & \psi_2/\chi_2 & 1 \\ \varphi_3/\chi_3 & \psi_3/\chi_3 & 1 \end{vmatrix} = 0.$$

If $\chi_1\chi_2\chi_3 = 0$, we can obtain—by interchanging or adding the columns—non-zero functions in the third column; and then, by division, we get an equation of the (S) form.

The application of this method of transforming equations into the (S) form may, in practical cases, involve certain difficulties, since it is necessary to guess the requisite substitutions (26.2). In particular it is difficult to verify whether such a substitution exists in a given case, since the Massau method gives no answer to this question. We shall show that there exist equations (26.1) which cannot be reduced to an (S) form and we shall give the criteria of the existence of that form for a given equation.

26.2. Proceeding to applications, let us reduce to the (S) form the equation

$$f_1(u)f_2(v)f_3(w)-1 = 0.$$

(Here the function $F(u, v, w)$ has a special form

$$F = f_1(u)f_2(v)f_3(w)-1.)$$

It turns out that we can do so by three different methods.

F i r s t m e t h o d. We substitute $x = f_1$, $y = 1/f_2$. Eliminating the variables u and v from the given equation $f_1f_2f_3-1 = 0$ we obtain

$$x\frac{1}{y}f_3-1 = 0,$$

or, multiplying by y,

$$xf_3-y = 0.$$

We thus have, together with the substitution formulas, three equations

$$\begin{aligned} x \qquad\quad -f_1 &= 0,\\ yf_2 \;-1 &= 0,\\ xf_3-y \qquad &= 0. \end{aligned}$$

The fulfilment of these equations by x and y is equivalent to the equation

$$\begin{vmatrix} 1 & 0 & -f_1 \\ 0 & f_2 & -1 \\ f_3 & -1 & 0 \end{vmatrix} = 0.$$

Adding the terms of the second column to the terms of the third column, and then dividing by $f_1(f_2-1)$ we obtain successively

$$\begin{vmatrix} 1 & 0 & -f_1 \\ 0 & f_2 & f_2-1 \\ f_3 & -1 & -1 \end{vmatrix} = 0, \qquad \begin{vmatrix} 1/f_1 & 0 & 1 \\ 0 & f_2/(f_2-1) & 1 \\ f_3 & 1 & 1 \end{vmatrix} = 0. \qquad (S_I)$$

The last equation is of the (S) form.

Second method. We substitute $x = f_1+f_2$, $y = f_1f_2$ in equation $f_1f_2f_3-1 = 0$; we obtain $f_3y-1 = 0$. From the expressions for x and y we have

$$y = f_1(x-f_1) \quad \text{or} \quad f_1x-y-f_1^2 = 0,$$

and similarly

$$y = f_2(x-f_2) \quad \text{or} \quad f_2x-y-f_2^2 = 0.$$

We thus obtain a system of three equations,

$$xf_1-y-f_1^2 = 0,$$
$$xf_2-y-f_2^2 = 0,$$
$$yf_3-1 = 0,$$

which implies that

$$\begin{vmatrix} f_1 & -1 & -f_1^2 \\ f_2 & -1 & -f_2^2 \\ 0 & f_3 & -1 \end{vmatrix} = 0.$$

Dividing by function f_3 (which is not constantly equal to zero because $f_1f_2f_3 = 1$) and interchanging columns two and three we obtain the (S) form:

$$\begin{vmatrix} f_1 & f_1^2 & 1 \\ f_2 & f_2^2 & 1 \\ 0 & -1/f_3 & 1 \end{vmatrix} = 0. \qquad (S_{II})$$

This equation is not equivalent to the equation $f_1f_2f_3-1 = 0$ because on expanding the determinant we obtain

$$f_1f_2^2-\frac{f_2}{f_3}+\frac{f_1}{f_3}-f_1^2f_2 = 0,$$

or

$$f_1f_2^2f_3-f_2+f_1-f_1^2f_2f_3 = 0,$$

or

$$(f_2-f_1)(f_1f_2f_3-1) = 0.$$

Equation (S) is therefore satisfied not only by the threes of numbers u, v, w which verify the given equation but also by threes u_0, v_0, w where $f_1(u_0) = f_2(v_0)$ and w is an arbitrary number. It will thus be seen that from a nomogram defined by an (S) equation we shall have to reject solutions u_0, v_0, w for which $f_1(u_0) = f_2(v_0)$ and w is arbitrary.

Third method. We substitute $x = f_1+f_2+f_3$, $y = f_1f_2+ +f_1f_3+f_2f_3$. The given equation is transformed into the equation

$$f_1(y-f_1f_2-f_1f_3)-1 = 0,$$

and then into

$$f_1[y-f_1(f_2+f_3)]-1 = 0 \quad \text{or} \quad f_1[y-f_1(x-f_1)]-1 = 0,$$

i.e.

$$-f_1^2x+f_1y+f_1^3-1 = 0. \tag{u}$$

We find that the reduction of u and w gives us the equations

$$\begin{gathered} f_2[y-f_2(f_1+f_3)]-1 = 0, \\ f_2[y-f_2(x-f_2)]-1 = 0, \\ -f_2^2x+f_2y+f_2^3+1 = 0. \end{gathered} \tag{v}$$

Similarly, by reducing u and v, we obtain

$$-f_3^2x+f_3y+f_3^3-1 = 0. \tag{w}$$

Equations (u), (v) and (w) give us the equation

$$\begin{vmatrix} f_1 & -f_1^2 & f_1^3-1 \\ f_2 & -f_2^2 & f_2^3-1 \\ f_3 & -f_3^2 & f_3^3-1 \end{vmatrix} = 0$$

and ultimately the (S) form

$$\begin{vmatrix} \dfrac{-f_1}{1-f_1^3} & \dfrac{-f_1^2}{1-f_1^3} & 1 \\ \dfrac{-f_2}{1-f_2^3} & \dfrac{-f_2^2}{1-f_2^3} & 1 \\ \dfrac{-f_3}{1-f_3^3} & \dfrac{f_3^2}{1-f_3^3} & 1 \end{vmatrix} = 0. \tag{S$_{III}$}$$

Expanding this determinant we obtain, as can easily be verified, the equation

$$(f_1-f_2)(f_2-f_3)(f_3-f_1)(f_1f_2f_3-1) = 0.$$

In addition to the threes u, v, w which satisfy the given equation $f_1f_2f_3-1 = 0$, equation (S_{III}) has the following solutions:

such threes u_0, v_0, w that $f_1(u_0) = f_2(v_0)$ and w is arbitrary,
such threes u_0, v, w_0 that $f_1(u_0) = f_3(w_0)$ and v is arbitrary,
such threes u, v_0, w_0 that $f_2(v_0) = f_3(w_0)$ and u is arbitrary.

As before, having drawn a nomogram defined by equation (S_{III}), we must reject those threes of numbers u_0, v_0, w for which $f_1(u_1) = f_2(v_0)$ and w is arbitrary, those threes u_1, v, w_1 for which $f_1(u_1) = f_3(w_1)$ and v is arbitrary and those threes u, v_2, w_2 for which $f_2(v_2) = f_3(w_3)$ and u is arbitrary.

We have obtained for the equation $f_1f_2f_3-1 = 0$ three Soreau forms, (S_I), (S_{II}) and (S_{III}), and consequently three nomograms.

The nomogram defined by equation (S_I) consists of three rectilinear scales with equations

$$x_u = \frac{1}{f_1(u)}, \quad x_v = 0, \quad x_w = f_3(w),$$

$$y_u = 0, \quad y_v = \frac{f_2(v)}{f_2(v)-1}, \quad y_w = 1.$$

The nomogram defined by equation (S_{II}) consists of three scales, one of them rectilinear and the other two lying on a common curve of the second order:

$$x_u = f_1, \quad x_v = f_2^2, \quad x_w = 0,$$

$$y_u = f_1^2, \quad y_v = f_2^2, \quad y_w = -1/f_3.$$

Indeed, the base of both the u-scale and the v-lattice is the parabola $y = x^2$ and the base of the w-scale is the straight line $x = 0$, i.e., the axis of the parabola.

The nomogram defined by equation (S_{III}) has three scales with a common base

$$x_u = \frac{-f_1}{1-f_1^3}, \qquad x_v = \frac{-f_2}{1-f_2^3}, \qquad x_w = \frac{-f_3}{1-f_3^3},$$

$$y_u = \frac{f_1^2}{1-f_1^3}, \qquad y_v = \frac{f_2^2}{1-f_2^3}, \qquad y_w = \frac{f_3^2}{1-f_3^3}.$$

It can easily be verified that the common base of these scales is a curve of the third order, for on dividing y by x we have (omitting the indices u, v and w)

$$y/x = -f;$$

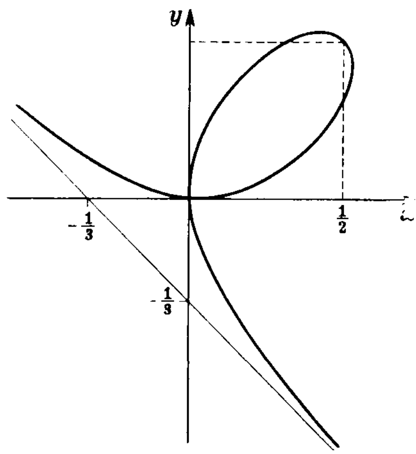

FIG. 140

substituting this in the first equation we obtain

$$x = \frac{y/x}{1+(y/x)^3}, \qquad \text{whence} \qquad x = \frac{x^2y}{x^3+y^3}$$

and finally

$$x^3 - xy + y^3 = 0.$$

This equation presents the so called *folium of Descartes* (Fig. 140). It is a curve with a double point at the origin of the system and an asymptote $x+y = -1/3$. It will be observed that we have transformed the equation $f_1f_2f_3 - 1 = 0$ into the forms (S_I), (S_{II}) and (S_{III}) exclusively by so called *rational* *operations*, i.e., by addition, subtraction, multiplication and division. It could be proved that the forms we have obtained are the only (S) forms of the given equation obtained by rational operations. This means that every collineation nomogram of the equation $f_1 f_2 f_3 = 1$ in which the scales u, v and w are scales of rational functions f_1, f_2 or f_3 is either a drawing defined by equation (S_I), (S_{II}) or (S_{III}), or a projective transformation of one of them.

26.3. Consider in an analogous way the equation

$$f_1(u)+f_2(v)+f_3(w) = 0.$$

Here again there are three different methods of reducing the equation to an (S) form.

F i r s t m e t h o d. We substitute $x = f_2(v)$, $y = f_3(w)$ in the given equation; we obtain $f_1+x+y = 0$, i.e., a system of equations,

$$\begin{aligned} x+y+f_1 &= 0, \\ x \quad -f_2 &= 0, \\ y-f_3 &= 0, \end{aligned}$$

whose determinant must be equal to zero:

$$\begin{vmatrix} 1 & 1 & f_1 \\ 1 & 0 & -f_2 \\ 0 & 1 & -f_3 \end{vmatrix} = 0.$$

Adding the terms of the first column to the terms of the second column, we obtain

$$\begin{vmatrix} 1 & 2 & f_1 \\ 1 & 1 & -f_2 \\ 0 & 1 & -f_3 \end{vmatrix} = 0;$$

dividing the terms of the first row by two and interchanging columns two and three, we finally obtain

$$\begin{vmatrix} 1/2 & f_1/2 & 1 \\ 1 & -f_2 & 1 \\ 0 & -f_3 & 1 \end{vmatrix} = 0. \qquad (S_I)$$

The nomogram consists of three scales on three parallel lines.

Second method. We substitute $x = f_1 f_2$, $y = f_1 + f_2$. We easily obtain a system of equations

$$x - f_1 y + f_1^2 = 0,$$
$$x - f_2 y + f_2^2 = 0,$$
$$y + f_3 = 0,$$

which implies that

$$\begin{vmatrix} 1 & -f_1 & f_1^2 \\ 1 & -f_2 & f_2^2 \\ 0 & 1 & f_3 \end{vmatrix} = 0.$$

Adding the terms of the third column to the terms of the first column, we obtain

$$\begin{vmatrix} 1+f_1^2 & -f_1 & f_1^2 \\ 1+f_2^2 & -f_2 & f_2^2 \\ f_3 & 1 & f_3 \end{vmatrix} = 0;$$

dividing the first row by $1+f_1^2$, the second row by $1+f_2^2$ and the third row by f_3 and interchanging columns one and three, we obtain an (S) form:

$$\begin{vmatrix} \dfrac{f_1^2}{1+f_1^2} & \dfrac{f_1}{1+f_1^2} & 1 \\ \dfrac{f_2^2}{1+f_2^2} & \dfrac{f_2}{1+f_2^2} & 1 \\ 1 & -\dfrac{1}{f_3} & 1 \end{vmatrix} = 0. \tag{S}$$

The scales u and v lie on the same curve with parametric equations

$$x = \frac{f^2}{1+f^2}, \quad y = \frac{f}{1+f^2}.$$

Eliminating parameter f, we obtain

$$\frac{x}{y} = f \quad \text{and} \quad y = \frac{x/y}{1+(x/y)^2} \quad \text{or} \quad y = \frac{xy}{x^2+y^2}.$$

This curve is thus of the second degree with the equation

$$x^2 - x + y^2 = 0 \quad \text{or} \quad (x-1/2)^2 + y^2 = (1/2)^2.$$

The third scale, w, is rectilinear with equations

$$x = 1, \quad y = -1/f_3.$$

It is a tangent to the circle $x^2 - x + y^2 = 0$.

The nomogram consists of two scales on a common curve of the second degree and of a third scale on a tangent to that curve.

Third method. We substitute $x = f_1 f_2 f_3$, $y = f_1 f_2 + f_1 f_3 + f_2 f_1$ in the given equation $f_1 + f_2 + f_3 = 0$. Eliminating f_2 and f_3 from those equations, we obtain

$$x = f_1(y - f_1 f_2 - f_1 f_3) \quad \text{or} \quad x = f_1[y - f_1(-f_1)],$$

i.e.

$$x - f_1 y - f_1^3 = 0. \tag{u}$$

On account of symmetry, the remaining two equations will be of the same form:

$$x - f_2 y - f_2^3 = 0, \tag{v}$$

$$x - f_3 y - f_3^3 = 0, \tag{w}$$

The result of the elimination of variables x, y and z from equations (u), (v) and (w) is the equation

$$\begin{vmatrix} 1 & -f_1 & -f_1^3 \\ 1 & -f_2 & -f_2^3 \\ 1 & -f_3 & -f_3^3 \end{vmatrix} = 0.$$

We thus have a third form for the equation $f_1 + f_2 + f_3 = 0$:

$$\begin{vmatrix} f_1 & f_1^3 & 1 \\ f_2 & f_2^3 & 1 \\ f_3 & f_3^3 & 1 \end{vmatrix} = 0. \tag{S$_{\text{III}}$}$$

In this case the nomogram consists of three scales lying on a curve of the third degree,

$$x = f, \quad y = f^3.$$

It is a parabola of the third degree, $y = x^3$, with the inflection point at the origin of the system (Fig. 141).

Equation (S_{III}), with the determinant expanded, has the form

$$(f_1-f_2)(f_1-f_3)(f_2-f_3)(f_1+f_2+f_3) = 0.$$

Since our problem concerns only the last factor being equal to zero, we must reject the following threes of numbers:

u_0, v_0, w, where $f_1(u_0) = f_2(v_0)$ and w is arbitrary,
u_1, v, w_1, where $f_1(u_1) = f_3(w_1)$ and v is arbitrary,
u, v_2, w_2, where $f_2(v_2) = f_3(w_2)$ and u is arbitrary.

Geometrically this means that at the intersection of a certain straight line with the base of the scales we must read the value

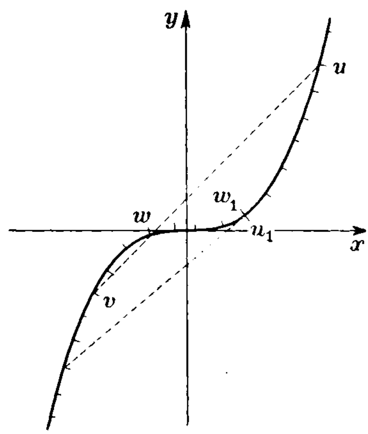

FIG. 141

of one variable only, e.g. u; if we wished to read the value of w corresponding to the pair $\bar{u}$, $\bar{v}$ for which the point $\bar{v}$ coincides with the point $\bar{u}$ on a common scale, we should have to draw a tangent to the curve at point $\bar{u} = \bar{v}$; the intersection point of the tangent with the curve determines the required value of $\bar{w}$.

