NOMOGRAPHY

Theory and Application

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INTRODUCTION

Nomography was introduced by Professor Maurice d'Ocagne, in 1884, in France. It has been developed chiefly by French and German mathematicians since that time in works given in the Bibliography. A debt to d'Ocagne is universally admitted by American writers. The full scope and power of several parts of the complete subject as he practiced it is scarcely known in the United States though we make heavy use of its elementary parts. His chief work was published in 1921, "Traite de Nomographie", and has a large number of practical problems. The subject matter, appearing in the Table of Contents in the last five pages of the book, is not divided or arranged into easily taught topics but follows mostly physical characteristics of charts. The Index is seventy-five percent authors' names and chart titles. Although at least one translation has been made, none appears ever to have been published. The Perpetual Calendar of Crepin (Figure 4—17) and the double rectification of a family of curves (Figure 9—5) of Colonel Lafay, both faithfully credited by d'Ocagne, seem the most common references actually drawn from his work, but his authorship and paternity of the broad subject of Nomography are unquestioned.

The magic and fun of Nomography arrived full-blown with d'Ocagne. There is apparently an inner satisfaction that comes with each use of an alignment diagram – as though somehow the operator were getting away with something that was quite smart and for which he could claim some portion of the credit. Then, of course, a nomogram saves a lot of time. This happy feeling has in no way diminished over close to one hundred years, and to this day a kind of half-chuckle accompanies each use. How to make nomograms is a different story and is usually the first question asked by each new operator.

The basic philosophy of this book is entirely different from the d'Ocagne treatment. It is intended for self-study as well as classroom use. The classical elements of the subject are presented through a thorough grounding in determinants, central projection, duality and all those basic graphical techniques so helpful in conceiving and interpreting nomographic strategy and operations. Empirical data and imperfect canonical forms have been included. Careful attention to the many examples worked out and numerous problems assigned will provide a student with a good foundation for extending the theory into new regions made possible by modern electronic developments. For suggestions as to what some of these might be, the reader is referred to Vol II listed in the Bibliography.

This book is the selection and final arrangement of material presented to undergraduate and graduate college students and found helpful in preparing nomograms and nomographic devices at the professional level for some twenty-five years. A considerable portion of the material is entirely new to the best of the writer's knowledge.

The author wishes to thank the Rome Air Development Center for preparing the first draft of this text and publishing it for Air Force use, and for similar treatment of the second volume of modern electronic applications of this material given in the reference as Final Report, Vol. II. Mr. Denis Maynard of Griffiss Air Force Base, Rome Air Development Center, is to be thanked for his encouragement and for implementing the first writing. Thanks should also go to the following persons each of whom, as students of the subject, designed with the writer a chart, drafted it and permitted its use in this text: W. L. Allison, M. E. Arnold, A. N. Aronsen, Glen Bennett, Roy Blackmer, R. K. Breese, Noel Davis, A. W. Eade, K. H. Epple, G. Eichenseer, John R. Fennessey, Ralph Franke, William R. Frazier, Jr., George Fuld, Phillip Gladding, Robert J. Hecht, J. H. Hughes, E. J. Kletsky, Richard M. McCullough, Paul Michaels, Dick Perley, William Rice, Paul M. Robinson, R. W. Stanhouse, Robert A. Vietch, and R. Weithoff.

A subdivision into main and appendix parts has been made because readers will vary so widely in background, knowledge and experience. It was believed that basic material in determinants should be at the back of the book so as not to clutter the development of earlier chapters. The applicability of material in Chapter 14 will become clearer as the knowledge and facility of the reader grows. To separate perfectly what should be in the front and what in the rear does not deserve further effort now. The present division is a working start. There will be readers to whom a considerable portion of the book is merely review; for others, most of it can present a challenge. The book aims to follow a middle course.

Problem work is important in a text of this kind so a considerable number of these appear in the appendix chapters as well as the earlier ones. These have often been selected for double or triple duty to provide practice and training in adjacent material and to support and illustrate the theory of other sections. Also, illustrative examples are used frequently.

Nomography cannot be practiced with "approximate" diagrams. It is not ethical to use the term "nomogram" for a diagram which looks like one but was put together surreptitiously. A nomographer's algebraic procedures should either have a clear mathematical pedigree or state clearly what their limitations are. No approximations appear in this book.

Since nomograms are made to be used by people, a great deal of common sense and good judgement is needed to minimize errors and fatigue and keep users in a happy frame of mind. Patient study of each chart for improvement of design and practical use is a hall-mark of first rate nomography. If this book can encourage and develop all these elements of professional attitude and provide each reader with a basis for carrying them through, it will have served a good purpose.

The best success in turning out good nomograms will accordingly come by drawing upon all the mathematical knowledge a person can muster. Ingenuity is in demand almost all the time, as the topics in the Appendix chapters show. The Perpetual Calendar. Example 4-8, Figure 4—17, is a good illustration of how a seemingly complex problem can get put into nomographic form. The type of chart used here is the simplest but the adaptation of the calendar problem to it is quite ingenious.

And so we come full circle back to the subject a brilliant young Professor pulled out of a hat almost a hundred years ago. We should compare the elementary ways in which it can be presented to the general level he used in his own presentation. It is true that Nomography can be studied in limited degree by using Euclidean geometry on each of a rather small number of diagrams. In contrast to such a limited attack, notions of dual correspondence of point and line, cross-ratio, etc., are projective in nature. Nomography is a lot more fun and more productive at the higher level where projective tools are used and the projective transformation is actually used as an operator upon a canonical form.

What makes Nomography especially challenging is that at any level the driving force is the actual construction of workable charts. This requires measurements with linear scales and since the canonical form has just been observed to be a projective element, the metric and projective disciplines are inextricably meshed. The separate elements of the determinant are admirably adapted to metric use, that is for measurements, while the arrangement of these parts in a canonical form has a projective significance amenable to projective transformation by determinant operator. Thus this curious and practical field might be thought of as an excursion through intermixed metric and projective territories. For this excursion, most readers will find it profitable to learn to operate the projective vehicle. That is one skill this book tries to give them. If a lesser skill is conveyed so that the person fascinated by nomograms feels competent to turn his hand to their accurate and practical design when needed, whether or not they are the most sophisticated of diagrams, this book will have served a good purpose. It will have brought him the enjoyment that comes with making and using them and the savings from their efficiency.

The author will appreciate having any errors brought to his attention or learning of any sections that seem hard to understand.

PART I

These first three chapters give the essential ingredients of the determinant approach to Nomography in both theory and practice. Although some reference to the appendices may be necessary for persons not used to determinants and other elementary processes, the basic procedures unique to Nomography are presented in these fifty pages. A thorough understanding of this material is advised to see the reader through the rest of the text.