Exercises

1. Reduce the equation of the second degree

$$z^2+uz+v = 0$$

to an (S) form by the Massau method (substitute $u = x$, $v = y$).

2. Reduce to an (S) form the equation

$$auv+buw+cvw = 0.$$

(Substitute $bu+cv = x$, $auv = y$.)

3. Reduce to an (S) form the equations

a. $w = \dfrac{3u+5v-1}{4u-v}$ (e.g., substitute $3u+5v = x$ and $4u-v = y$),

b. $\tan w = \dfrac{uv}{u+v}$,

c. $\sin w = \sin u \sin v$,

d. $\dfrac{\sin w}{u} + \dfrac{\cos w}{v} = k$ (substitute $1/v = x$ and $1/u = y$),

e. $w = \dfrac{u^2-v}{v^2-u}$ (substitute $x = w$, $y = wv^2+v$).

§ 27. Curvilinear nomograms for the equations $f_1(u)f_2(v)f_3(w) = 1$, $f_1(u)+f_2(v)+f_3(w) = 0$, $f_1(u)f_2(v)f_3(w) = f_1(u)+f_2(v)+f_3(w)$

27.1. The equation $f_1f_2f_3 = 1$ can be reduced to three different (S) forms: (S_I), (S_{II}) and (S_{III}). Each of them is the equation of a certain nomogram. The (S_I) form leads to the well-known nomogram with three rectilinear scales defined by equations

$$x_1 = \frac{1}{f_1(u)}, \qquad x_2 = 0, \qquad x_3 = f_3(w),$$

$$y_1 = 0, \qquad y_2 = \frac{f_2(v)}{f_2(v)-1}, \qquad y_3 = 1.$$

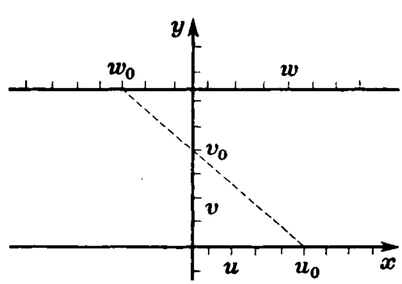

FIG. 142

The u-scale is rectilinear with base on the x-axis, the v-scale is rectilinear with base on the y-axis, the w-scale lies on a line parallel to the x-axis (Fig. 142).

It can easily be seen that we have here an N-shaped collineation nomogram, discussed in detail in § 12. As can easily be guessed, we use this nomogram in cases where only one function assumes very large values; we then draw it on the oblique straight line, i.e., here on the y-axis.

The (S_{II}) form

$$\begin{vmatrix} f_1 & f_1^2 & 1 \\ f_2 & f_2^2 & 1 \\ 0 & -1/f_3 & 1 \end{vmatrix} = 0$$

defines the following scale equations:

$$\text{(u)} \quad \begin{aligned} x_1 &= f_1, \\ y_1 &= f_1^2, \end{aligned} \qquad \text{(v)} \quad \begin{aligned} x_2 &= f_2, \\ y_2 &= f_2^2, \end{aligned} \qquad \text{(w)} \quad \begin{aligned} x_3 &= 0, \\ y_3 &= -1/f_3. \end{aligned}$$

The scales (u) and (v) lie on a common curve (parabola), $y = x^2$, and the w-scale lies on the straight line $x = 0$ intersecting the parabola at two points: at the origin of the system (0, 0, 1) and at the point at infinity (0, 1, 1).

Every projective transformation of the plane (x, y) turns the parabola into a curve of the second degree (an ellipse, a parabola or a hyperbola), and changes the straight line that intersects it at two points into a straight line having two points in common with the curve, i.e., into a chord.

A nomogram containing a curve is of course more difficult to execute than a nomogram with only rectilinear scales, and it would not be advisable to draw it in cases where a nomogram with three rectilinear scales satisfies the required accuracy conditions in the given intervals. To verify whether, for a given equation (S_{II}), the nomogram gives sufficient variability of the unit of the scale, it is useful to construct a nomogram for the equation $uvw = 1$.

On the basis of elementary transformations of a determinant we have

$$\begin{vmatrix} u & u^2 & 1 \\ v & v^2 & 1 \\ 0 & -1/w & 1 \end{vmatrix} = \begin{vmatrix} u & 1+u^2 & 1 \\ v & 1+v^2 & 1 \\ 0 & 1-1/w & 1 \end{vmatrix} = \begin{vmatrix} 1 & u & 1+u^2 \\ 1 & v & 1+v^2 \\ 1 & 0 & 1-1/w \end{vmatrix}$$

and consequently the form

$$\begin{vmatrix} \dfrac{1}{1+u^2} & \dfrac{u}{1+u^2} & 1 \\ \dfrac{1}{1+v^2} & \dfrac{v}{1+v^2} & 1 \\ \dfrac{1}{1-1/w} & 0 & 1 \end{vmatrix} = 0.$$

This equation shows that the base of the first two scales is a circle, since we have in succession

$$x = \frac{1}{1+u^2}, \quad y = \frac{u}{1+u^2}, \quad y = ux,$$

$$x = \frac{1}{1+(y/x)^2}, \quad \frac{x}{x^2+y^2} = 1,$$

$$(x-0{\cdot}5)^2+y^2 = \tfrac{1}{4}.$$

Proceeding to the construction of the nomogram we observe that the w-scale will be a projective scale because

$$x = \frac{1}{1-1/w} = \frac{w}{w-1}.$$

The limit for $w \to \infty$ is $x = 1$. As regards the variables u and v, the limit for $u \to \infty$ and $v \to \infty$ is the origin of the system (0, 0). Fig. 143 shows that a nomogram of this shape would be more convenient than a nomogram with three parallel scales if, for the given interval, funtion f_1 assumed values from 0 to ∞, funtion f_2—values from $-\infty$ to 0, and function f_3—very small negative values. As we know, a nomogram with three rectilinear scales would then have infinite dimensions.

Form (S_{III})

$$\begin{vmatrix} \dfrac{-f_1}{1-f_1^2} & \dfrac{f_1^2}{1-f_1^3} & 1 \\ \dfrac{-f_2}{1-f_2^2} & \dfrac{f_2^2}{1-f_2^3} & 1 \\ \dfrac{-f_3}{1-f_3^2} & \dfrac{f_3^2}{1-f_3^3} & 1 \end{vmatrix} = 0$$

leads to a drawing composed of three scales on one curve, called the *folium of Descartes*, $x^3+y^3-xy = 0$.

It is a so-called *unicursal curve,* i.e., a curve whose points can be assigned in a one-to-one and analytic manner to the points of a straight (projective) line or to the elements of a pencil of straight lines. It is sufficient to take a pencil of lines with the vertex at the origin of the system, which is a double point for the curve, and to assign to each straight line of the pencil the point at which it intersects the curve for the third time (point (0, 0) is then reckoned twice and consequently it has two corresponding straight lines: the x-axis and the y-axis).

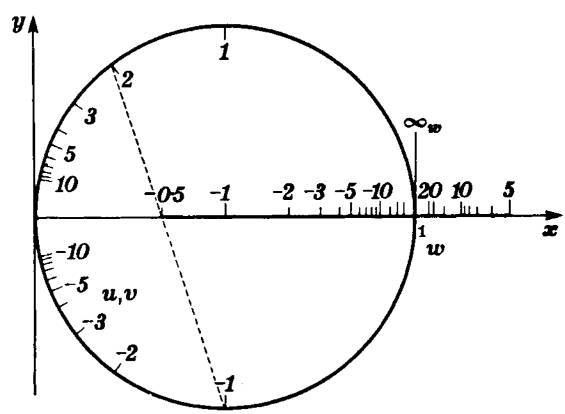

FIG. 143

In order to discuss the cases where the equations $f_1 f_2 f_3 = 1$ are more conveniently represented by this nomogram than by nomograms of the preceding two types let us draw the scale of the function $uvw = 1$.

All the scales have the same parametric equations

$$x = u/(u^3-1), \qquad y = u^2/(u^3-1).$$

Obviously, as $u \to \infty$, the corresponding points tend to the double point along an arc tangent to the y-axis (Fig. 144); it is to the same point that the points assigned to numbers close to zero tend along an arc tangent to the x-axis. For u close to unity the corresponding points tend to infinity along arcs approaching the line $x-y+1/3 = 0$ asymptotically.

This drawing will be a suitable nomogram in the case where, in the given interval, one function assumes very large positive values, the second—very large negative values and the third—very small negative values; in this case the preceding type also gives good results.

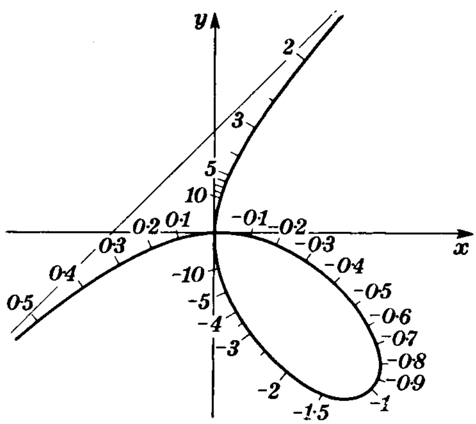

FIG. 144

27.2. Equation

$$f_1+f_2+f_3 = 0$$

can be represented in the forms (S_I), (S_{II}) and (S_{III}).

As we know from § 10, a nomogram based on equation (S_I) consists of three functional scales $u' = f_1(u)$, $v' = f_2(v)$ and $w' = f_3(w)$ on three parallel lines. It can thus be used only for functions f_1, f_2 and f_3 which are bounded in given intervals.

A nomogram based on equation (S_{II}), i.e. with scales

$$x = \frac{f_i^2}{1+f_i^2}, \qquad y = \frac{f_i}{1+f_i^2} \qquad \text{for} \qquad i = 1, 2,$$

$$x = 1, \qquad y = -\frac{1}{f_3},$$

consists of two scales on a common curve of the second degree

$$(x-\tfrac{1}{2})^2+y^2 = (\tfrac{1}{2})^2$$

and a rectilinear scale on a tangent to the curve.

In order to find out which equations are conveniently represented by such a nomogram, let us make a drawing for the equation

$$u+v+w = 0.$$

The nomogram obtained is shown in Fig. 145. It can be seen that this form is convenient if, in the given intervals, two functions f_i have very large absolute values and the third has very small

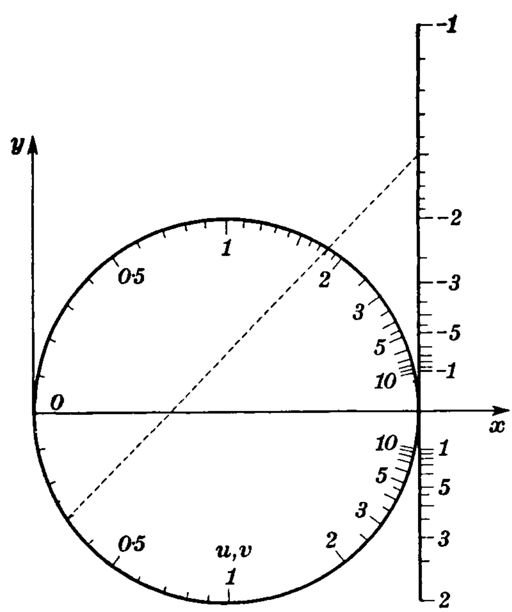

FIG. 145

absolute values. Then of course the circle must be replaced by an ellipse with the major axis (or only a diameter) parallel to the rectilinear scale on the tangent.

In the case of form (S_{III}) we have three scales on the curve $y = x^3$.

Let us take again the equation $u+v+w = 0$. Each of the three scales has the parametric equations

$$x = u, \qquad y = u^3. \tag{*}$$

This nomogram (Fig. 146a) can be used in cases where great accuracy is required for those values of two variables to which

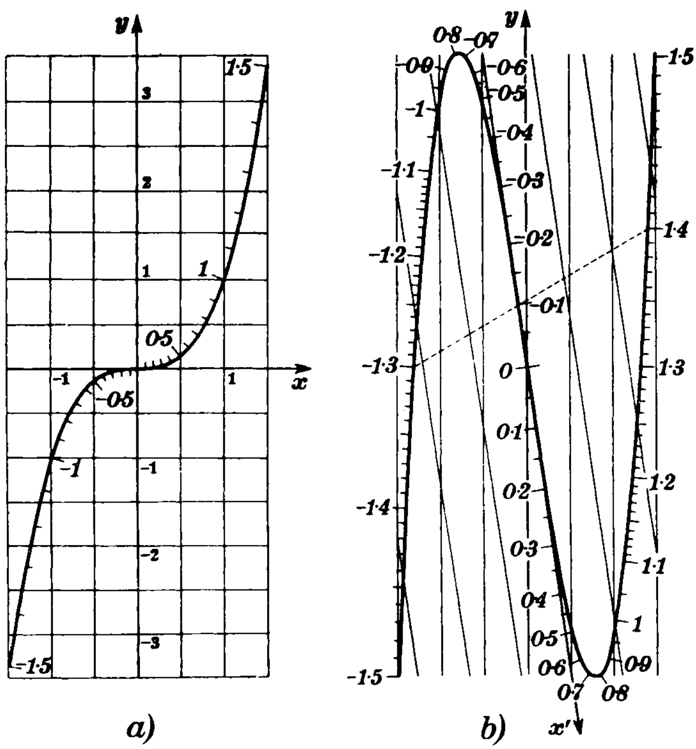

FIG. 146

large values of two functions correspond; the third function should then have small values. In those cases we transform the plane in an affine manner so as to have a large angle between the positive parts of the axes x and y; then the parts of the scales corresponding to the absolutely large values of two functions will get nearer to each other (Fig. 146b).

Remark. Nomogram 146b can be drawn more accurately by finding the equation of the curve in orthogonal coordinates. Accordingly, let us take the direction of, say, the tangent intersecting the curve at point $u = 2$ as the direction of the ξ-axis, and the direction of the tangent at point $u = 2$ as the direction of the η-axis.

The slope of the first tangent will be obtained by writing its equation in the form

$$y-y_0 = 3x_0^2(x-x_0)$$

and substituting in it $x = 2$, $y = 8$. We obtain

$$8-x_0^3 = 3x_0^2(2-x_0),$$

and thus

$$x_0^3-3x_0^2+4 = (x_0-2)^2(x_0+1) = 0,$$

i.e., $x_0 = -1$. The required slope is thus equal to 3. Since the other slope is $3 \cdot 2^2 = 12$, according to the notation introduced in § 4.3, point P, (in our case a point at infinity) has coordinates (1, 12, 0), point Q (also at infinity) has coordinates (1, 3, 0) and point A has coordinates (0, 0, 1); the straight line AP has an equation $y = 12x$, or, in homogeneous coordinates,

$$-12x_1+x_2 = 0,$$

the line AQ has an equation $y = 3x$ or

$$-3x_1+x_2 = 0,$$

and the line PQ has of course an equation $x_3 = 0$.

The transformation will thus be defined by equations

$$\xi = \frac{-12x_1+x_2}{x_3} = -12x+y,$$

$$\eta = \frac{-3x_1+x_2}{x_3} = -3x+y.$$

The scale equations are of the form

$$\xi = u^3-12u,$$
$$\eta = u^3-3u.$$

27.3. Equation $f_1 f_2 f_3 = 1$ can easily be reduced to the form $\varphi_1+\varphi_2+\varphi_3 = 0$, for it is sufficient to substitute $\varphi_i = \log f_i$.

Similarly, equation $f_1+f_2+f_3=0$ can be reduced to the form $\psi_1\psi_2\psi_3=1$ by substituting $f_i=\log\psi_i$, i.e. $\psi_i=10^{f_i}$. Consequently, in each of the two types of equations we have six different nomograms, not equivalent to one another by projection. It will be observed, however, that the substitution $\varphi_i=\log f_i$, for instance, can be applied only to those intervals of the independent variable for which function f_i is positive; the second substitution, $\psi_i=10^{f_i}$, assigns to all real values of f_i only a part of the set of the values of ψ_i, namely the positive part. Therefore, if we constructed, after the second substitution, a curvilinear nomogram for example, then all the values of f_1 from $-\infty$ to $+\infty$ would be placed on part of the curve only; the unlimited real axis would be contracted to a part of a curve of the second or third degree. Such contraction is of course useful in certain cases.

By a suitable substitution, also the equation

$$f_1+f_2+f_3=f_1f_2f_3 \tag{27.1}$$

can be reduced to the preceding types. Take function ω_i, such that

$$\cot\omega_i=f_i. \tag{27.2}$$

From the well-known formula

$$\tan(\omega_1+\omega_2+\omega_3)=\frac{\tan\omega_1+\tan\omega_2+\tan\omega_3-\tan\omega_1\tan\omega_2\tan\omega_3}{1-\tan\omega_2\tan\omega_3-\tan\omega_1\tan\omega_3-\tan\omega_1\tan\omega_2}$$

by substitution (27.2) we obtain in view of (27.1) the equation

$$\tan(\omega_1+\omega_2+\omega_3)=0,$$

and consequently

$$\omega_1(u)+\omega_2(v)+\omega_3(w)=0. \tag{27.3}$$

Similarly, equation (27.1) can be changed into the equation

$$\psi_1\psi_2\psi_3=1.$$

Obviously, any such transformation from one form into another involves a deformation of the functional scales of the given functions. Equation (27.1) can also be represented by a nomogram directly, i.e., by writing a form (S) which would contain only the functions f_i and expressions made up from them by addition, subtraction, multiplication and division.

Accordingly, let us substitute

$$x = f_1+f_2, \quad y = f_1 f_2-1; \tag{27.4}$$

by eliminating f_1 and f_2 from equation (27.1) we obtain

$$x+f_3 = f_3(y+1) \quad \text{or} \quad x-f_3 y = 0,$$

and by eliminating f_1 and f_2 from the substitutions (27.4) we obtain the equations

$$xf_1-y-(f_1^2+1)= 0, \quad xf_2-y-(f_2^2+1) = 0.$$

The last three equations give us form (S) for equation (27.1):

$$\begin{vmatrix} f_1 & -1 & -1-f_1^2 \\ f_2 & -1 & -1-f_2^2 \\ 1 & -f_3 & 0 \end{vmatrix} = 0$$

or

$$\begin{vmatrix} \dfrac{2}{f_1^2+4} & \dfrac{f_1}{f_1^2+4} & 1 \\ \dfrac{2}{f_2^2+4} & \dfrac{f_2}{f_2^2+4} & 1 \\ \dfrac{2}{3} & \dfrac{1}{3f_3} & 1 \end{vmatrix} = 0.$$

The first two scales lie, as can easily be found, on the circumference $(x-1/4)^2+y^2 = 1/16$ and third on the straight line $x = 2/3$, which does not intersect the circle at real points (Fig. 147).

For example, for the equation $uvw = u+v+w$ we have, in the circle, a scale of fast decreasing units as u (or v) tends to infinity, and a projective scale on the straight line.

For equation (27.1) we can also construct a nomogram with three scales on one curve. To give it the necessary Soreau form let us substitute

$$x = f_1 f_2 f_3, \quad y = f_2 f_3+f_1 f_3+f_1 f_2. \tag{27.5}$$

Substituting the value $f_3 = x/f_1 f_2$ in equation (27.1) and in the second equation of (27.5), we obtain

$$x = f_1+f_2+\frac{x}{f_1 f_2} \quad \text{or} \quad \frac{f_1}{x} f_2^2+\left(\frac{f_1^2}{x}-f_1\right) f_2+1 = 0,$$

$$f_1 f_2+\frac{x}{f_1 f_2}(f_1+f_2) = y \quad \text{or} \quad \frac{f_1}{x} f_2^2+\left(\frac{1}{f_1}-\frac{y}{x}\right) f_2+1 = 0.$$

By subtraction we obtain

$$\frac{f_1^2}{x}-f_1 = \frac{1}{f_1}-\frac{y}{x} \quad \text{or} \quad \left(\frac{1}{f_1}+f_1\right) x-y-f_1^2 = 0.$$

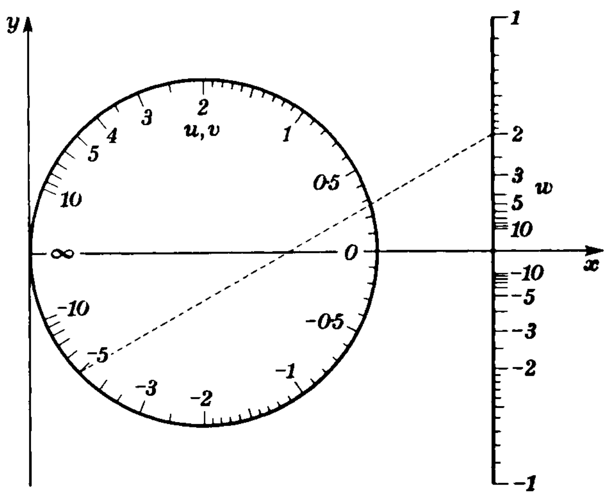

Fig. 147

On account of the symmetry of formulas (27.1) and (27.5) we also have

$$\left(\frac{1}{f_2}+f_2\right) x-y-f_2^2 = 0, \quad \left(\frac{1}{f_3}+f_3\right) x-y-f_3^2 = 0,$$

i.e. an (S) equation

$$\begin{vmatrix} f_1+1/f_1 & -1 & -f_1^2 \\ f_2+1/f_2 & -1 & -f_2^2 \\ f_3+1/f_3 & -1 & -f_3^2 \end{vmatrix} = 0.$$

The scale equations will be of the form

$$x = f_i + 1/f_i, \quad y = f_i^2 \quad \text{for} \quad i = 1, 2, 3. \tag{27.6}$$

They lie on a curve of the third degree whose equation is

$$x = \sqrt{y} + 1/\sqrt{y} \quad \text{or} \quad x^2 y = (y+1)^2. \tag{27.7}$$

Figure 148 represents a nomogram for the equation

$$uvw = u+v+w.$$

It will be seen that representing equation (27.1) by means of a nomogram of this kind is useful only if one function, say f_1, assumes very small values and the other two assume very large values.

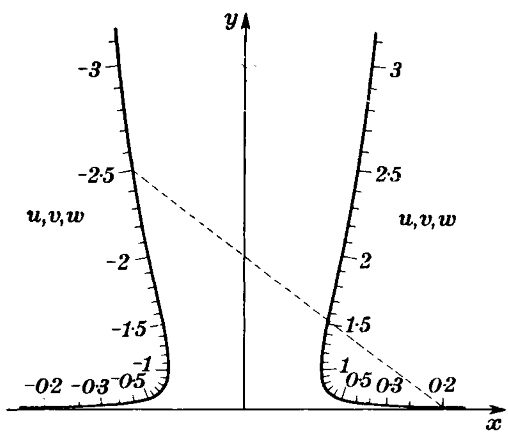

FIG. 148

We have found only curvilinear nomograms for equation (27.1). In § 28 we shall prove the non-existence, for that equation, of a nomogram with three rectilinear scales involving scales of functions f_i or functions constructed from f_i by means of the four arithmetical operations.

R e m a r k. Curve (27.7) has an isolated point, $(0, -1)$, which does not belong to any of the scales of (27.6), since the ordinates of points belonging to the scales are positive.

§ 28. The nomographic order of an equation. Kind of nomogram. Critical points

28.1. Equation

$$F(x, y, z) = 0$$

can be represented by a collineation nomogram only if it can be written in a Soreau form (§ 20, equation (26.1)).

A direct reduction of a given equation to form (S) involves as a rule very complicated calculations—we shall deal with this in § 32. It is often possible to simplify our investigations considerably by performing certain preliminary operations, aimed at reducing the left side of the equation $F(x, y, z) = 0$ to the form of a so called nomographic polynomial.

We shall define first a *nomographic monomial.* It is a product

$$af(x).g(y).h(z).k(u)$$

of factors depending on one variable only.

For example, the function $3 \sin x . \log(1+y) \tan z$ is a nomographic monomial of three variables, while the function $(x+y)z$ is not a nomographic monomial.

A *nomographic polynomial* is a finite sum of nomographic monomials.

The following functions are examples of nomographic polynomials:

$$xe^x . \log y+y \tan z+z+1,$$

$$(x+y+z)^2 = x^2+y^2+z^2+2xy+2yz+2xz,$$

$$\log (z+1)^{x-y} = x \log(z+1)-y \log(z+1).$$

It can be proved that, say, $\sqrt{x^2+y^2+z^2}$ is not a nomographic polynomial, but $\sin(x^2+y^2+z^2)$ is a nomographic polynomial because

$$\sin (x^2+y^2+z^2) = \sin x^2 \cos y^2 \cos z^2-\sin x^2 \sin y^2 \sin z^2+$$
$$+\cos x^2 \sin y^2 \cos z^2+ \cos x^2 \cos y^2 \sin z^2.$$

In the nomographic polynomial

$$a_1f_1(x)\, g_1(y)\, h_1(z)+a_2f_2(x)\, g_2(y)\, h_2(z)+ \dots +a_nf_n(x)\, g_n(y)\, h_n(z) \quad (28.1)$$

there occur n functions f_i of the variable x, n functions g_i of the variable y and n functions h_i of the variable z. In certain cases the polynomial can be written in a simpler form. For instance, in the nomographic polynomial:

$$w(x, y) = x^2y^2+5xy^2+y^2+x^2+3xy+8x+6y+8,$$

which is of the form

$$\sum_{i=1}^{8} a_i f_i(x) g_i(y),$$

we have the following functions of the variable x:

$$f_1 = f_4 = x^2, \quad f_2 = f_5 = f_6 = x \quad \text{and} \quad f_3 = f_7 = f_8 = 1,$$

and the following functions of the variable y:

$$g_1 = g_2 = g_3 = y^2, \quad g_5 = g_4 = y \quad \text{and} \quad g_1 = g_6 = g_8 = 1.$$

We thus have three different functions of the variable x and three different functions of the variable y. It will be seen, however, that $w(x, y)$ can be written in a simpler way by introducing the following functions:

$$f_1^* = f_1+5f_2+2f_3 = x^2+5x+2,$$

and

$$f_2^* = f_2+2f_3 = x+2,$$

as well as

$$g_1^* = g_1+g_4 = y^2+1$$

and

$$g_1^* = g_5+g_4 = y+1,$$

because

$$w(x, y) = f_1g_1+3f_2g_2 = (x^2+5x+2)(y^2+1)+3(x+2)(y+1).$$

To express the above in general terms let us adopt the following definition.

Functions $f_i(x)$, $i = 1, 2, \ldots, n$, defined in a common interval $\langle a, b\rangle$ are *linearly independent in a wider sense* if the identity

$$c_0+c_1f_1+c_2f_2+\ldots+c_nf_n = 0$$

implies that all c_i are equal to zero.

Function $f_l(x)$ is *linearly dependent* on functions $f_1, f_2, \ldots, f_n$ *in a wider sense* if there exist constants c_i for $i = 0, 1, \ldots, n$

which are not all equal to zero (i.e. $\sum c_i^2 \neq 0$) and are such that

$$f_l(x) = c_0 + c_1 f_1(x) + \ldots + c_n f_n(x).$$

A system of functions $f_1, \ldots, f_n$ is termed a *base*.

Obviously, if f_l is linearly dependent in a wider sense on functions f_i for $i = 1, 2, \ldots, k$, then functions $f_1, f_2, \ldots, f_n, f_l$ are linearly dependent. As regards the base, we shall assume it to be linearly independent.

Assume that among the functions f_i occurring in the nomographic polynomial (28.1) there are k functions $f_1, \ldots, f_k$ on which the remaining ones are linearly dependent, that among the functions g_i there are m functions $g_1, \ldots, g_m$ on which the remaining ones are linearly dependent in a wider sense and that among the functions h_i there are p functions $h_1, \ldots, h_p$ on which the remaining ones are linearly dependent in a wider sense.

Then the nomographic polynomial (28.1) can be written in the form

$$\sum b_{ijl} f_i(x) g_j(y) h_l(z) \tag{28.2}$$

where i varies from 1 to k, j varies from 1 to m and l varies from 1 to p.

It may occur of course that in the new form some coefficients b_{ijl} will be equal to zero, and consequently we shall have $k' < k$ instead of k functions f_i, $m' < m$ instead of m functions g_j, or $p' < p$ instead of p functions h_l.

In that case we shall repeat the process and obtain a form in which a smaller number of functions f_i, g_j, h_l occur.

Obviously, after a finite number of such steps, we shall obtain a form in which no further simplifications of this kind will be possible. The ultimate form will contain k_0 linearly independent functions of the variable x, m_0 linearly independent functions of the variable y and p_0 linearly independent functions of the variable z.

It can be shown that numbers k_0, m_0 and p_0 do not depend on the choice of the base in the individual steps of the procedure.

The sum $k_0+m_0+p_0$ is termed the *nomographic order of the polynomial.*

Function $f_1(x)+f_2(y)+f_3(z)$ which depends in an essential way on three variables, is a nomographic polynomial of the third nomographic order. The left side of the Cauchy equation $f_1(u)g_3(w)+f_2(v)h_3(w)+1$ (§ 16) is, in the case of linear independence of functions g_3 and h_3, a nomographic polynomial of the fourth nomographic order. Function

$$\begin{vmatrix} \varphi_1(x) & \psi_1(x) & 1 \\ \varphi_2(x) & \psi_2(x) & 1 \\ \varphi_3(x) & \psi_3(x) & 1 \end{vmatrix}$$

is a nomographic polynomial of order $\leqslant 6$.

The *nomographic order of the equation*

$$F(x, y, z) = 0$$

is the least order of the nomographic polynomial $w(x, y, z)$ occurring in the equivalent equation

$$w(x, y, z) = 0.$$

In a three-dimensional domain in which there are no zeros of functions $\varphi_3(x)$, $\psi_3(y)$, $\chi_3(z)$ equation (14.3) is of nomographic order $\leqslant 6$.

The definitions of the nomographic polynomial and of its order can be extended to functions of more variables. Thus for instance

$$xy^2z \log u+2^x u^2+y^2 \sin y+\log(u+1) \sin y$$

is a nomographic polynomial of four variables, x, y, z, u, of order 8, since we have here two functions of the variable x, namely 2^x and x, three functions of the variable y, namely y^2, $y^2 \sin y$ and $\sin y$, one function of the variable z, namely z, and three functions of the variable u, namely $\log u$, u^2 and $\log(u+1)$.

28.2. Collineation nomograms are divided into classes according to the number of curvilinear scales: if the number of curvilinear scales appearing in the nomogram is equal to k, we call it a *nomogram of kind k*.

The nomograms with three rectilinear scales which were dealt with in § 10–14 are of kind 0 because they do not contain any curvilinear scale; nomograms which can be used to represent equations of the Cauchy type are of kind one because they contain one curvilinear and two rectilinear scales. Nomograms for the Clark equation are of kind two or three according to whether the third scale is rectilinear or curvilinear (two lie on a curve of the second degree).

Obviously, a collineation nomogram can be at most of kind three since we have only three scales.

28.3. Suppose that for a given equation

$$F(x, y, z) = 0$$

there exists a pair of numbers x_0, y_0 such that the equation

$$f(x_0, y_0, z) = 0$$

with one variable, z, is satisfied for every value of z from a certain interval. The pair of numbers x_0, y_0 will then be termed a *neutral pair* for the given equation. For example, the equation

$$xy+z = 0$$

has a neutral pair $x = 0$ and $z = 0$ since the expression

$$0y+0$$

is equal to zero for any value of y. If this equation is written in the form

$$1/z+1/xy = 0,$$

then, if we want to consider also very large values of the variables, we must regard the pair $z = \infty$ and $x = \infty$ and the pair $z = \infty$ and $y = \infty$ as neutral pairs, since for any number $y \neq 0$ there exist such numbers x_n and z_n tending to infinity that the equation

$$1/z_n+1/x_n y = 0$$

is satisfied; for any number $x \neq 0$ there exist such numbers x_n and z_n tending to infinity that the equation

$$1/z'_n+1/xy_n = 0$$

is satisfied.