CHAPTER 1

THE DETERMINANT IN NOMOGRAPHY

1–1. *The Equation of a Straight Line in Determinant Form.* The equation of a straight line determined by two points $P_1(X_1, Y_1)$ and $P_2(X_2, Y_2)$ is

$$\frac{Y - Y_1}{Y_2 - Y_1} = \frac{X - X_1}{X_2 - X_1} \qquad (1\text{–}1)$$

Hence the condition that three points P_1, P_2, P_3 be collinear is

$$\frac{Y_3 - Y_1}{Y_2 - Y_1} = \frac{X_3 - X_1}{X_2 - X_1} \qquad (1\text{–}2)$$

or

$$X_2Y_3 - X_2Y_1 - X_1Y_3 - X_3Y_2 + X_3Y_1 + X_1Y_2 = 0. \qquad (1\text{–}3)$$

This turns out to be the expansion of

$$\begin{vmatrix} X_1 & Y_1 & 1 \\ X_2 & Y_2 & 1 \\ X_3 & Y_3 & 1 \end{vmatrix} = 0 \qquad (1\text{–}4)$$

which is likewise the *condition that the three points be collinear.*

1–2. *Parametric Representation of a Relation Between Two Variables.* The equation $y = f(x)$ says that for any given value of x the value of y is determined.

Another way of expressing the relation between y and x is by a pair of equations which tie y and x together through a third variable, u. These are of the form

$$y = y(u) \qquad x = x(u). \qquad (1\text{–}5)$$

Since u is in turn a function of x, $u = u(x)$, we get, on substituting into (1–5)

$$y = f(x). \qquad (1\text{–}6)$$

This is called "eliminating the parameter."

Example 1–1. Given the equation $y = u^2$; $x = u$, eliminate the parameter u and express y directly as a function of x.

Answer: $y = x^2$. The curve is a parabola.

1–3. *Plotting Parameter Values on the Curve.* Since any value of u gives rise to a value of x and one of y and determines a point of the curve, that value of u can be attached to the curve at the point that has x and y for its coordinates. This creates a scale in u. Convenient values of u can be chosen permitting easy reading of it along this curved scale.

Example 1–2. Figure 1–1. Plot the scales of the following three parametric functions in terms of their respective parameters:

a) $X_1 = 0$; $Y_1 = U$

b) $X_2 = 6$; $Y_2 = V$

c) $X_3 = 3$; $Y_3 = \frac{W}{2}$

Example 1–3. Figure 1–2. Plot the following three functions in terms of their respective parameters:

a) $X_1 = 0$; $Y_1 = U/5$

b) $X_2 = 3$; $Y_2 = \frac{60 - W}{10}$

c) $X_3 = \dfrac{3}{\frac{V}{2} + 1}$; $Y_3 = \dfrac{6}{\frac{V}{2} + 1}$

1–4. *Determinant Equation of a Straight Line in Parametric Form.* Write three parametric equations.

$X_1 = U_1$; $Y_1 = U_2$
where U_1 and U_2 are functions of parameter U,

$X_2 = V_1$; $Y_2 = V_2$
V_1 and V_2 are functions of parameter V,

$X_3 = W_1$; $Y_3 = W_2$
and W_1 and W_2 are functions of parameter W. (1–7)

These determine three curves in U and V and W which can be imagined to appear as in Figure 1–3. Substituting (1–7) into (1–4) one obtains

$$\begin{vmatrix} U_1 & U_2 & 1 \\ V_1 & V_2 & 1 \\ W_1 & W_2 & 1 \end{vmatrix} = 0. \qquad (1\text{–}8)$$

Expansion of this determinant produces some function

$$F(U, V, W) = 0. \qquad (1\text{–}9)$$

Values of U, V and W which satisfy (1–9) satisfy (1–8).

Through (1–7) they give rise to X, Y coordinates which satisfy (1–4) and are collinear. Hence, they also give rise to U, V, W points which are collinear. Conversely, *any straight line cuts the U, V, W scales of Figure 1–3 in points which satisfy equation (1–9).*

The given equation (1–9), its determinant form (1–8) and the interpretation of the form in the manner (1–7) are basic equipment of nomography. Because of their importance the above steps are now repeated using definite quantities. These examples should be well-understood.

Example 1–4. The three parametric equations from Example 1–2 whose graphs appear in Figure 1–1 were

$$X_1 = 0;\ Y_1 = U$$

$$X_2 = 6;\ Y_2 = V$$

$$X_3 = 3;\ Y_3 = \frac{W}{2} \qquad (1\text{–}10)$$

Put these into the determinant (1–4) to obtain

$$\begin{vmatrix} 0 & U & 1 \\ 6 & V & 1 \\ 3 & \frac{W}{2} & 1 \end{vmatrix} = 0. \qquad (1\text{–}11)$$

The expansion of this determinant is

$$U + V = W. \qquad (1\text{–}12)$$

Then values of U, V and W which satisfy (1–12) satisfy (1–11) and by (1–10) give rise to X, Y coordinates satisfying (1–4). Hence these are X, Y coordinates for points U, V, W which lie on a straight line. Conversely *any straight line cuts Figure 1–1 in U, V, W values which satisfy U + V = W.*

Example 1–5. The three parametric equations from example 1–3 whose graphs appear in Figure 1–2 were

$$X_1 = 0;\ Y_1 = \frac{U}{5}$$

$$X_2 = 3;\ Y_2 = \frac{60 - W}{10}$$

$$X_3 = \frac{3}{\frac{V}{2} + 1};\ Y_3 = \frac{6}{\frac{V}{2} + 1}. \qquad (1\text{–}13)$$

Put these into determinant (1–4) to obtain

$$\begin{vmatrix} 0 & \frac{U}{5} & 1 \\ 3 & \frac{60 - W}{10} & 1 \\ \frac{3}{\frac{V}{2} + 1} & \frac{6}{\frac{V}{2} + 1} & 1 \end{vmatrix} = 0. \qquad (1\text{–}14)$$

On expansion this is found to be the equation

$$U \cdot V = W. \qquad (1\text{–}15)$$

The values of U, V, and W which satisfy (1–15) satisfy (1–14) and by (1–13) give rise to X, Y coordinates which satisfy (1–4). Hence these are X, Y coordinates for points U, V, W which lie on a straight line. Conversely, *any straight line across Figure 1–2 will cut the U, V, W scales at values which satisfy the equation U · V = W.*

Example 1–6, Figure 1–4. 1) Find by expansion the equation represented by the determinant form

$$\begin{vmatrix} -\frac{U}{2} & \frac{\sqrt{3}}{2}U & 1 \\ V & 0 & 1 \\ +\frac{W}{2} & \frac{\sqrt{3}}{2}W & 1 \end{vmatrix} = 0. \qquad (1\text{–}16)$$

2) Plot the U, V, W scales parametrically presented here, assuming that $X_1 = -U/2$; $Y_1 = \sqrt{3}/2\ U$, etc., following the form of (1–4). 3) Draw a straight line across the chart and show that the values in which it cuts these scales satisfy the equation represented. 4) Record these U, V, W values along this straight line.