What corresponds in a nomogram to a neutral pair of numbers? If C_1, C_2 and C_3 are lines on which the scales lie, then (Fig. 149) the following two cases may occur:

1. C_3 is a curve; then point x_0 of scale C_1 must be identical with point y_0 of scale C_2 if the threes of points x_0, y_0, z with the variable point z running over the curve C_3 are to be always collinear. In this case the point $x_0 = y_0$ of intersection of curves C_1 and C_2 is called a *critical point* of the nomogram.

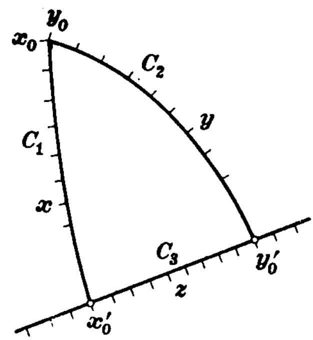

FIG. 149

2. C_3 is a straight line; then the pair of points of intersection of C_1 with C_3 and of C_2 with C_3 may be a neutral pair, x_0', y_0', for then every point representing number z forms a collinear three with the pair x_0', y_0'.

It can easily be seen that, conversely, every point of intersection of scales C_i and C_k has a neutral pair of numbers corresponding to it; similarly, in the case where C_j is a straight line intersecting the remaining two scales C_i and C_k at points P_i and P_k, the numbers corresponding to those points form a neutral pair of the equation. This follows from our assumption of the one-to-one correspondence between the values of the variable and the points of the corresponding scale.

Equation

$$(1-1/u)(1-1/v)(1-1/w) = 1$$

has six neutral pairs:

1. $u = 0,\ v = 1,$ 3. $v = 0,\ u = 1,$ 5. $w = 0,\ u = 1,$

2. $u = 0,\ w = 1,$ 4. $v = 0,\ w = 1,$ 6. $w = 0,\ v = 1.$

For example, in order to verify if the first pair is neutral it is sufficient to write the equation in the form

$$\left(1-\frac{1}{v}\right)\left(1-\frac{1}{w}\right)=\frac{1}{1-1/u} \quad \text{or} \quad \left(1-\frac{1}{v}\right)\left(1-\frac{1}{w}\right)=\frac{u}{u-1}.$$

The nomogram for this equation is made up of three rectilinear scales, u, v and w, which, when ordered in a cyclic manner, have their zero points each at point *1* of the preceding one (Fig. 150). Obviously, each vertex is a critical point of the nomogram and it is easy to verify that each pair of vertices represents one neutral pair of numbers (e.g., the pair 0_u, 1_v).

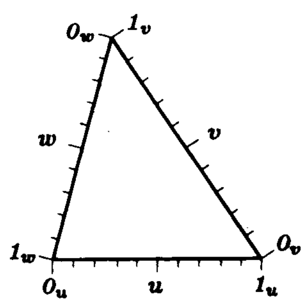

FIG. 150

Similarly, it is easy to verify that the equation

$$1/u+1/v = 1/w$$

has three neutral pairs

1. $u = 0$, $v = 0$, 2. $u = 0$, $w = 0$, 3. $v = 0$, $w = 0$.

These pairs correspond to only one critical point of the nomogram (Fig. 151).

In addition, let us observe that if the plane of a nomogram undergoes a projective transformation, a critical point is made to coincide with a critical point. On this ground we easily verify that an N-shaped nomogram has three critical points although the equation has six neutral pairs, and a nomogram with three parallel scales has three neutral pairs but one critical point.

It is easy to verify that the equation $f_1 f_2 f_3 = f_1 + f_2 + f_3$ does not possess a single real neutral pair, which makes it clear why in the nomogram of this equation (shown in Fig. 147) the straight line on which the w-scale lies does not intersect the curve. If there existed an intersection point, it would have to correspond to a neutral pair. Similarly, in the second nomogram for the equation (Fig. 148) the curve of the third degree does not possess

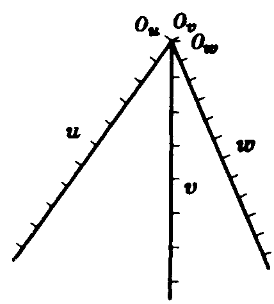

FIG. 151

a double point. Finally, it is obvious why we cannot have a nomogram of three straight lines for this equation—since every two straight lines on a plane intersect at a critical point of the nomogram.

Exercises

1. Find the nomographic order of the Cauchy, Clark and Soreau (type I and III) equations.

2. Indicate the critical points of the Clark nomogram.

§ 29. Equations of the third nomographic order

Assume that function $F(x, y, z)$ depends in an essential way on each of the three variables, x, y and z. This means that there exist numbers x_0, y_0 such that $F(x_0, y_0, z)$ is not constant, there exist numbers y_0, z_0 such that $F(x, y_0, z_0)$ is not constant and there exist numbers x_0, z_0 such that $F(x_0, y, z_0)$ is not constant. If function $F(x, y, z)$ can be reduced to the form of a nomographic polynomial, then, by hypothesis, at least three functions, $f_1(x)$, $f_2(y)$ and $f_3(z)$ must occur in that polynomial. Assume that we have only these three functions. Equation $F(x, y, z) = 0$ is then of the third nomographic order.

The most general form of an equation of the third nomographic order is the equation

$$a_{111} f_1 f_2 f_3 + a_{011} f_2 f_3 + a_{101} f_1 f_3 + a_{110} f_1 f_2 + a_{100} f_1 + a_{010} f_2 + \\ + a_{001} f_3 + a_{000} = 0. \quad (29.1)$$

Substituting $u = f_1(x)$, $v = f_2(y)$, $w = f_3(z)$ we obtain the form

$$a_{111} uvw + a_{011} vw + a_{101} uw + a_{110} uv + a_{100} u + a_{010} v + \\ + a_{001} w + a_{000} = 0. \quad (29.1')$$

The coefficients have been denoted here by letter a with three indices which run over the values 0 and 1; the first index is 1 or 0 according to whether u occurs in the component or not; the meaning of the second and the third index is analogous.

We shall prove the following theorem:

Every relation of form (29.1′) *can be reduced by suitable homographic substitutions,*

$$u = \frac{\alpha_1 u' + \alpha_2}{\alpha_3 u' + \alpha_4}, \qquad v = \frac{\beta_1 v' + \beta_2}{\beta_3 v' + \beta_4}, \qquad w = \frac{\gamma_1 w' + \gamma_2}{\gamma_3 w' + \gamma_4},$$

to one of the three canonical forms:

$$u'v'w' = 1, \quad \text{(I)}$$

$$u' + v' + w' = 0, \quad \text{(II)}$$

$$u'v'w' = u' + v' + w'. \quad \text{(III)}$$

We shall carry out the proof in two parts.

1. We shall first reduce (29.1′) to an equation in which the second degree terms have zero coefficients.

If $a_{111} \neq 0$, we can write the equation in the form

$$uvw + b_1 vw + b_2 uw + b_3 uv + c_1 u + c_2 v + c_3 w + d = 0 \quad (29.2)$$

or

$$(u+b_1)(v+b_2)(w+b_3) + \\ (c_1 - b_2 b_3) u + (c_2 - b_1 b_3) v + (c_3 - b_1 b_2) w + d = 0,$$

which, by translating $u' = u + b_1$, $v' = v + b_2$, $w' = w + b_3$, becomes

$$u'v'w' + b_1 u' + b_2 v' + b_3 w' + c = 0. \quad (29.3)$$

If $a_{111} = 0$ but one of the coefficients a_{ijk} is different from zero, we first perform the homographic substitution $u' = u^{(-1)^{i+1}}$, $v' = v^{(-1)^{k+1}}$ and $w' = w^{(-1)^{j+1}}$, reducing the equation in this way to form (29.2), e.g., if $a_{100} \neq 0$, then, by the homographic substitution $u' = u$, $v' = 1/v$, $w' = 1/w$, we obtain the equation

$$a_{100}u' + a_{011}\frac{1}{v'} \cdot \frac{1}{w'} + a_{101}u'\frac{1}{w'} + a_{110}u'\frac{1}{v'} + a_{010}\frac{1}{v'} + \\ + a_{001}\frac{1}{w'} + a_{000} = 0,$$

i.e., multiplying by $v'w'/a_{100}$, an equation of form (29.2).

Thus, in every case, equation (29.1′) can be reduced by homographic substitutions to a form containing no terms of the second degree.

2. In the second part of our proof we shall reduce equation (29.3) to one of the canonical forms.

a. If at least one of the numbers b_1, b_2, b_3 is equal to zero, this can be done in a very elementary manner. For example, let $b_3 = 0$ and $cb_1b_2 \neq 0$; we have the identity

$$u'v'w' + b_1u' + b_2v' + c = u'v'\left(w' - \frac{b_1b_2}{c}\right) + \frac{b_1b_2}{c}\left(u' + \frac{c}{b_1}\right)\left(v' + \frac{c}{b_2}\right),$$

which immediately gives

$$\frac{b_1u'}{b_1u'+c} \cdot \frac{b_2v'}{b_2v'+c}\left(w' - \frac{b_1b_2}{c}\right) + \frac{b_1b_2}{c} = 0.$$

This equation becomes an equation of form (I) if we substitute

$$u'' = \frac{u'}{b_1u'+c}, \qquad v'' = \frac{v'}{b_2v'+c}, \qquad w'' = -cw' + b_1b_2.$$

If, besides $b_3 = 0$, we also have $c = 0$, equation

$$u'v'w' + b_1u' + b_2v' = 0$$

can be written in form (II) since dividing by $u'v'$ we obtain

$$w' + b_1\frac{1}{v'} + b_3\frac{1}{u'} = 0,$$

and consequently $w'' + u'' + v'' = 0$, where $u'' = b_2/u'$ and $v'' = b_1/v'$.

b. In order to investigate the case of $c \neq 0$ and $b_1 b_2 b_3 \neq 0$ we shall introduce the notion of *singular homography*. This is what we call a correspondence between x and y defined by the equation

$$axy+bx+cy+d = 0 \qquad (\mathrm{h}_0)$$

if the coefficients a, b, c, and d satisfy the equality

$$ad = bc. \qquad (*)$$

(If $ad \neq bc$, equation (h_0) can be written in the form

$$y = -\frac{bx+d}{ax+c},$$

and this defines homography in the usual meaning of the term, since the determinant

$$\begin{vmatrix} -b & -d \\ a & c \end{vmatrix} = ad-bc$$

is different from zero.)

If $a \neq 0$, the singular homography equation is of the form

$$(ax+c)(y+b/a) = 0;$$

if $a = 0$, then either $b = 0$, or $c = 0$, i.e., the equation becomes

$$cy+d = 0 \quad \text{or} \quad bx+d = 0.$$

In all cases, a certain value x_0 (or y_0) of one variable can have any value of the other variable correspond to it; for we can see that, in the case of $a \neq 0$, for $x_0 = -c/a$ we can have an arbitrary y and for $y_0 = -b/a$ we can have an arbitrary x; in the case of $a = 0$, for $y_0 = -d/c$ we can have an arbitrary x and for $x_0 = -d/b$ we can have an arbitrary y.

Now let us find for equation (29.3) a value w_0'—we shall call it the *singular value* of the equation—for which the equation

$$w_0' u'v'+b_1 u'+b_2 v'+b_3 w_0'+c = 0$$

will define singular homography between the variables u' and v' (generally, for a fixed value of w', equation (29.3) defines ordinary homography).

The value w'_0 will be obtained, in view of condition (*), from the equation

$$w'_0(b_3 u'_0+c)-b_1 b_2 = 0,$$

i.e., from the equation

$$b_3 w_0'^2+cw'_0-b_1 b_2 = 0. \tag{w}$$

Similarly, the singular value of variable u' is a number u'_0 for which equation (29.3), i.e., the equation

$$u'_0 v'w'+b_2 v'+b_3 w'+b_1 u'_0+c = 0,$$

defines singular homography; the condition for that is the equality

$$u'_0(b_1 u'_0+c)-b_2 b_3 = 0$$

or

$$b_1 u_0'^2+cu'_0-b_2 b_3 = 0. \tag{u}$$

We shall similarly obtain the singular value of the variable v' from the equation

$$b_2 v_0'^2+cv'_0-b_1 b_3 = 0. \tag{v}$$

Relations (u), (v) and (w) are equations of the second degree with a common discriminant

$$\Delta = c^2+4b_1 b_2 b_3.$$

Let us first consider the case of $\Delta = 0$, i.e., $c^2 = -4b_1 b_2 b_3$. Let us effect a translation by substituting

$$u'' = u'+\frac{c}{2b_1}, \qquad v'' = v'+\frac{c}{2b_2}, \qquad w'' = w'+\frac{c}{2b_3};$$

we then obtain

$$\begin{aligned} u'v'w'+b_1 u'+b_2 v'+b_3 w' \\ &= \left(u''-\frac{c}{2b_1}\right)\left(v''-\frac{c}{2b_2}\right)\left(w''-\frac{c}{2b_3}\right)+ \\ &\quad +b_1\left(u''-\frac{c}{2b_1}\right)+b_2\left(v''-\frac{c}{2b_2}\right)+b_3\left(w''-\frac{c}{2b_3}\right)+c \\ &= u''v''w''-\frac{c}{2b_1}v''w''-\frac{c}{2b_2}u''w''-\frac{c}{2b_3}u''v''+ \end{aligned}$$

$$+\left(b_1+\frac{c^2}{4b_2b_3}\right)u''+\left(b_2+\frac{c^2}{4b_1b_3}\right)v''+\left(b_3+\frac{c^2}{4b_1b_2}\right)w''-$$

$$-\frac{c^3}{8b_1b_2b_3}-3\frac{c}{2}+c.$$

By hypothesis we have the free term $c^2 = -4b_1b_2b_3$, and consequently the coefficients of u'', v'' and w'' are equal to zero; we can thus write

$$u''v''w''-\frac{c}{2b_1}v''w''-\frac{c}{2b_2}u''w''-\frac{c}{2b_3}u''v'' = 0.$$

Dividing by $u''v''w''/c$ we obtain

$$\frac{1}{2b_1u''}+\frac{1}{2b_2v''}+\frac{1}{2b_3w''}-\frac{1}{c} = 0,$$

and substituting

$$u''' = \frac{1}{2b_1u''}, \qquad v''' = \frac{1}{2b_2v''},$$

$$w''' = \frac{1}{2b_3w''}-\frac{1}{c} = \frac{c-2b_3w''}{2b_3cw''}$$

we finally obtain

$$u'''+v'''+w''' = 0,$$

i.e. form (II).

If the discriminant Δ is different from 0, we shall base the method of reducing equation (29.3) to a canonical form on an identity which contains the pairs of singular values of the equation, i.e., by equations (u), (v) and (w), the numbers

$$u_0' = \frac{-c-\sqrt{\Delta}}{2b_1}, \qquad u_1' = \frac{-c+\sqrt{\Delta}}{2b_1},$$

$$v_0' = \frac{-c-\sqrt{\Delta}}{2b_2}, \qquad v_1' = \frac{-c+\sqrt{\Delta}}{2b_2},$$

$$w_0' = \frac{-c-\sqrt{\Delta}}{2b_3}, \qquad w_1' = \frac{-c+\sqrt{\Delta}}{2b_3}.$$

It is the identity

$$(\sqrt{\Delta}-c)(u-u_0)(v-v_0)(w-w_0)+(\sqrt{\Delta}+c)(u-u_1)\times$$
$$\times(v-v_1)(w-w_1) = 2\sqrt{\Delta}(uvw+b_1u+b_2v+b_3w+c). \qquad (29.4)$$

This identity can be verified either by direct algebraic calculation or, on the basis of the unicity of the Taylor expansion, by finding the partial derivatives of both sides of the equation.