Answer: $\frac{1}{U} + \frac{1}{V} = \frac{1}{W}$

Example 1–7, Figure 1–5. 1) Find by expansion the equation represented by the determinant form

$$\begin{vmatrix} \frac{3}{1 + U^2} & \frac{3U}{1 + U^2} & 1 \\ \frac{+3}{1 + V^2} & \frac{-3V}{1 + V^2} & 1 \\ \frac{3}{1 + W} & 0 & 1 \end{vmatrix} = 0. \qquad (1\text{–}17)$$

2) Determine the shape of the U curve before plotting, eliminate the parameter U and identify the resulting function. 3) Plot the U, V, W scales pre-

sented parametrically here, assuming that $X_1 = 3/1+U^2$. 4) Verify that the diagram works.

Answer: $U \cdot V = W$

The U-curve is a circle.

1–5. *Limitations of the Method thus Far; Its Promise.* All that has been shown thus far is that when three parametric functions are plotted and a determinant formed from them as in the examples, following the pattern of (1–4) and (1–7), any straight line will meet these curves in values which are solutions to the equation obtained by expanding the resulting determinant (1–8). The converse problem asks how, if the equation is given, one gets (when possible) the determinant for it. This is the subject of the following chapter. Necessary and sufficient conditions for doing this are discussed in the Appendix but are not very useful.

A student will sometimes ask if functions other than straight lines can be put into a determinant form analogous to (1–4) and hence that, say, a circle or other curve, rather than a straight line may be made the "join" of values which satisfy the expansion of the determinant. This is discussed in the Appendix.

1–6. *Scale Multipliers or Scale Factors.* The nomographer is always interested in certain ranges of the variables in the equation he has at hand. The range of one or all of these variables may be infinite, but usually a variable's range is finite, dictated by the practical limits of the problem it came from. At the same time, the amount of space that can be devoted to the scale for such a variable is limited by the size of the diagram the nomographer wishes to make and the way the scale spreads out over it. Hence, it becomes necessary to devise ways of enlarging or reducing any scale until it uses the available space well.

A scale $\quad X_1 = 0 \qquad Y_1 = U \qquad 0 \leqq U \leqq 10$

is vertical and ten units long. It can keep its pattern but be made twenty inches long by the device

$$X_1 = 0 \qquad\qquad Y_1 = 2U,$$

that is, by introducing a multiplier 2 as shown. In general, a scale

$$X_1 = U_1 \qquad\qquad Y_1 = U_2$$

where U_1, U_2 are functions of U, can be written arbitrarily with multipliers inserted

$$X_1 = aU_1 \qquad\qquad Y_1 = bU_2$$

and then values of a or b or both arrived at which spread this scale out advantageously in X and Y. If the available length in X for the scale is L_X, and the range in U is

$$U_l \leqq U \leqq U_h$$

then one has

$$L_X = a(U_1(U_h) - U_1(U_l)); \quad a = L_X/(U_1(U_h) - U_1(U_l)) \tag{1–18}$$

Similarly for

$$L_Y = b(U_2(U_h) - U_2(U_l)), \quad b = L_Y/(U_2(U_h) - U_2(U_l))$$

Example 1–8, Figure 1–6. A nomographer has the equation U + V = W in determinant form,

$$\begin{vmatrix} 0 & uU & 1 \\ 15 & vV & 1 \\ \dfrac{u \cdot 15}{u+v} & \dfrac{uv\,W}{u+v} & 1 \end{vmatrix} = 0 \tag{1–19}$$

where u and v are constants whose value can be chosen. He wishes to have a chart for this equation which is 15 inches wide and 20 inches high. Ranges of the variables are $0 \leqq U \leqq 10$; $0 \leqq V \leqq 20$. This chart consists of three vertical uniform scales placed respectively at X = 0, X = 15, and $X = \dfrac{u \cdot 15}{u+v}$ and with scale multipliers, u, v, and $w = \dfrac{uv}{u+v}$. The width of the chart is already fifteen inches. The U scale can be stretched out to cover twenty inches if u is given the value 2, while the V scale will cover twenty inches if v is assigned the value 1. Then the multiplier of the scale W becomes 2/3 and it lies 10 inches from the U scale.

Expansion of the determinant (1–19) shows that this diagram is for the equation U + V = W.

Example 1–9, Figure 1–7. A nomographer has the equation $U \cdot V = W$ in determinant form:

$$\begin{vmatrix} 0 & uU & 1 \\ 15 & 20 - wW & 1 \\ \dfrac{15}{1 + \dfrac{w}{u}V} & \dfrac{15}{1 + \dfrac{w}{u}V} & 1 \end{vmatrix} = 0 \tag{1–20}$$

and he wishes a rectangular chart for it 15 inches wide and 20 inches high. Ranges of the variables are $0 \leqq U \leqq 200$, $0 \leqq W \leqq 10$. The chart consists of vertical, uniform scales in U and W, U increasing up, W down, 15 inches apart, and a diagonal scale in V from lower left (V=00) to upper right (V=0). Then u = 1/10, w = 2. Figure 1–7.

PROBLEMS

In each of the following eleven problems three curves are given in parametric form. In each problem:

1) Replace the X and Y coordinates in the determinant of equation (1–4) by the expressions in the variables given below.

2) Expand this determinant and find the equation between the three variables.

3) Plot and calibrate a short portion of each of the three curves and check these on the completed figure accompanying each problem.

4) For each curve, wherever possible, eliminate the parameter, express the curve in the form Y = Y(X) and identify the kind of curve.

5) Show that the diagram "works" by drawing one or more straight lines across its face and noting that the values where it cuts the scales satisfy the equation between those variables found in step 2). (Use the complete figure for this.)

PROBLEM 1–1. Ordinary addition. Figure 1–8.

$$X_1 = 0 \qquad Y_1 = 10 \cdot U$$
$$X_2 = 15 \qquad Y_2 = 20 \cdot V$$
$$X_3 = 5 \qquad Y_3 = 6.67 \cdot W$$

The chart scale-stems fill a space 15 × 20 inches. Show that expansion of the determinant yields the equation U + V = W.

PROBLEM 1–2. General quadratic. Figure 1–9.

$$X_1 = 0 \qquad Y_1 = 10B$$
$$X_2 = 15 \qquad Y_2 = 10A$$
$$X_3 = \frac{15}{x+1} \qquad Y_3 = \frac{-10\,x^2}{x+1}$$

The chart scale-stems fill a space 15 × 20 inches. Show that the expansion of the determinant yields

$$x^2 + Ax + B = 0.$$

PROBLEM 1–3. Parabolic nomogram for multiplication. Figure 1–10.

$$X_1 = -V \qquad Y_1 = V^2$$
$$X_2 = U \qquad Y_2 = U^2$$
$$X_3 = 0 \qquad Y_3 = W$$

The chart scale-stems fill a space 6 × 9 inches. Show that the expansion of the determinant gives UV(U + V) = W(U + V),(U ≠ − V) or U · V = W.