Thus, if $\Delta \neq 0$ and $b_1b_2b_3 \neq 0$, we can write equation (29.3) in the form

$$c_1(u'-u_0')(v'-v_0')(w'-w_0')+c_2(u'-u_1')(v'-v_1')(w'-w_1') = 0.$$

If $\Delta > 0$, then the singular pairs u_0', u_1', v_0', v_1', w_0', w_1' are different numbers; consequently, substituting

$$u'' = \frac{u'-u_0'}{u'-u_1'}, \qquad v'' = \frac{v'-v_0'}{v'-v_1'}, \qquad w'' = -\frac{c_1}{c_2}\cdot\frac{w'-w_0'}{w'-w_1'},$$

we finally obtain

$$u''v''w'' = 1,$$

i.e., canonical form (I).

If $\Delta < 0$, we have complex numbers u_0', u_1', ... and $\sqrt{\Delta}$ in identity (29.4). Let us write this clearly:

$$\left(i\sqrt{-\Delta}-c\right)\left(u'+\frac{c}{2b_1}+\frac{i\sqrt{-\Delta}}{2b_1}\right)\left(v'+\frac{c}{2b_2}+\frac{i\sqrt{-\Delta}}{2b_2}\right)\times$$
$$\left(w'+\frac{c}{2b_3}+\frac{i\sqrt{-\Delta}}{2b_3}\right)+(i\sqrt{-\Delta}+c)\left(u'+\frac{c}{2b_1}-\frac{i\sqrt{-\Delta}}{2b_1}\right)\times$$
$$\times\left(v'+\frac{c}{2b_2}-\frac{i\sqrt{-\Delta}}{2b_2}\right)\left(w'+\frac{c}{2b_3}-\frac{i\sqrt{-\Delta}}{2b_3}\right) = 0.$$

In order to simplify the calculations let us write again this equation assuming

$$x\sqrt{-\Delta} = 2b_1u'+c, \qquad y\sqrt{-\Delta} = 2b_2v'+c, \qquad z\sqrt{-\Delta} = 2b_3w'+c;$$

reducing by $\left(\sqrt{-\Delta}\right)^3 8b_1b_2b_3$ we obtain

$$(i\sqrt{-\Delta}-c)(x+i)(y+i)(z+i)+(i\sqrt{-\Delta}+c)(x-i)(y-i)(z-i) = 0.$$

Performing the operations partly, we obtain

$$\left[\left(-cx-\sqrt{-\Delta}\right)+i\left(\sqrt{-\Delta}\,x-c\right)\right]\left[(yz-1)+i(y+z)\right]+$$
$$+\left[\left(cx+\sqrt{-\Delta}\right)+i\left(\sqrt{-\Delta}\,x-c\right)\right]\left[(yz-1)+i(-y-z)\right]=0.$$

We can now omit the real part, which, as follows from identity (29.4) by comparison with the right side, is equal to zero. The imaginary part will have the coefficient

$$2\left(\sqrt{-\Delta}x-c\right)(yz-1)-2(y+z)\left(cx+\sqrt{-\Delta}\right)=0.$$

We thus have ultimately the equation

$$\frac{\sqrt{-\Delta}x-c}{cx+\sqrt{-\Delta}}(yz-1)-(y+z)=0,$$

or, substituting

$$u'''=\frac{\sqrt{-\Delta}x-c}{cx+\sqrt{-\Delta}}, \qquad v'''=y, \qquad w'''=z,$$

a canonical equation of form (III):

$$u'''v'''w'''-u'''-v'''-w'''=0.$$

The theorem on reducing the general equation of the third nomographic order to a canonical form by means of homographic substitutions is thus finally proved, since a superposition of a finite number of homographic substitutions is a homographic substitution (§ 9).

Remark 1. In practical problems it is convenient to have the calculations needed to reduce a given equation to a canonical form divided into two parts in the same way as in the proof:

1. We reduce the equation to the simplified form (29.3);
2. We calculate the discriminant Δ and then find the singular values u_0, u_1, v_0, v_1, w_0, w_1 writing the index "0" if we have $\sqrt{\Delta}$ with the sign $+$ and the index "1" if we take $\sqrt{\Delta}$ with the sign $-$.

a. If $\Delta>0$, we use identity (29.4) obtaining at once canonical form (I).

b. If $\Delta < 0$, we also use identity (29.4), but, knowing that the real part of the left side is equal to zero, we calculate only the imaginary part; in performing the operations we multiply pairs of factors, $(\sqrt{\Delta}-c)(u-u_0)$ and $(v-v_0)(w-w_0)$, because this brings us quicker to canonical form (III).

Remark 2. Singular elements can also be found directly for the form (29.1′), i.e., without reducing the equation to a simplified form; in our proof, however, to make the calculations simpler, we have introduced form (29.3), because here the discriminant has a very simple structure and is obviously seen to be the same for all three equations, (u), (v) and (w). If the equation were left in the general form, the proof that the discriminants are equal would be lengthy. The reduction to a simpler form facilitates the calculations in one more respect: it shows clearly when we can pass to the canonical form directly without finding the singular values and when it is necessary to use identity (29.4).

EXAMPLE 1. Reduce to a canonical form the equation

$$\log z = \frac{-x\sin y - x + \sin y - 3}{x\sin y - x + 2\sin y - 3}.$$

Solution. 1. Substituting $u = x$, $v = \sin y$ and $w = \log z$, we obtain the equation

$$uvw+2vw-uw+uv+u+v-3w+3 = 0.$$

Joining the first four terms we obtain a simplified form,

$$(u+2)(v-1)(w+1)+u-2v+2w+u-v-3w+5 = 0,$$
$$(u+2)(v-1)(w+1)+2u-3v-w+5 = 0,$$
$$u'v'w'+2u'-3v'-w'-1 = 0,$$

where $u = u'-2$, $v = v'+1$ and $w = w'-1$.

2. In the second part of the solution we find the singular values:

a. $u'v'w'-3v'-w'+(2u'-1) = 0$, $u'(2u'-1)-3 = 0$,
$2u'^2-u'-3 = 0$, whence $u'_0 = -1$, $u'_1 = 3/2$.

b. $v'u'w'+2u'-w'-(3v'+1) = 0$, $-v'(3v'+1)+2 = 0$,
$3v'^2+v'-2 = 0$, whence $v'_0 = -1$, $v'_1 = 2/3$.

c. $w'u'v'+2u'-3v'-(w'+1) = 0$, $-w'(w'+1)+6 = 0$,
$w'^2+w'-6 = 0$, whence $w'_0 = -3$, $w'_1 = 2$.

Since the discriminant Δ is equal to $(-1)^2+4.2(-3)(-1) = 25$, identity (29.4) gives in this case

$$(5+1)(u'+1)(v'+1)(w'+3)+(5-1)(u'-3/2)(v'-2/3)(w'-2)=0$$

or

$$\frac{3}{2}\cdot\frac{u'+1}{u'-3/2}\cdot\frac{v'+1}{v'-2/3}\cdot\frac{w'+3}{w'-2}=1.$$

Coming back to the original variables, i.e., writing

$$u' = u+2 = x+2, \qquad v' = v-1 = \sin y-1,$$
$$w' = 1+w = 1+\log z$$

we finally obtain canonical form (I):

$$9\,\frac{x+3}{2x+1}\cdot\frac{\sin y}{3\sin y-5}\cdot\frac{4+\log z}{-1+\log z}=1.$$

EXAMPLE 2. Write in a canonical form the equation

$$z=\frac{\sqrt{x}-5y^2-6}{y^2\sqrt{x}+2}.$$

Solution. 1. Substituting $u=\sqrt{x}$, $v=y^2$ and $w=z$, we can see that the equation

$$uvw-u+5v+2w+6=0$$

already has a simplified form.

2. We seek the singular values:

a. $uvw+5v+2w+(-u+6)=0$, $u(-u+6)-10=10$, whence $u_0=3-i$, $u_1=3+i$.

b. $vuw-u+2w+(5v+6)=0$, $v(5v+6)+2=0$, whence $v_0=-3/5-i/5$, $v_1=-3/5+i/5$.

c. $wuv-u+5v+(2w+6)=0$, $w(2w+6)+5=0$, whence $w_0=-3/2-i/2$, $w_1=-3/2+i/2$.

By identity (29.4) we can write our equation in the form

$$(2i-6)(u-3+i)(v+3/5+i/5)(w+3/2+i/2)+$$
$$+(2i+6)(u-3-i)(v+3/5-i/5)(w+3/2-i/2)=0.$$

To simplify the calculations, substitute

$$u'=u-3, \qquad v'=5v+3, \qquad w'=2w+3;$$

we then obtain

$$(i-3)\,(u'+i)\,(v'+i)\,(w'+i)+(i+3)\,(u'-i)\,(v'-i)\,(w'-i) = 0$$

or

$$[(-3u'-1)+i(u'-3)]\,[(v'w'-1)+i(v'+w')]+$$
$$+[(3u'+1)+i(u'-3)]\,[(v'w'-1)+i(-v'-w')] = 0.$$

Calculating the imaginary part we finally obtain

$$2(u'-3)\,(v'w'-1)+2(-3u'-1)\,(v'+w') = 0$$

or

$$\frac{u'-3}{3u'-1}\,(v'w'-1)-v'-w' = 0.$$

We have reduced the equation to form (III):

$$\frac{\sqrt{x}-6}{3\sqrt{x}-8}\,(5y^2+3)\,(2z+3) = \frac{\sqrt{x}-6}{3\sqrt{x}-8}+(5y^2+3)+(2z+3).$$

EXAMPLE 3. Write in a canonical form the equation

$$\log z = \frac{3\cos y-2\sin x-12}{\sin x\cos y+6}.$$

S o l u t i o n. 1. Writing $u = \sin x$, $v = \cos y$, $w = \log z$, we have

$$uvw+6w-3v+2u+12 = 0.$$

This is a simplified form.

2. We find the singular values:

a. $uvw-3v+6w+(2u+12) = 0$, $u(2u+12)+18 = 0$,
$u^2+6u+9 = 0$, whence $u_0 = u_1 = -3$.

b. $vuw+2u+6w+(-3v+12) = 0$, $v(-3v+12)-12 = 0$,
$v^2-4v+4 = 0$, whence $v_0 = v_1 = 2$.

c. $wuv+2u-3v+(6w+12) = 0$, $w(6w+12)+6 = 0$,
$w^2+2w+1 = 0$, whence $w_0 = w_1 = -1$.

2. We substitute

$$u' = u+3, \quad v' = v-2, \quad w' = w+1$$

and obtain

$$(u'-3)\,(v'+2)\,(w'-1)+2(u'-3)-3(v'+2)+6(w'-1)+12$$
$$= u'v'w'-u'v'+2u'w'-3v'w' = 0.$$

Dividing by $u'v'w'$ we obtain the canonical form

$$-\frac{3}{u'}+\frac{2}{v'}-\frac{1}{w'}+1=0,$$

$$\frac{3}{u'}-\frac{2}{v'}+\frac{1-w'}{w'}=0,$$

$$\frac{3}{3+\sin x}+\frac{2}{2-\cos y}+\frac{-\log z}{1+\log z}=0.$$

EXAMPLE 4. Reduce to a canonical form the equation

$$x^2y+3x^2z^3+2yz^3+5z^3=0.$$

Solution. Here the transformations are elementary; we assume

$$u=x^2, \quad v=y, \quad w=z^3$$

and obtain in succession

$$uv+3uw+2vw+5w=0,$$

$$\frac{uv}{w}+3u+2v+5=0,$$

$$\frac{1}{w}+5\left(\frac{1}{u}+\frac{3}{5}\right)\left(\frac{1}{v}+\frac{2}{5}\right)-\frac{6}{5}=0,$$

$$5\left(\frac{1}{u}+\frac{3}{5}\right)\left(\frac{1}{v}+\frac{2}{5}\right)\frac{1}{1/w-6/5}+1=0,$$

$$\left(\frac{6}{u}+3\right)\left(\frac{5}{v}+2\right)\frac{w}{5-6w}+1=0;$$

substituting the original variables, we finally obtain

$$\left(\frac{5}{x^2}+3\right)\left(\frac{5}{y}+2\right)\frac{z^3}{5-6z^3}+1=0.$$

Exercises

Reduce the following equations to a canonical form:

1. $uvw+3uv+4vw-5uw+u-v+8w-1=0.$

2. $-2uvw+uv-uw+vw+1=0.$

3. $uvw=u+v-w.$

4. $uvw+u+v+2w+2=0.$

§ 30. Equations of the fourth nomographic order

An equation of the fourth order (in the sense of nomography) contains four linearly independent functions: one function of the variable x, namely $f_1(x)$, one function of the variable y, namely $f_2(y)$, and two functions of the variable z, namely $\varphi_3(z)$ and $\psi_3(z)$. It is an equation of the form

$$\alpha_{12}\varphi_3+\beta_{12}\psi_3+\gamma_{12} = 0,$$

where α_{12}, β_{12} and γ_{12} are nomographic polynomials with variables f_1 and f_2. Let us write this in a homogeneous form,

$$\alpha_{12}\varphi_3+\beta_{12}\psi_3+\gamma_{12}\chi_3 = 0, \tag{30.1}$$

which, divided by χ_3, can give us of course the preceding form (thus equation (30.1) is not of the fifth order although five functions occur in it, since it can be reduced to the fourth order).

The coefficients in equation (30.1) are nomographic polynomials:

$$\alpha_{12} = a_{11}f_1f_2+a_{21}f_1+a_{31}f_2+a_{41},$$
$$\beta_{12} = a_{12}f_1f_2+a_{22}f_1+a_{32}f_2+a_{42},$$
$$\gamma_{12} = a_{13}f_1f_2+a_{23}f_1+a_{33}f_2+a_{43}.$$

It will be observed that the general Clark equation,

$$f_1f_2\varphi_3+(f_1+f_2)\psi_3+\chi_3 = 0,$$

is, under the assumption of linear independence of φ_3 and ψ_3, an equation of the fourth nomographic order. Similarly, the Cauchy equation

$$f_1\varphi_3+f_2\psi_3+\chi_3 = 0,$$

is a special equation of the fourth nomographic order.

It will thus be seen that every equation of the fourth nomographic order can be represented, by a homographic transformation of one of the variables f_1 and f_2, in the Clark form or in the Cauchy form.

THEOREM. *Every equation of the fourth nomographic order can be reduced to one of the canonical forms,*

$$F_1F_2\Phi_3+(F_1+F_2)\Psi_3+X_3 = 0 \quad (\textit{the Clark equation}),$$
$$F_1\Phi_3+F_2\Psi_3+X_3 = 0 \quad (\textit{the Cauchy equation}),$$

where F_1 is a homographic function of the variable f_1, F_2 is a linear function of the variable f_2, and Φ_3, Ψ_3 and X_3 are linear functions of the variables φ_3, ψ_3 and χ_3.

P r o o f. Write equation (30.1) in the form

$$(a_{11}\varphi_3+a_{12}\psi_3+a_{13}\chi_3)f_1f_2+(a_{21}\varphi_3+a_{22}\psi_3+a_{23}\chi_3)f_1+$$
$$+(a_{31}\varphi_3+a_{32}\psi_3+a_{33}\chi_3)f_2+(a_{41}\varphi_3+a_{42}\psi_3+a_{43}\chi_3) = 0$$

and denote the coefficients of f_1f_2, f_1 and f_2 and the free term by L_1, L_2, L_3 and L_4 respectively; we thus have

$$\begin{aligned} a_{11}\varphi_3+a_{12}\psi_3+a_{13}\chi_3 &= L_1, \\ a_{21}\varphi_3+a_{22}\psi_3+a_{23}\chi_3 &= L_2, \\ a_{31}\varphi_3+a_{32}\psi_3+a_{33}\chi_3 &= L_3, \\ a_{41}\varphi_3+a_{42}\psi_3+a_{43}\chi_3 &= L_4. \end{aligned} \tag{l}$$

In our further considerations we shall use the fact that the determinant of this system is equal to zero, i.e., that

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & L_1 \\ a_{21} & a_{22} & a_{23} & L_2 \\ a_{31} & a_{32} & a_{33} & L_3 \\ a_{41} & a_{42} & a_{43} & L_4 \end{vmatrix} = 0; \tag{Cl}$$

this follows from the fact that (l), as a system of four linear equations with coefficients a_{ik} and free terms L_k, is solved by φ_3, ψ_3 and χ_3, none of which is equal to zero by hypothesis.

Expanding the determinant according to the last column we obtain equation (Cl) in the form

$$A_1L_1-A_2L_2+A_3L_3-A_4L_4 = 0, \tag{Cl}$$

where A_1, A_2, A_3 and A_4 are subdeterminants of the terms L_1, L_2, L_3 and L_4.