PROBLEM 1–4. Cubic with a constant term. Figure 1–11.

$$X_1 = 0 \qquad Y_1 = 10A$$
$$X_2 = 15 \qquad Y_2 = 10B$$
$$X_3 = \frac{15x}{x + x^2} \qquad Y3 = \frac{-10(-0.4 + x^3)}{x + x^2}$$

The chart scale-stems fill a space 15 × 20 inches. For this problem, consider only the curve C = −0.4. Show that the expanded determinant gives

$$x^3 + Ax^2 + Bx - 0.4 = 0.$$

PROBLEM 1–5. Air flow problem. Figure 1–12.

The chart scale-stems fill a space 20 × 15 inches, divided as shown.

This problem requires a *compound* diagram, as the figure shows — a combination of three diagrams. Two of these compute the quantities.

$$A = -\,K^{-3}\frac{L_T}{L_P}$$

The ranges on these will be *found* to be

$$\text{A from 0 to } -\frac{1}{6000}$$
$$\text{B from 0 to } -125$$

On this basis, the scales for A, B and V are given by the equations shown below. A and B, being negative quantities, are thus made to plot in the upward direction. There is no need to draw in these two scales since they are merely intermediate scales on the use of the chart.

The student should check the operation of the chart by one or more collineations. With A and B known, this problem then proceeds like the rest.

$$X_1 = 0$$
$$X_2 = 20$$
$$X_3 = \frac{20(88{,}300{,}000)}{0.12 \cdot V^4 + 88{,}300{,}000}$$

$$Y_1 = -88{,}300{,}000\ A$$
$$Y_2 = -0.12\ B$$
$$Y_3 = \frac{0.12(88{,}300{,}000)V}{0.12 \cdot V^4 + 88{,}300{,}000}$$

Show that the expanded determinant gives

$$V = 52.8\left[\frac{1}{A}(1 - B/3V)\right]^{1/3}$$

PROBLEM 1–6. Trajectory range. Figure 1–13.

This problem has scale-stems filling a space 20 × 15 inches. It is also one of those requiring a compound diagram—one made up of two diagrams in this case, namely one for $W = R_O + R_G$ and the other as stated. Since the ranges on R_O and R_G are known, the range of W can be found and the solution obtained for the remainder of the chart as specified.

$$X_1 = 0$$
$$X_2 = 4.58$$
$$X_3 = \frac{4.58}{0.00959\sqrt{H} + 1}$$

$$Y_1 = 0.06S$$
$$Y_2 = -0.0023W + 15$$
$$Y_3 = \frac{0.000205H + 15}{0.00959\sqrt{H} + 1}$$

Show that the expanded determinant gives

$$\frac{\sqrt{H} \cdot S}{4} - \frac{+H}{11.19} = W = R_O + R_G$$

PROBLEM 1–7. Compound pendulum. Figure 1–14.

$X_1 = 0$	$Y_1 = 2L$
$X_2 = -7.5/A$	$Y_2 = 2A$
$X_3 = 7.5/B$	$Y_3 = 2B$

The chart scale-stems fill a space 15 × 20 inches. Show that the expanded determinant gives

$$L = (A^2 + B^2)/(A + B).$$

PROBLEM 1–8. Two-dimensional probability density. Figure 1–15.

$$X_1 = 0$$
$$X_2 = 15$$
$$X_3 = \frac{1215}{2s^2 + 81}$$

$$Y_1 = 4.15 \log_e{}^y = 20$$
$$Y_2 = 0.0222r^2$$
$$Y_3 = \frac{2s^2(20 - 4.15 \log_e{}^y 2\pi s^2)}{2s^2 + 81}$$

The chart scale-stems fill a space 15 × 20 inches. Show that the expanded determinant gives

$$y = \frac{1}{2\pi s^2} \cdot e \left\{\frac{-r^2}{2s^2}\right\}.$$

The latter term is an exponential.

PROBLEM 1–9. Complex variables. Figure 1–16.

$X_1 = W$	$Y_1 = -1$	where:
$X_2 = 0$	$Y_2 = -1/V^2$	$V = \tanh \eta/2$
$X_3 = U$	$Y_3 = U^2$	$U = \tan \varepsilon/2$

The chart scale-stems fill a space 15 × 20 inches. Show that the expanded determinant gives

$$W + V^2U + WU^2V^2 - U = 0.$$

PROBLEM 1–10. Altered quadratics. Figure 1–17.

$$X_1 = 15$$
$$X_2 = 0$$
$$X_3 = \frac{15x^2}{x^2 + 1}$$

$$Y_1 = a$$
$$Y_2 = b$$
$$Y_3 = \frac{-x}{x^2 + 1}$$

The chart scale-stems fill a space 15 × 20 inches. Show that the expanded determinant gives

$$a \cdot x + b/x + 1 = 0.$$

PROBLEM 1–11. Logarithmic mean. Figure 1–18.

$X_1 = 0$	$Y_1 = M$
$X_2 = \frac{6}{\ln y_1}$	$Y_2 = \frac{y_1}{\ln y_1}$
$X_3 = \frac{6}{\ln y_2}$	$Y_3 = \frac{y_2}{\ln y_2}$

The chart scale-stems fill a space 7.5 × 9 inches. Show that the expanded determinant gives

$$M = (y_2 - y_1)/(\ln y_2 - \ln y_1).$$

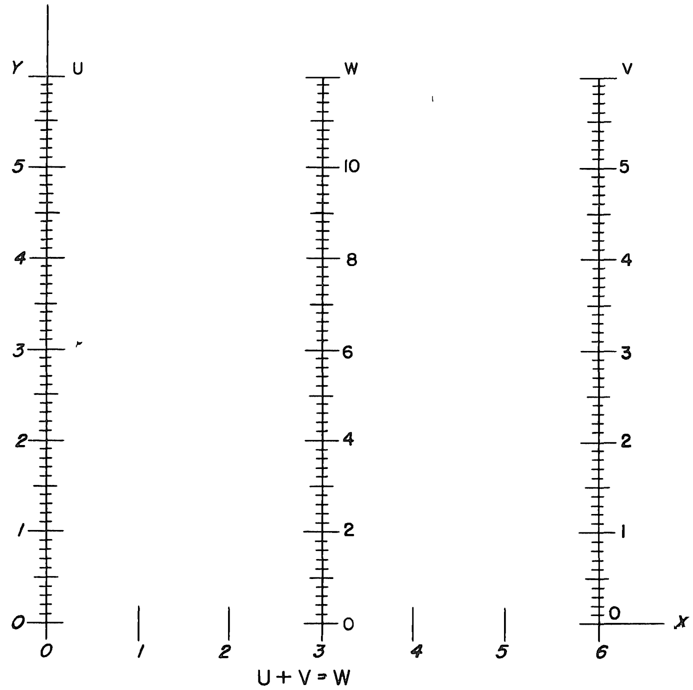


Figure 1–1.

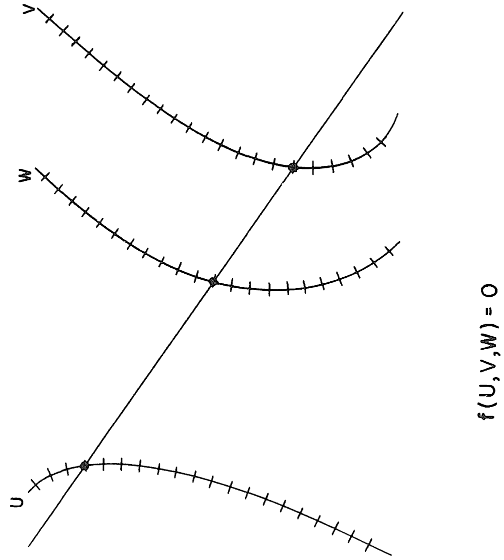


Figure 1–3.

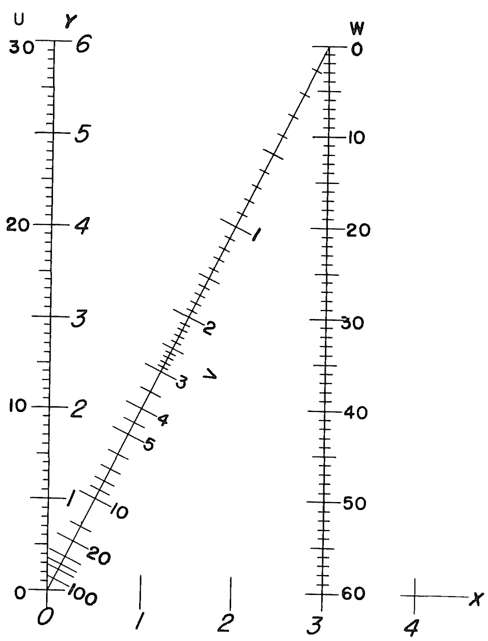


Figure 1–2.

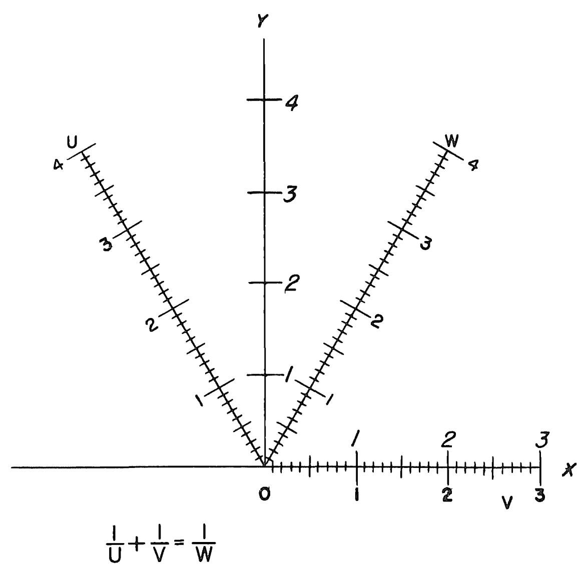


Figure 1–4.

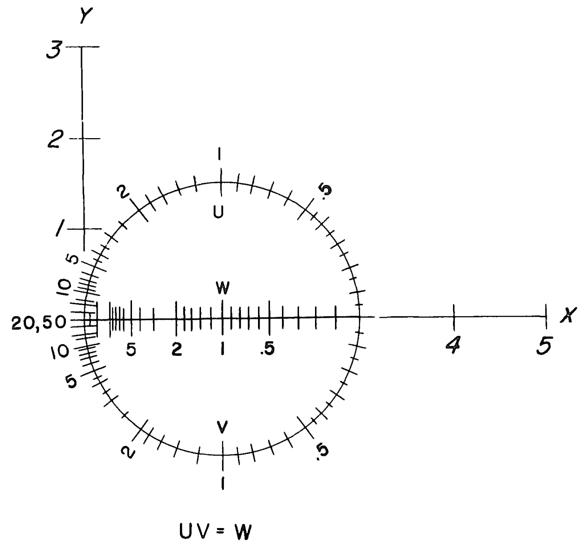


Figure 1–5.

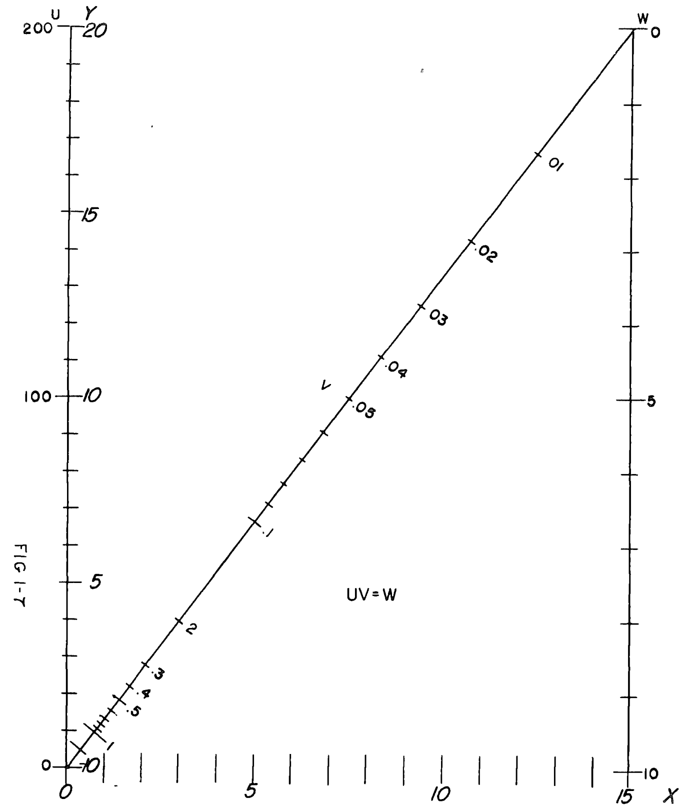


Figure 1–7.

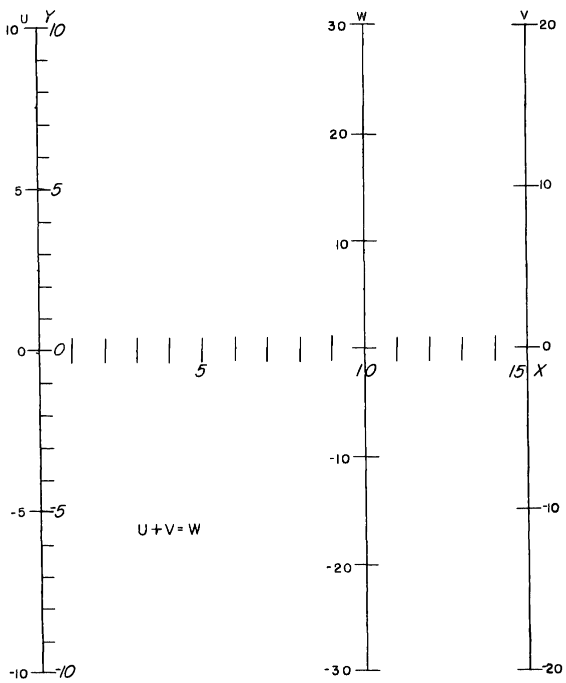


Figure 1–6.