Equation (Cl) is termed the *Clark identity.*

Proceeding to the proof proper let us ask whether there exists a homographic transformation

$$f_1 = \frac{\alpha f_1^* + \beta}{\gamma f_1^* + \delta} \tag{p}$$

$\left(\text{i.e., such that} \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \neq 0\right)$ which turns equation (30.1) into a Clark equation.

Let us write equation (30.1) in the form

$$L_1 f_1 f_2 + L_2 f_1 + L_3 f_2 + L_4 = 0$$

and perform substitution (p); multiplying by the denominator we obtain the equation

$$L_1(\alpha f_1^* + \beta) f_2 + L_2(\alpha f_1^* + \beta) + L_3 f_2(\gamma f_1^* + \delta) + L_4(\gamma f_1^* + \delta) = 0,$$

$$(\alpha L_1 + \beta L_3) f_1^* f_2 + (\alpha L_2 + \gamma L_4) f_1^* + (\beta L_1 + \delta L_3) f_2 + (\beta L_2 + \delta L_4) = 0.$$

This equation will have a Clark form if the coefficient of f_1^* is equal to the coefficient of f_2:

$$\alpha L_2 + \gamma L_4 = \beta L_1 + \delta L_3.$$

As can be seen from the Clark identity, the last equation will be satisfied for all values of φ_3, ψ_3 and χ_3 if we take

$$\alpha = A_2, \quad \beta = A_1, \quad \gamma = A_4 \quad \text{and} \quad \delta = A_3.$$

Let us discuss the following cases:

a. $\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = A_2 A_3 - A_1 A_4 \neq 0.$

b. $A_2 A_3 = A_1 A_4$, but A_i are not all equal to zero.

c. $A_1 = A_2 = A_3 = A_4 = 0.$

In case a., as has been verified, the homography

$$f_1 = \frac{A_2 f_1^* + A_1}{A_4 f_1^* + A_3}$$

transforms equation (30.1) into a Clark equation.

In case b., without asking ourselves whether there exists another homographic transformation (i.e., such that $\alpha : \beta : \gamma : \delta \neq A_2 : A_1 : A_4 : A_3$), let us verify whether it is possible by translating

$$f_1 = f_1^* + m, \quad f_2 = f_2^* + n$$

to reduce the given equation to a Cauchy form. We should then have the equation

$$L_1(f_1^* + m)(f_2^* + n) + L_2(f_1^* + m) + L_3(f_2^* + n) + L_4 = 0,$$

or

$$L_1 f_1^* f_2^* + (nL_1 + L_2) f_1^* + (mL_1 + L_3) f_2^* + mL_2 + nL_3 + \\ + L_4 + mnL_1 = 0. \quad \text{(Ca)}$$

For example, if $A_4 \neq 0$, it will be seen that assuming

$$m = A_2/A_4, \quad n = -A_3/A_4$$

we shall have by the Clark identity and the assumption $A_2A_3 = A_1A_4$ and $A_4 \neq 0$

$$mL_2+nL_3+L_4+mnL_1 = \frac{A_2}{A_4}L_2-\frac{A_3}{A_4}L_3+L_4-\frac{A_2A_3}{A_4^2}L_1$$
$$= -\frac{1}{A_4}(A_1L_1-A_2L_2+A_3L_3-A_4L_4) = 0$$

for every value of functions φ_3, ψ_3 and χ_3.

Now assuming in equation (Ca)

$$nL_1+L_2 = \frac{-A_3L_1+A_4L_2}{A_4} = \Psi_3, \quad mL_1+L_3 = \frac{A_2L_1+A_4L_3}{A_4} = \Phi_3$$

and

$$f_1^{**} = 1/f_1^*, \quad f_2^{**} = 1/f_2^*,$$

we obtain

$$\Phi_3 f_1^{**}+\Psi_3 f_2^{**}+L_1 = 0,$$

i.e., an equation of the Cauchy form.

If $A_4 = 0$, then by the assumption of $A_2A_3 = A_1A_4$ we should have $A_2 = 0$ or $A_3 = 0$. Assume that, say, $A_2 = 0$. The Clark identity becomes

$$A_1L_1-A_3L_3 = 0.$$

At least one of the coefficients A_1 and A_3 is different from zero; e.g., if $A_3 \neq 0$, then

$$L_3 = \frac{A_1}{A_3}L_1$$

and equation (30.1) becomes

$$L_1f_1f_2+L_2f_1+\frac{A_1}{A_3}L_1f_2+L_4 = 0$$

or

$$L_1\left(f_1+\frac{A_1}{A_3}\right)f_2+L_2\left(f_1+\frac{A_1}{A_3}\right)+L_4-L_2\frac{A_1}{A_3} = 0.$$

Assuming that

$$f_1 + A_1/A_3 = 1/f_1^*$$

we have

$$\left(L_4 - L_2 \frac{A_1}{A_3}\right) f_1^* + L_1 f_2 + L_2 = 0,$$

i.e., also an equation of the Cauchy form.

We follow a similar procedure if $A_2 \neq 0$ and $A_3 \neq 0$.

If $A_4 = A_3 = A_2 = 0 \neq A_1$, then equation (30.1) is an equation of the third nomographic order, since the Clark identity implies $L_1 = 0$ for all values of the third variable.

In case c., for $A_1 = A_2 = A_3 = A_4 = 0$, we shall prove that equation (30.1) is an equation of the third nomographic order.

Accordingly, let us assume that at least one of the subdeterminants of the matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \tag{t}$$

is different from zero; e.g., let

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \alpha_{33} \neq 0. \tag{c}$$

Let us write the system of equations

$$a_{11}\varphi_3 + a_{12}\psi_3 + a_{13}\chi_3 = L_1,$$

$$a_{21}\varphi_3 + a_{22}\psi_3 + a_{23}\chi_3 = L_2,$$

$$a_{31}\varphi_3 + a_{32}\psi_3 + a_{33}\chi_3 = L_3.$$

Since the determinant of this system is equal to zero,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0,$$

multiplying these equations by the subdeterminants α_{13}, α_{23}, α_{33} of the terms of the last column we obtain zero:

$$0 = \alpha_{33}L_3 - \alpha_{23}L_2 + \alpha_{13}L_1,$$

whence by the assumption (c) that $a_{33} \neq 0$

$$L_3 = (a_{23}L_2 - a_{13}L_1)/a_{33}.$$

Similarly, a system of equations

$$a_{11}\varphi + a_{12}\psi + a_{13}\chi = L_1,$$
$$a_{21}\varphi + a_{22}\psi + a_{23}\chi = L_2,$$
$$a_{41}\varphi + a_{42}\psi + a_{43}\chi = L_4,$$

in view of its determinant being equal to zero, gives us

$$L_4 = (a'_{23}L_2 - a'_{13}L_1)/a_{33},$$

where

$$a'_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{41} & a_{42} \end{vmatrix} \quad \text{and} \quad a'_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{41} & a_{42} \end{vmatrix}.$$

Equation (30.1) assumes the form

$$a_{33}L_1 f_1 f_2 + a_{33}L_2 f_1 + (a_{23}L_2 - a_{13}L_1)f_2 + (a'_{23}L_2 - a'_{13}L_1) = 0$$

or, on dividing by L_1,

$$a_{33} f_1 f_2 + a_{33}\frac{L_2}{L_1} f_1 + a_{23}\frac{L_2}{L_1} f_2 - a_{13} f_2 + a'_{23}\frac{L_2}{L_1} - a'_{13} = 0.$$

It is an equation of the third nomographic order because the third variable z occurs only in the function L_2/L_1.

If all the subdeterminants of the second row of table (t) were equal to zero, then—as we know—its rows would be proportional and the expressions L_1, L_2, L_3 and L_4 would of course also be proportional:

$$L_1 : L_2 : L_3 : L_4 = c_1 : c_2 : c_3 : c_4;$$

equation (30.1) would then assume the form

$$\frac{L_1}{c_1}(c_1 f_1 f_2 + c_2 f_1 + c_3 f_2 + c_4) = 0.$$

It would be a singular relation between the variables x, y and z: arbitrary values of x and y would always have the same number z corresponding to them, namely the root of the equation $L_1 = 0$; and to numbers z for which $L_1(z) \neq 0$ there would correspond pairs of numbers x and y satisfying the equation

$$c_1 f_1 f_2 + c_2 f_1 + c_3 f_2 + c_4 = 0.$$

Clearly, it is only the last equation that could profitably be represented by a nomogram (a double scale).

The theorem on reducing an equation of the fourth nomographic order to the Clark form or to the Cauchy form has thus been proved for all cases.

EXAMPLE 1. Reduce to a canonical form the equation:

$$(xy+z+y+1)\sqrt{z+1}+(xy-x+3y+2)\sqrt{z+2}+(2xy+3) = 0.$$

We have $\varphi_3 = \sqrt{z+1}$, $\psi_3 = \sqrt{z+2}$ and $\chi_3 = 1$.

We write the equation in the form

$$xy(\varphi_3+\psi_3+2\chi_3)+x(\varphi_3-\psi_3+0\chi_3)+$$
$$+y(\varphi_3+3\psi_3+0\chi_3)+(\varphi_3+2\psi_3+3\chi_3) = 0$$

with coefficients depending on z:

$$L_1 = \varphi_3+\psi_3+2\chi_3,$$
$$L_2 = \varphi_3-\psi_3,$$
$$L_3 = \varphi_3+3\psi_3,$$
$$L_4 = \varphi_3+2\psi_3+3\chi_3.$$

The Clark identity for this example will have the form

$$\begin{vmatrix} 1 & 1 & 2 & L_1 \\ 1 & -1 & 0 & L_2 \\ 1 & 3 & 0 & L_3 \\ 1 & 2 & 3 & L_4 \end{vmatrix} = 0$$

or

$$12L_1-4L_2+0L_3-8L_4 = 0.$$

On the strength of the theorem which has been proved the homographic substitution

$$x = \frac{4x^*+12}{8x^*+0} = \frac{x^*+3}{2x^*}$$

reduces the equation to the Clark form

$$L_1(x^*+3)y+L_2(x^*+3)+L_3y.2x^*+L_4.2x^* = 0,$$
$$(L_1+2L_3)x^*y+(2L_4+L_2)x^*+3L_1y+3L_2 = 0,$$
$$(3\varphi_3+7\psi_3+2\chi_3)x^*y+(3\varphi_3+3\psi_3+6\chi_3)x^*+$$
$$+(3\varphi_3+3\psi_3+6\chi_3)y+3\varphi_3-3\psi_3 = 0.$$

Reverting to the variables x and z we finally obtain

$$(3\sqrt{z+1}+7\sqrt{z+2}+2)\frac{3}{2x-1}y+(3\sqrt{z+1}+$$

$$+3\sqrt{z+2}+6)\left(\frac{3}{2x-1}+y\right)+3\sqrt{z+1}-3\sqrt{z+2}=0.$$

As we see, this is a Clark form.

EXAMPLE 2. Reduce to a canonical form the equation

$$(\sin x\cos y+2\sin x+3\cos y-1)\sin z+$$

$$+(2\sin x\cos y-\sin x+\cos y+3)\sin 2z+$$

$$+(3\sin x\cos y+4\sin x+7\cos y-1)\sin 3z=0.$$

Solution. Let us substitute

$$\sin x=u,\qquad \cos y=v,\qquad \sin z=w_1,$$

$$\sin 2z=w_2,\qquad \sin 3z=w_3$$

and let us represent the left side as a polynomial with variables u and v. We obtain

$$L_1uv+L_2u+L_3v+L_4=0,$$

where

$$\begin{aligned} L_1&=w_1+2w_2+3w_3,\\ L_2&=2w_1-w_2+4w_3,\\ L_3&=3w_1+w_2+7w_3,\\ L_4&=-w_1+3w_2-w_3.\end{aligned}\qquad (1)$$

The Clark identity assumes the form

$$\begin{vmatrix} 1 & 2 & 3 & L_1\\ 2 & -1 & 4 & L_2\\ 3 & 1 & 7 & L_3\\ -1 & 3 & -1 & L_4\end{vmatrix}=0L_1-0L_2+0L_3-0L_4=0.$$

Since all the coefficients in the Clark identity are equal to zero, the equation is of the third nomographic order. Equations (1) give us

$$L_3=L_1+L_2\quad\text{and}\quad L_4=L_1-L_2;$$

we thus have the form

$$L_1 uv + L_2 u + (L_1 + L_2) v + (L_1 - L_2) = 0$$

or

$$L_1(uv + v + 1) + L_2(u + v - 1) = 0,$$

$$uv + v + 1 + \frac{L_2}{L_1} u + \frac{L_2}{L_1} v - \frac{L_2}{L_1} = 0,$$

which is indeed an equation of the third nomographic order because only three linearly independent functions occur in it:

$$u = \sin x, \quad v = \cos y \quad \text{and} \quad \frac{L_1}{L_2} = \frac{2\sin z - \sin 2z + 4\sin z}{\sin z + 2\sin 2z + 3\sin 3z}.$$

EXAMPLE 3. Reduce to a canonical form the equation

$$(xy - x)z + (xy + x + y + 1)z^2 + (xy - x + y - 1)z^3 = 0.$$

Solution. We write the equation in the form

$$xy(z + z^2 + z^3) + x(-z + z^2 - z^3) + y(z^2 + z^3) + (z^2 - z^3) = 0.$$

We have here

$$z + z^2 + z^3 = L_1,$$
$$-z + z^2 - z^3 = L_2,$$
$$z^2 + z^3 = L_3,$$
$$z^2 - z^3 = L_4,$$

whence the Clark identity is given by the equality

$$\begin{vmatrix} 1 & 1 & 1 & L_1 \\ -1 & 1 & -1 & L_2 \\ 0 & 1 & 1 & L_3 \\ 0 & 1 & -1 & L_4 \end{vmatrix} = 2L_1 + 2L_2 - 2L_3 - 2L_4 = 0.$$

Condition $A_2 A_3 - A_1 A_4 = 0$ being satisfied, the equation can be represented in the Cauchy form by means of a translation

$$x = x^* + A_2/A_4 = x^* - 1 \quad \text{and} \quad y = y^* - A_3/A_4 = y^* + 1,$$

$$(x^* - 1)(y^* + 1) L_1 + (x^* - 1) L_2 + (y^* + 1) L_3 + L_4 = 0,$$

$$x^* y^* L_1 + x^*(L_1 + L_2) + y^*(-L_1 + L_3) + L_4 - L_1 - L_2 + L_3 = 0,$$

but $L_1+L_2-L_3-L_4 = 0$, and so

$$x^*y^*L_1+x^*(L_1+L_2)+y^*(L_3-L_1) = 0,$$

$$(L_3-L_1)\frac{1}{x+1}+(L_1+L_2)\frac{1}{y-1}+L_1 = 0,$$

$$z\frac{1}{x+1}+2z^2\frac{1}{1-y}+(z+z^2+z^3) = 0.$$

The last equation has the Cauchy canonical form.

Exercises

Reduce to canonical forms the following equations:

1. $3uvw+3vw-uw+2uw^2-2w^2+6uv-w-6v-2u+2 = 0.$
2. $\cos z\sin^2 y-\sin z\cos x\cos^2 y+\cos x\cos z-\sin z\cos^2 y+\cos z+1 = 0.$
3. $uvw+uvw^2+5uw^2+3uw+3vw-vw^2-5w^2-6uv+3w-26u+2v+14 = 0.$
4. $\sin^2 x\cos y-\sin^3 z+\sin^3 y\sin x-\sin x-\sin^2 y+\sin x\cos y\sin^3 z = 0.$

§ 31. Criteria of nomogrammability of a function

Function $F(x, y, z)$ is said to be *nomogrammable* if there exist functions

$$X_i(x),\quad Y_i(y)\quad \text{and}\quad Z_i(z),\quad i = 1, 2, 3$$

for which we have the identity

$$F(x, y, z) = \begin{vmatrix} X_1(x) & X_2(x) & X_3(x) \\ Y_1(y) & Y_2(y) & Y_3(y) \\ Z_1(z) & Z_2(z) & Z_3(z) \end{vmatrix}. \tag{31.0}$$

In § 26 we proved the nomogrammability of functions

$$F(x, y, z) = f_1(x)f_2(y)f_3(z)-1$$

and of functions

$$(f_2-f_1)(f_1f_2f_3-1)$$

and

$$(f_1-f_2)(f_2-f_3)(f_3-f_1)(f_1f_2f_3-1),$$

but the Massau method, which was used, gave no answer in the case where the corresponding functions G and H (26.2) could not be found.