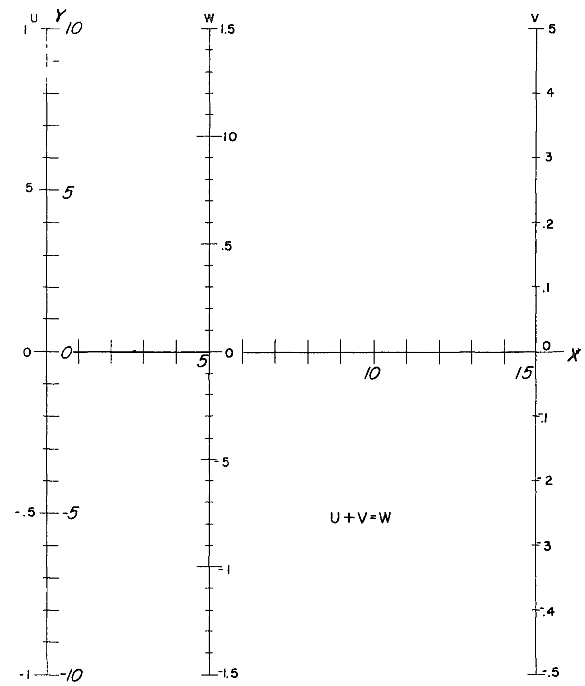


Figure 1–8.

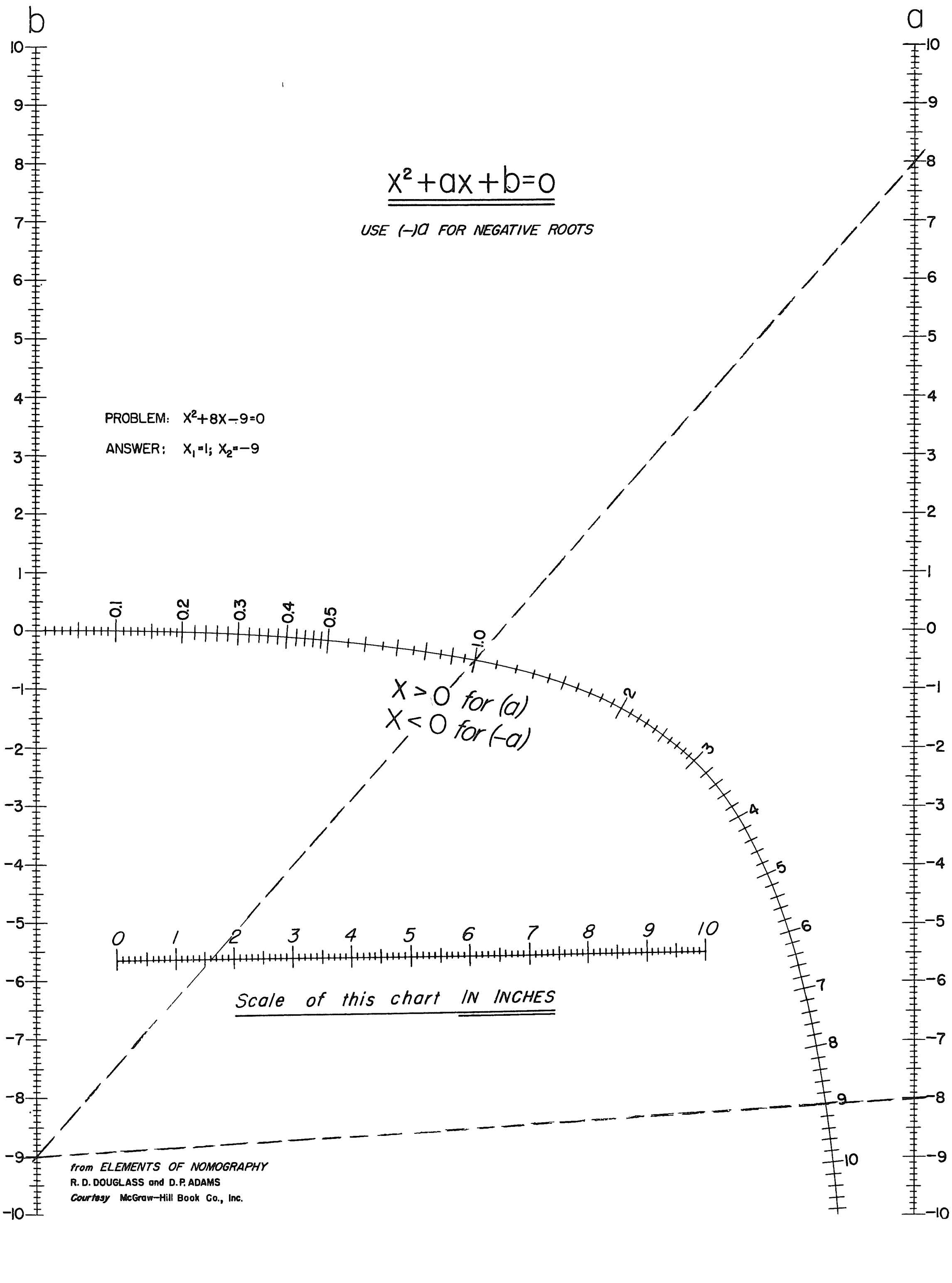

b
a
X²+ax+b=o
USE (–)a FOR NEGATIVE ROOTS
PROBLEM: X²+8X–9=0
ANSWER: X₁=1; X₂=–9
X > 0 for (a)
X < 0 for (–a)
Scale of this chart IN INCHES
from ELEMENTS OF NOMOGRAPHY
R. D. DOUGLASS and D. P. ADAMS
Courtesy McGraw-Hill Book Co., Inc.

Figure 1–9.