In this section we shall give the necessary and sufficient conditions of nomogrammability of a function.

31.1. Let us first deal with the particular case of function F depending on two variables.

The necessary and sufficient condition for a function of two variables $F(x, y)$ to be of the form

$$F(x, y) = \begin{vmatrix} X_1(x) & X_2(x) \\ Y_1(y) & Y_2(y) \end{vmatrix} \tag{31.1}$$

in a rectangular domain D is the identity

$$F(x, y) = \frac{F(x, b) \, . \, F(a, y)}{F(a, b)} + \frac{F(x, b') \, . \, F(a', y)}{F(a', b')}$$

for two pairs, a, b' and a', b, such that

$$F(a, b') = F(a', b) = 0 \quad \textit{and} \quad F(a, b) \neq 0 \neq F(a', b'). \tag{31.2}$$

Necessity: Assume that there exists a representation of form (31.1) and consider a curve C_x with parametric equations (Fig. 152)

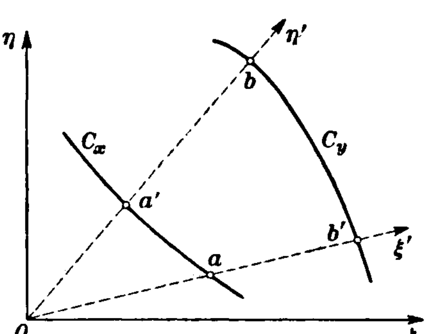

FIG. 152

$$\xi = X_1(x), \qquad \eta = X_2(x), \tag{C_x}$$

and a curve C_y with parametric equations

$$\xi = Y_1(y), \qquad \eta = Y_2(y), \tag{C_y}$$

Let ξ' and η' be different straight lines starting from the origin O of the system, let the line ξ' intersect the curves C_x and C_y at points corresponding to the values $x = a$ and $y = b'$ and let the line y' intersect the curves C_x and C_y at points corresponding to the values $x = a'$ and $y = b$.

It will be observed that the values of function $F(x, y)$ are proportional to the area of a triangle with one vertex at the origin of the system, the second lying on the curve C_x and the third on the curve C_y. Let us change the system ξ, η into the system of axes ξ', η'. The abscissas of the points corresponding to the values a' and b will be equal to zero, and the ordinates of the points corresponding to the values a and b' will also be equal to zero. We shall thus obtain

$$F(a', y) = \begin{vmatrix} 0 & X_2(a') \\ Y_1(y) & Y_2(y) \end{vmatrix} = -X_2(a')Y_1(y),$$

$$F(a, y) = \begin{vmatrix} X_1(a) & 0 \\ Y_1(y) & Y_2(y) \end{vmatrix} = X_1(a)Y_2(y),$$

$$F(x, b) = \begin{vmatrix} X_1(x) & X_2(x) \\ 0 & Y_2(b) \end{vmatrix} = Y_2(b)X_1(x),$$

$$F(x, b') = \begin{vmatrix} X_1(x) & X_2(x) \\ Y_1(b') & 0 \end{vmatrix} = -Y_1(b')X_2(x).$$

These equations imply that

$$F(x, y) = X_1(x)\,Y_2(y) - X_2(x)\,Y_1(y)$$
$$= k_1 F(x, b) F(a, y) + k_2 F(x, b') F(a', y).$$

The constants k_1 and k_2 will be found by substituting $x = a$, $y = b$ and then $x = a'$, $y = b'$:

$$F(a', b') = 0 + k_2 F(a', b') F(a', b'),$$

$$F(a, b) = k_1 F(a, b) F(a, b) + 0;$$

we obtain

$$k_1 = \frac{1}{F(a, b)} \quad \text{and} \quad k_2 = \frac{1}{F(a', b')},$$

and thus finally

$$F(x, y) = \frac{F(x, b)\,F(a, y)}{F(a, b)} + \frac{F(x, b')\,F(a', y)}{F(a', b')}$$

$$= \begin{vmatrix} F(x, b) & -\dfrac{F(x, b')}{F(a', b')} \\ F(a', y) & \dfrac{F(a, y)}{F(a, b)} \end{vmatrix}.$$

The sufficiency of the condition is obvious.

Remark 1. The proof can be conducted in a purely algebraic way; however, the way which has been chosen here shows the naturalness of assumption (31.2). Moreover, it is obvious that if there exists one representation of form (31.1), then there exist infinitely many such representations.

Using this theorem we can decide whether an equation of the form

$$z = F(x, y) \tag{31.3}$$

is a Cauchy equation. Obviously if a function $F(x, y)$ has a representation (31.1) and one of the "partial" functions

$$F(x, b'), \quad F(x, b), \quad F(a', y), \quad F(a, y)$$

is constant, then equation (31.3) is a Cauchy equation.

31.2. Let us now consider a function of three variables $F(x, y, z)$ which can be represented in the determinant form (31.0). This form defines in a three-dimensional space ξ, η, ζ three curves C_x C_y and C_z with parametric equations

$$\xi = X_1(x), \quad \eta = X_2(x), \quad \zeta = X_3(x), \tag{C$_x$}$$

$$\xi = Y_1(y), \quad \eta = Y_2(y), \quad \zeta = Y_3(y), \tag{C$_y$}$$

$$\xi = Z_1(z), \quad \eta = Z_2(z), \quad \zeta = Z_3(z). \tag{C$_z$}$$

As in the case of two variables, equality (31.0) signifies that the values of function $F(x, y, z)$ are proportional to the volumes of tetrahedrons having one of their vertices at the origin O of the system and the remaining ones, successively, on curves C_x, C_y and C_z.

Let us choose numbers a, b and c in such a way that the straight lines joining the origin O of the system with points $A \in C_x$, $B \in C_y$ and $C \in C_z$, corresponding to the values $x = a$, $y = b$ and $z = c$ respectively, will not lie in one plane (Fig. 153). We thus have an inequality

$$F(a, b, c) \neq 0. \qquad (31.4)$$

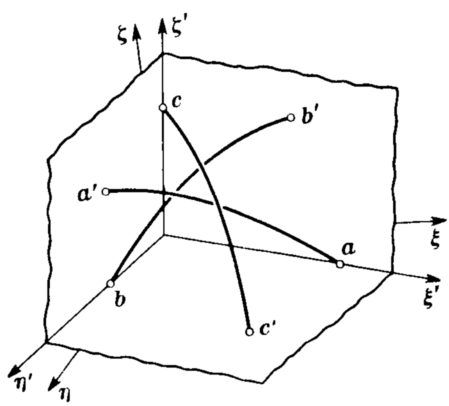

FIG. 153

Denote by a', b', c' such numbers that

$$F(a', b, c) = 0, \quad F(a, b', c) = 0, \quad F(a, b, c') = 0. \qquad (31.5)$$

Take the straight line OA as the ξ'-axis, the straight line OB as the η'-axis and the straight line OC as the ζ'-axis (Fig. 153). Let us also assume that in our system of coordinates none of the points A', B', C', corresponding to the values $x = a'$, $y = b'$ and $z = c'$, lies on any of the axes of the system. This means that

$$F(a, b', c') \neq 0, \quad F(a', b, c') \neq 0, \quad F(a', b', c) \neq 0. \qquad (31.6)$$

Assume that function $F(x, y, z)$ has a representation of form (31.0), numbers a, b, c, a', b', c' being such that conditions (31.4), (31.5) and (31.6) are satisfied.

The values of functions $X_i(x)$, $Y_i(y)$ and $Z_i(z)$ are found by substituting values in equation (31.0) as follows:

$$F(x, b, c) = \begin{vmatrix} X_1(x) & X_2(x) & X_3(x) \\ 0 & X_2(b) & 0 \\ 0 & 0 & Z_3(c) \end{vmatrix} = c_1 X_1(x), \tag{31.7}$$

$$F(x, b', c) = \begin{vmatrix} X_1(x) & X_2(x) & X_3(x) \\ Y_1(b') & 0 & Y_3(b') \\ 0 & 0 & Z_3(c) \end{vmatrix} = c_2 X_2(x), \tag{31.8}$$

$$F(x, b, c') = \begin{vmatrix} X_1(x) & X_2(x) & X_3(x) \\ 0 & Y_2(b) & 0 \\ Z_1(c') & Z_2(c') & 0 \end{vmatrix} = c_3 X_3(x), \tag{31.9}$$

$$F(a', y, c) = \begin{vmatrix} 0 & X_2(a') & X_3(a') \\ Y_1(y) & Y_2(y) & Y_3(y) \\ 0 & 0 & Z_3(c) \end{vmatrix} = c_4 Y_1(y), \tag{31.10}$$

$$F(a, y, c) = \begin{vmatrix} X_1(a) & 0 & 0 \\ Y_1(y) & Y_2(y) & Y_3(y) \\ 0 & 0 & Z_3(c) \end{vmatrix} = c_5 Y_2(y), \tag{31.11}$$

$$F(a, y, c') = \begin{vmatrix} X_1(a) & 0 & 0 \\ Y_1(y) & Y_2(y) & Y_3(y) \\ Z_1(c') & Z_2(c') & 0 \end{vmatrix} = c_6 Y_3(y), \tag{31.12}$$

$$F(a', b, z) = \begin{vmatrix} 0 & X_2(a') & X_3(a') \\ 0 & Y_2(b) & 0 \\ Z_1(z) & Z_2(z) & Z_3(z) \end{vmatrix} = c_7 Z_1(z), \tag{31.13}$$

$$F(a, b', z) = \begin{vmatrix} X_1(a) & 0 & 0 \\ Y_1(b') & 0 & Y_3(b') \\ Z_1(z) & Z_2(z) & Z_3(z) \end{vmatrix} = c_8 Z_2(z), \tag{31.14}$$

$$F(a, b, z) = \begin{vmatrix} X_1(a) & 0 & 0 \\ 0 & Y_2(b) & 0 \\ Z_1(z) & Z_2(z) & Z_3(z) \end{vmatrix} = c_9 Z_3(z). \tag{31.15}$$

Expanding determinant (31.0) and taking into consideration equations (31.7)–(31.15) we obtain

$$\begin{aligned} F(x, y, z) = {} & k_1 F(x, b, c)\, F(a, y, c)\, F(a, b, z) + \\ & + k_2 F(x, b', c)\, F(a, y, c')\, F(a, b, z) + \\ & + k_3 F(x, b, c')\, F(a', y, c)\, F(a, b', z) + \\ & + k_4 F(x, b, c')\, F(a, y, c)\, F(a', b, z) + \\ & + k_5 F(x, b', c)\, F(a', y, c)\, F(a, b, z) + \\ & + k_6 F(x, b, c)\, F(a, y, c')\, F(a, b', z). \end{aligned} \tag{31.16}$$

In order to find the coefficients $k_1, \ldots, k_6$, let us substitute successively the threes of numbers abc, $ab'c'$, $a'bc'$, $a'b'c$, $a'b'c'$

$$\begin{aligned} F(a, b, c) &= k_1 F(a, b, c)\, F(a, b, c)\, F(a, b, c), \\ F(a, b', c') &= -k_6 F(a, b, c)\, F(a, b', c')\, F(a, b', c'), \\ F(a', b, c') &= -k_4 F(a', b, c')\, F(a, b, c)\, F(a', b, c'), \\ F(a', b', c) &= -k_5 F(a', b', c)\, F(a', b', c)\, F(a, b, c), \\ F(a', b', c') &= (k_2 + k_3)\, F(a', b', c)\, F(a, b', c')\, F(a', b, c'); \end{aligned}$$

we obtain

$$\begin{aligned} k_1 &= \frac{1}{F(a, b, c)^2}, \\ k_4 &= \frac{-1}{F(a, b, c)\, F(a', b, c')}, \\ k_5 &= \frac{-1}{F(a, b, c)\, F(a', b', c)}, \\ k_6 &= \frac{1}{F(a, b, c)\, F(a, b', c')}. \end{aligned} \tag{31.17}$$

The right side of equation (31.16) can be written in the form of determinant (31.0) if and only if the coefficients k_i satisfy the equation

$$k_1 k_2 k_3 + k_4 k_5 k_6 = 0. \tag{31.18}$$

Hence, taking into account equations (31.17), we obtain a second equation containing k_2 and k_3:

$$k_2 k_3 = \frac{1}{F(a, b, c)\, F(a, b', c')\, F(a', b, c')\, F(a', b', c)}.$$

The coefficients k_2 and k_3 are thus the roots of the equation

$$F(a,b,c)\,F(a,b',c')\,F(a',b,c',)\,F(a',b',c)k^2 - \\ -F(a,b,c)\,F(a',b',c')k+1 = 0. \qquad (31.19)$$

We have proved the following theorem:

A necessary and sufficient condition for a function of three variables $F(x, y, z)$ to be nomogrammable is the identity

$$F(x,y,z) = \begin{vmatrix} F(x,b,c) & F(x,b',c) & F(x,b,c') \\ \dfrac{k_5}{k_1}F(a',y,c) & F(a,y,c) & \dfrac{k_2}{k_4}F(a,y,c') \\ k_4F(a',b,z) & \dfrac{k_4k_6}{k_2}F(a,b',z) & k_1F(a,b,z) \end{vmatrix}$$

in which

$$F(a,b,c') = 0, \quad F(a,b',c) = 0, \quad F(a',b,c) = 0,$$
$$F(a,b',c') \neq 0, \quad F(a',b,c') \neq 0, \quad F(a',b',c) \neq 0,$$

and the numbers $k_1, \ldots, k_6$ *are defined by equations* (31.17) *and* (31.19).

EXAMPLE. Write in form (31.0) the function

$$F(x,y,z) = 3x^2yz^2 - xy^2z^2 - xy^2z + x^2yz + xy^2 - xz^2 - y^2z - \\ -2xyz - yx^2 - 3yz - xz + y^2 + x - z + 1.$$

Take

$$a = 0, \quad b = 0, \quad c = 0.$$

We obtain

$$F(x,b,c) = x+1, \quad \text{i.e.,} \quad a' = -1,$$
$$F(a,y,c) = y^2+1, \quad \text{i.e.,} \quad b' = i,$$
$$F(a,b,z) = 1-z, \quad \text{i.e.,} \quad c' = 1,$$

and

$$F(a,b,c) = 1, \quad F(a',b',c') = 4i,$$
$$F(a,b',c) = -3i, \quad F(a',b,c') = 1, \quad F(a',b',c) = -i.$$

Hence we find

$$k_1 = 1, \quad k_4 = -1, \quad k_5 = -i, \quad k_6 = -\tfrac{1}{3}i,$$
$$k_2+k_3 = -\tfrac{4}{3}i, \quad k_2 \,.\, k_3 = -\tfrac{1}{3} \quad \text{and} \quad k_2 = -i.$$

Substituting the above in equation (33.20) we find that

$$F(x, y, z) = \begin{vmatrix} x+1 & -ix^2 & -x \\ iy & y^2+1 & -3iy \\ -z^2 & -iz & 1-z \end{vmatrix}$$

$$= \begin{vmatrix} 1+x & x^2 & x \\ y & y^2+1 & 3y \\ z^2 & z & 1-z \end{vmatrix}.$$

§ 32. Criterion of Saint Robert

Considerations regarding the representation of a given relation containing three variables by means of collineation (or linear lattice) nomograms are of algebraic nature. There are a great many cases in which other methods, based on the differential calculus, give quicker results. Those methods use a few elementary theorems, known from the study of differential and integral calculus. One of them is the theorem on transforming a function of many variables into a sum of components each of which is dependent on one variable only.

THEOREM 1. *A function $F(x, y, z)$ can be represented in the form*

$$F(x, y, z) = f_1(x)+f_2(y)+f_3(z) \tag{32.1}$$

if and only if all mixed partial derivatives of second order of function F are equal to zero, i.e., if and only if the equations

$$\frac{\partial^2 F}{\partial y \partial z} = \frac{\partial^2 F}{\partial x \partial z} = \frac{\partial^2 F}{\partial x \partial y} = 0 \tag{32.2}$$

are satisfied for all x and y of a certain space domain.