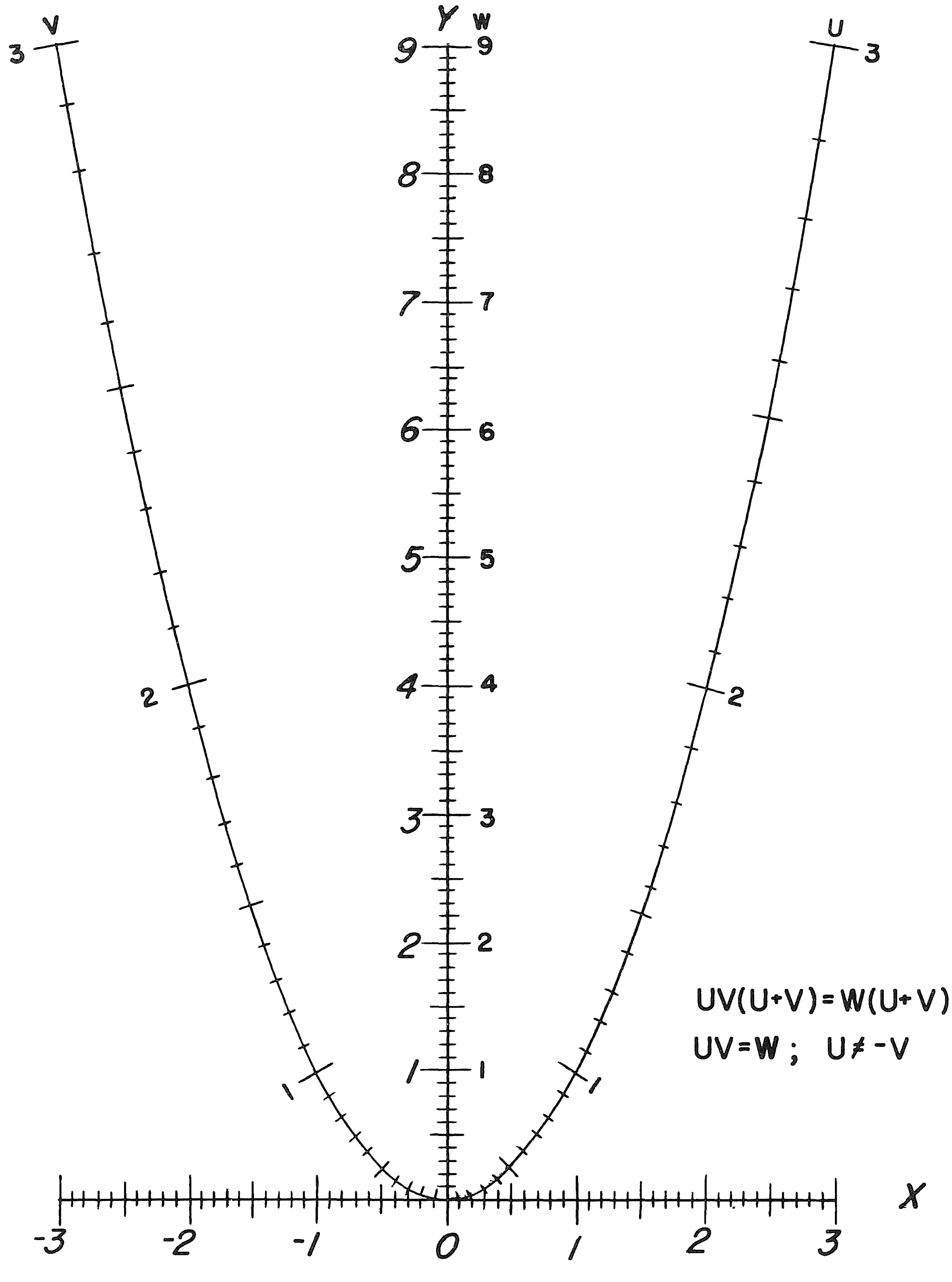
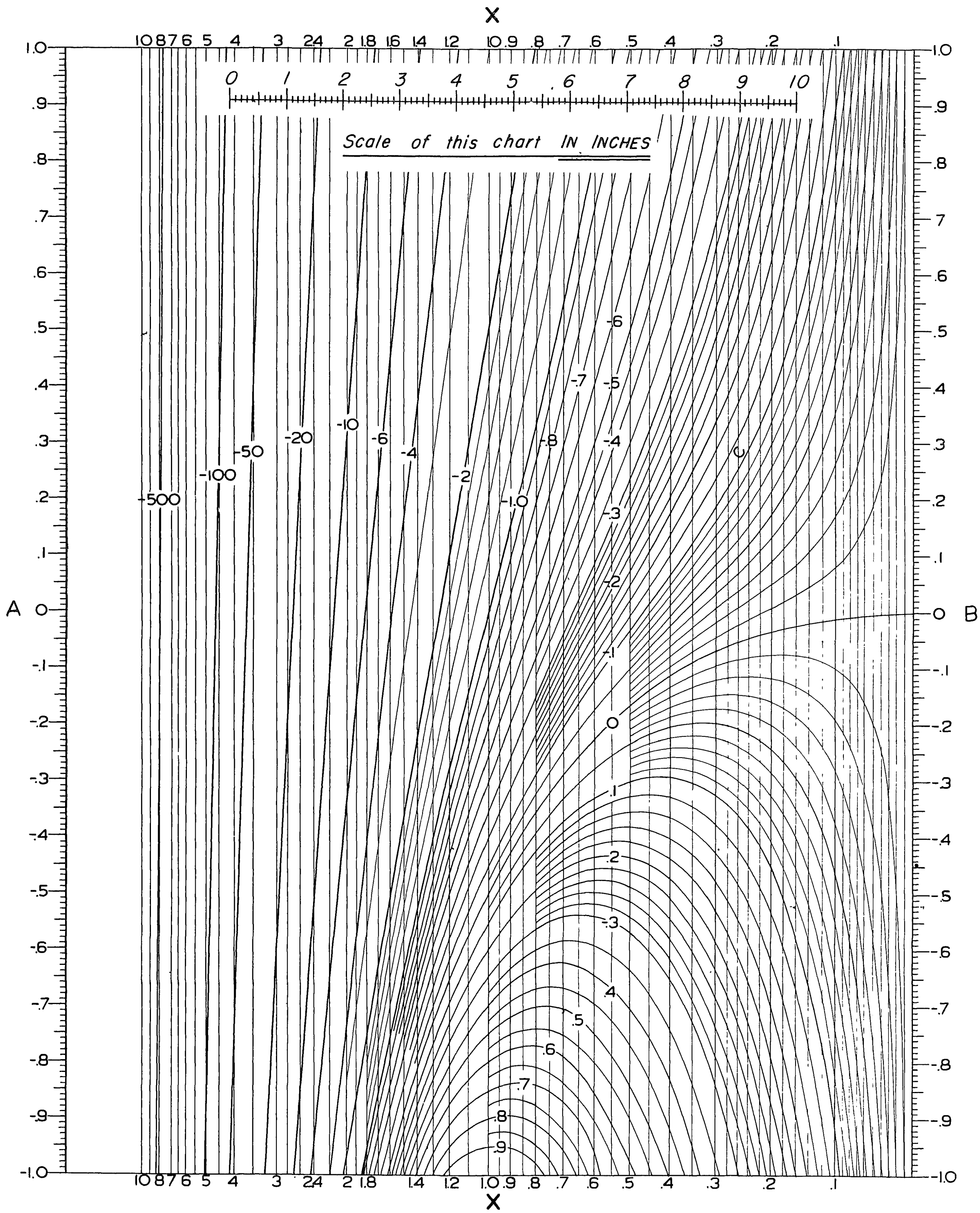
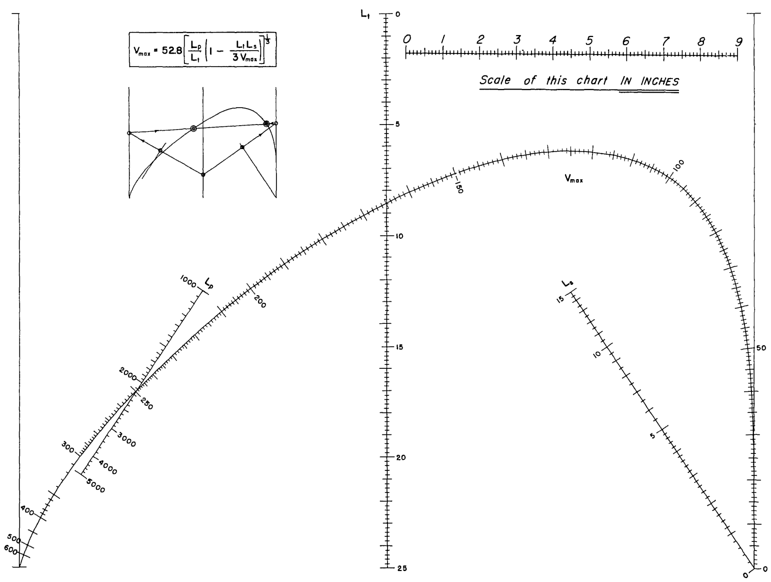