Proof of n e c e s s i t y. If equation (32.1) is satisfied, then of course equations (32.2) hold.

The s u f f i c i e n c y of the condition will be proved first for a function of two variables.

If

$$\frac{\partial^2 G(x, y)}{\partial x \partial y} = 0 \quad \text{for} \quad x_1 < x < x_2,\ y_1 < y < y_2, \tag{32.3}$$

then, as we know, we shall obtain by integration with respect to y

$$\partial G/\partial x = c,$$

where c is a function independent of y, i.e. $c = \varphi(x)$; integrating with respect to x we have

$$G = \int \varphi(x)\, dx + c_1,$$

where c_1 is independent of x: $c_1 = f_2(y)$. We finally obtain

$$G = \int \varphi(x)\, dx + c_1 = f_1(x) + f_2(y).$$

If $F(x, y, z)$ is a function of three variables satisfying conditions (32.2), then, just as for two variables, from equation $\partial^2 F/\partial x \partial y = 0$ we have

$$\partial F/\partial x = \varphi(x, z);$$

however, since

$$\frac{\partial(\partial F/\partial x)}{\partial z} = 0,$$

function φ does not depend on z, whence

$$F = \int \varphi(x) dx + \psi(y, z) = f_1(x) + \psi(y, z).$$

Since from the last equation we obtain by differentiation

$$\frac{\partial^2 F}{\partial y \partial z} = \frac{\partial^2 f_1}{\partial y \partial z} + \frac{\partial^2 \psi}{\partial y \partial z} = \frac{\partial^2 \psi}{\partial y \partial z},$$

we have, by assumption (32.2) and the validity of the theorem for two variables,

$$\psi(y, z) = f_2(y) + f_3(z),$$

whence

$$F(x, y, z) = f_1(x) + f_2(y) + f_3(z).$$

For example, the equation

$$uvw - u - v - w = 0$$

cannot be written in the form

$$w = f_1(u) + f_2(v),$$

because, finding w, we have

$$w = \frac{u+v}{uv-1} = \psi(u, v)$$

and

$$\frac{\partial\psi(u, v)}{\partial u} = \frac{uv-1-(u+v)v}{(uv-1)^2} = \frac{-v^2-1}{(uv-1)^2},$$

$$\frac{\partial^2\psi(u, v)}{\partial u\partial v} = \frac{-(uv-1)^2\,2v+(v^2+1)\,2(uv-1)\,u}{(uv-1)^4}$$

$$= \frac{-2uv^2+2v+2v^2u+2u}{(uv-1)^3} = 2\frac{u+v}{(uv-1)^3} \neq 0.$$

As we know, this implies that canonical equation (III) cannot be represented by a nomogram with three parallel scales, the w-scale being regular.

THEOREM 2. *A function* $F(x, y, z)$ *is a product*

$$g_1(x)\,g_2(y)\,g_3(z)$$

if and only if the mixed partial derivatives of second order of function $\ln F(x, y, z)$ *are equal to zero.*

This is an obvious conclusion from the first theorem because

$$\ln F = \ln g_1 + \ln g_2 + \ln g_3.$$

On the basis of this theorem we can prove that the third canonical equation (§ 27) cannot be written in the form

$$w = g_1(u)\,g_2(v). \tag{32.4}$$

Indeed, let us write

$$w = \frac{u+v}{uv-1} \quad \text{and} \quad \ln w = \ln(u+v) - \ln(uv-1)$$

and find the partial derivatives

$$\frac{\partial \ln w}{\partial u} = \frac{1}{u+v} - \frac{v}{uv-1},$$

$$\frac{\partial^2 \ln w}{\partial u\partial v} = \frac{-1}{(u+v)^2} - \frac{(uv-1)1-vu}{(uv-1)^2} = \frac{1}{(uv-1)^2} - \frac{1}{(u+v)^2} \neq 0.$$

Since $\dfrac{\partial^2 \log w}{\partial u \partial v} \neq 0$, representation (32.4) for canonical equation (III) does not exist. This implies that we cannot draw an N-shaped nomogram with a regular w-scale.

THEOREM 3. *The equation $F(x, y, z) = 0$ can be written in the Cauchy form if and only if for a certain pair of variables, say x and y, the equality*

$$\frac{\partial F}{\partial x} : \frac{\partial F}{\partial y} = \varphi_1(x)\, \varphi_2(y)\, \varphi_3(z) \tag{32.5}$$

holds for any values x, y, z satisfying the given equation.

This is the so-called *criterion of Saint Robert.*

The necessity of the condition is obvious, since if

$$F(x, y, z) = f_1(x)\, g_3(z) + f_2(y)\, h_3(z) + 1$$

then

$$\frac{\partial F}{\partial x} = f_1'\, g_3, \qquad \frac{\partial F}{\partial y} = f_2'\, h_3$$

and

$$\frac{\partial F}{\partial x} : \frac{\partial F}{\partial y} = f_1'(x) \frac{1}{f_2'(y)} \cdot \frac{g_3(z)}{h_3(z)},$$

i.e., F_x'/F_y' is a product of three functions, each of them depending on one variable only.

To show the sufficiency of the condition let us assume that equation (32.4) is satisfied and let us write it in the form

$$\frac{1}{\varphi_1(x)} \cdot \frac{\partial F}{\partial x} - \varphi_2(y)\, \varphi_3(z) \frac{\partial F}{\partial y} = 0.$$

It is a partial differential equation of the first order. In order to integrate it we write the ordinary differential equation

$$\frac{dx}{1/\varphi_1(x)} = \frac{dy}{-\varphi_2(y)\, \varphi_3(z)} = \frac{dz}{0}.$$

Since function φ_1 is not constantly equal to zero, $dz = 0$, i.e. $z = c$ is a first integral. Now let us take

$$\varphi_1(x)\, dx = -\frac{dy}{\varphi_2(y) C}, \qquad \text{where} \qquad C = \varphi_3(z).$$

Writing

$$\int \varphi_1(x)\,dx = f_1(x), \qquad \int \frac{dy}{\varphi_2(y)} = f_2(y),$$

we have, in view of $\varphi_3(z) = C$, a new first integral

$$f_1(x)+\varphi_3(z)\,f_2(y) = C_1.$$

The general integral will be obtained by taking an arbitrary function ψ and writing

$$C_1 = \psi(c),$$

i.e.,

$$f_1(x)+\varphi_3(z)\,f_2(y) = \psi(z).$$

Dividing by $-\psi(z)$, we can see that equation $F(x, y, z)$ must have a Cauchy form,

$$-f_1(x)\,\frac{1}{\psi(z)} - f_2(y)\,\frac{\varphi_3(z)}{\psi(z)} + 1 = 0.$$

Relation

$$\frac{\partial F}{\partial x} : \frac{\partial F}{\partial y} = \varphi_1(x)\,\varphi_2(y) \tag{32.6}$$

is a particular case of assumption (32.5). As follows from the proof, function $F(x, y, z)$ is then of the form

$$f_1(x)+f_2(y) = \psi(z)$$

and consequently the collineation nomogram is composed of three parallel scales and the lattice nomogram of three pencils of parallel lines (the Lalanne nomogram).

By theorem 2 condition (32.6) can be replaced by the equation

$$\frac{\partial^2 \left[\ln\left(\dfrac{\partial F}{\partial x} : \dfrac{\partial F}{\partial y}\right)\right]}{\partial x \partial y} = 0.$$

This is the so-called *equation of Saint Robert.*

EXAMPLE 1. Find whether there exists a substitution

$$x = f_1(u), \qquad y = f_2(v) \tag{32.7}$$

which reduces the Clark equation

$$xyg_3(z)+(x+y)\,h_3(z)+1 = 0, \tag{32.8}$$

to the Cauchy form.

We shall assume that the quotient $h_3(z)/g_3(z) = f_3(z)$ is not a constant, since otherwise equation (32.8) would not be of the fourth nomographic order.

Let us substitute in the Clark equation the unknown functions f_1 and f_2 and let us write

$$F(u, v, z) = f_1(u) f_2(v) g_3(z) + \big(f_1(u) + f_2(v)\big) h_3(z) + 1.$$

If functions (32.7) existed, the expression

$$\frac{\partial F}{\partial u} : \frac{\partial F}{\partial v} = \frac{f_1' f_2 g_3 + f_1' h_3}{f_1 f_2' g_3 + f_2' h_3} = \frac{f_2 g_3 + h_3}{f_1 g_3 + h_3} \cdot \frac{f_1'}{f_2'} = \frac{f_2 + f_3}{f_1 + f_3} \cdot \frac{f_1'}{f_2'},$$

in which $f_3 = h_3 : g_3$, would have to be a product of factors dependent on one variable each. Since f_1'/f_2' has a form like that, it is sufficient to find when

$$\frac{f_2 + f_3}{f_1 + f_3}$$

is also a product of this type. By theorem 2 the necessary and sufficient condition is that the mixed partial derivatives of second order of the expression

$$G = \ln (f_2 + f_3) - \ln (f_1 + f_3)$$

be equal to zero; since

$$\frac{\partial G}{\partial u} = -\frac{f_1'}{f_1 + f_3}, \qquad \frac{\partial G}{\partial v} = \frac{f_2'}{f_2 + f_3},$$

$$\frac{\partial^2 G}{\partial u \partial z} = \frac{f_1' f_3'}{(f_1 + f_3)^2}, \qquad \frac{\partial^2 G}{\partial v \partial z} = \frac{-f_2' f_3'}{(f_1 + f_3)^2}$$

we should thus have

$$f_1' f_3' = 0 \quad \text{and} \quad f_2' f_3' = 0,$$

whence, in view of $f_3' \neq 0$, we should obtain equations $f_1' = 0$ and $f_2' = 0$, which is impossible because neither of the functions f_1 and f_2 can be constant.

Consequently a Clark equation cannot be transformed into a Cauchy equation by means of substitution (32.7).

The application of the criterion of Saint Robert often involves very tedious calculations. To simplify our considerations let us observe that the equation

$$F(x, y, z) = 0$$

represents in general a surface in a three dimensional space; it will be of the Cauchy type if every plane $z = z_0$ intersects that surface along a straight line. This interpretation shows that the expression

$$-\frac{\partial F}{\partial x} : \frac{\partial F}{\partial y},$$

as the slope of that straight line, is a number dependent only on variable z. Calculating this coefficient we must of course take into consideration the fact that the variables x and y of the straight line satisfy the equation $F = 0$.

If the equation $F = 0$ is not of the Cauchy form, we seek a function (32.7) which, substituted in the equation, gives that form to it:

$$F\big(f_1(u)\, f_2(v), z\big) = F^*(u, v, z).$$

The substitution $z = f_3(w)$ is of course of no importance since it has no part in bringing the equation to the required form.

EXAMPLE 2. Reduce to a canonical form the equation

$$xy^2z^2 + x^2yz + xyz - xz^2 + yz + xy = 0. \qquad (32.9)$$

We have here

$$\frac{\partial F}{\partial x} : \frac{\partial F}{\partial y} = \frac{y^2z^2 + 2xyz + yz - z^2 + y}{2xyz^2 + x^2z + xz + z + x},$$

which, when we take equation (32.9) into account, assumes the value

$$\frac{\partial F}{\partial x} : \frac{\partial F}{\partial x} = \frac{xyz - yz/x}{xyz^2 + xz^2/y} = \frac{(x^2yz - yz)y}{(xy^2z^2 + xz^2)x} = \frac{x^2-1}{x^2} \cdot \frac{y^2}{y^2+1} \cdot \frac{1}{z};$$

equation (32.9) thus satisfies the Saint Robert condition. Integrating it by the ordinary method we obtain

$$\frac{x^2}{x^2-1} \cdot \frac{\partial F}{\partial x} - \frac{y^2}{y^2+1} \cdot \frac{1}{z} \cdot \frac{\partial F}{\partial y} = 0,$$

$$\frac{x^2-1}{x^2}\, dx = -\frac{y^2+1}{y^2}\, z\, dy = \frac{dz}{0},$$

$$x + 1/x = (-y + 1/y)z + \psi(z), \qquad (32.10)$$

where $\psi(z)$ is a function independent either of x or of y. We shall obtain its value from equation (32.9) if we take a concrete number for y. For example, let $y = 1$; equation (32.10) gives

$$x+1/x = \psi(z),$$

and equation (32.9) gives

$$x^2z+xz+z+x = 0$$

or

$$(x^2+1)z = -x(z+1), \qquad x+1/x = -(z+1)z.$$

We have finally obtained $\psi(z) = -1-1/z$; substituting this into (32.10) we have

$$(x+1/x)+(y-1/y)z+(1+1/z) = 0.$$

This equation is easily seen to be identical with equation (32.9).

The search for differential criteria for the Clark equation and the Soreau equations leads to very complex relations. Research in this field, initiated by Gronvall (1) and continued by Bitner (2) and Smirnov (3), has not resulted in a form which could be used in practical problems.

The problems here given concern only equations with three variables. Analogous problems for relations containing more than three variables are of less practical importance, because equations occurring in technology and natural sciences are in general easily replacable—through suitable substitutions—by a system of equations each of which is a relation of three variables.

---

(1) T. A. Gronvall, *Sur les équations entre trois variables représentables par des nomogrammes à points alignes*, Journal de Mathématiques Pures et Appliquées 8 (1912), p. 59.

(2) H. A. Bitner, *Necessary and sufficient conditions for anamorphosability of functions of three variables* (in Russian), Nomograficheskii Sbornik (1935), pp. 77–104, and *On the problem of general anamorphosis*, Uchenyye Zapiski 28 (1939), pp. 7–14.

(3) S. V. Smirnov, *On the problem of the general anamorphosis* (in Russian), Doklady Akademii Nauk SSSR, vol. LXV, 1949.

Exercises

1. Reduce to the Cauchy form the equations

a. $x^3z+y^2z^2+3x^2z-4yz^2+3xz+4z^2+z+1 = 0.$

b. $xyz^3+2xyz^2+yz^3+2xyz-xz^2-yz^2-2xz+yz-xy+x-y-z+1 = 0.$

2. On the basis of the Saint Robert criterion state the condition which must be satisfied by the coefficients of the general equation of the fourth nomographic order (301) if the equation is to be reducible to the Cauchy canonical form.

BIBLIOGRAPHY

H. A. GLAGOLEV, Course of Nomography (in Russian), Moscow 1943.

H. A. GLAGOLEV, Theoretical Foundations of Nomography (in Russian), 1936.

R. P. HOELSCHER, J. N. ARNOLD, S. H. PIERCE, Graphie Aids in Engineering Computation, New York–Toronto–London 1952.

V. HRUŠKA, Nomogramy s jednou průsvitkou, Praha 1947.

V. HRUŠKA, Počet grafický a graficko-mechanický, Praha 1952.

H. JOHNSON LEE, Nomography and Empirical Equations, New York 1952.

B. KONORSKI, Die Grundlagen der Nomographie, Berlin 1922.

H. KÖRWIEN, Graphisches Rechnen (Nomographie), Leipzig 1949.

A. S. LEVENS, Nomography, New York 1952.

P. LUCKEY, Nomographie, Leipzig 1949.

R. MEHMKE, Leitfaden graphischen Rechnens, Wien–Leipzig 1924.

W. MEYER ZUR CAPELLEN, Leitfaden zur Nomographie, Berlin 1953.

A. MÜLLER, Nomographie für die technische Praxis, Leipzig 1952.

B. A. NEVSKII, A Guide-Book to Nomography (in Russian), Moscow–Leningrad 1951.

M. D'OCAGNE, Principes usuels de nomographie avec application à divers problèmes concernant l'artillerie et l'aviation, Paris 1920.

M. D'OCAGNE, Traité de Nomographie (Théorie des abaques-Applications). Paris 1921.

M. W. PENTKOVSKII, Nomography (in Russian), Moscow–Leningrad 1949.

V. PLESKOT, Spojnicové nomogramy, Praha 1946.

M. SCHWERDT, Lehrbuch der Nomographie, Berlin 1924.

R. SOREAU, Nomographie ou Traité des Abaques, Paris 1921.

INDEX