Figure 1–10.



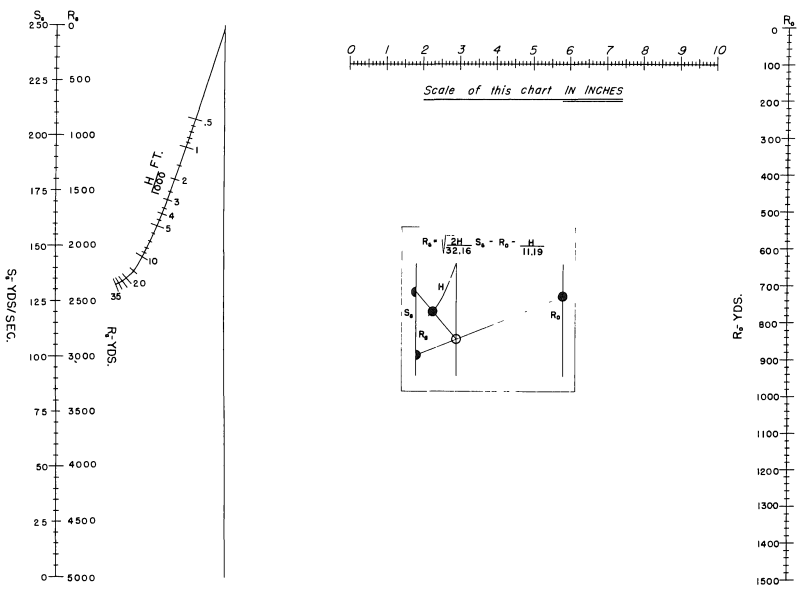
$X^3 + AX^2 + BX + C = 0$

Figure 1–11.



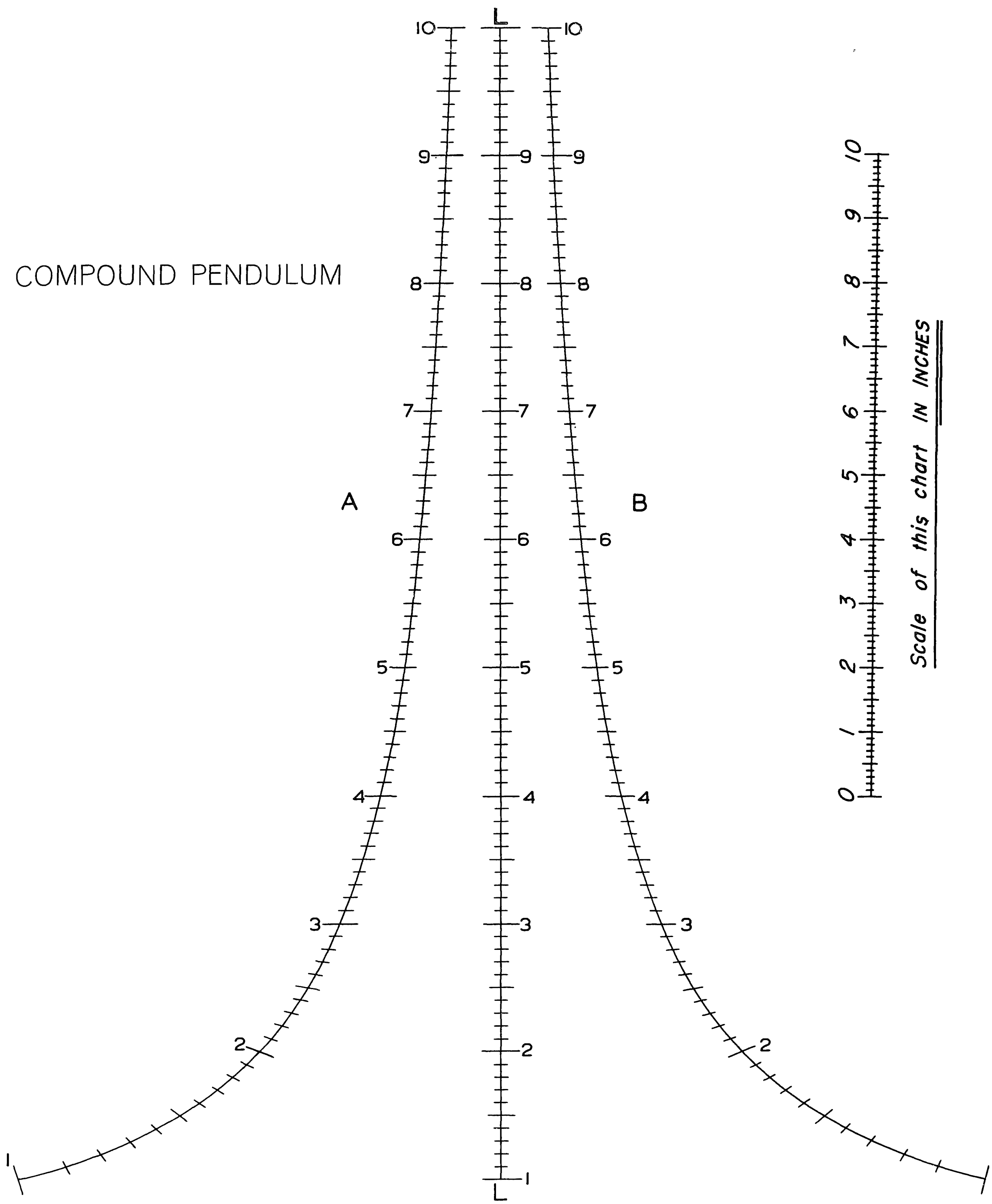
AIR FLOW PROBLEM

Figure 1–12.



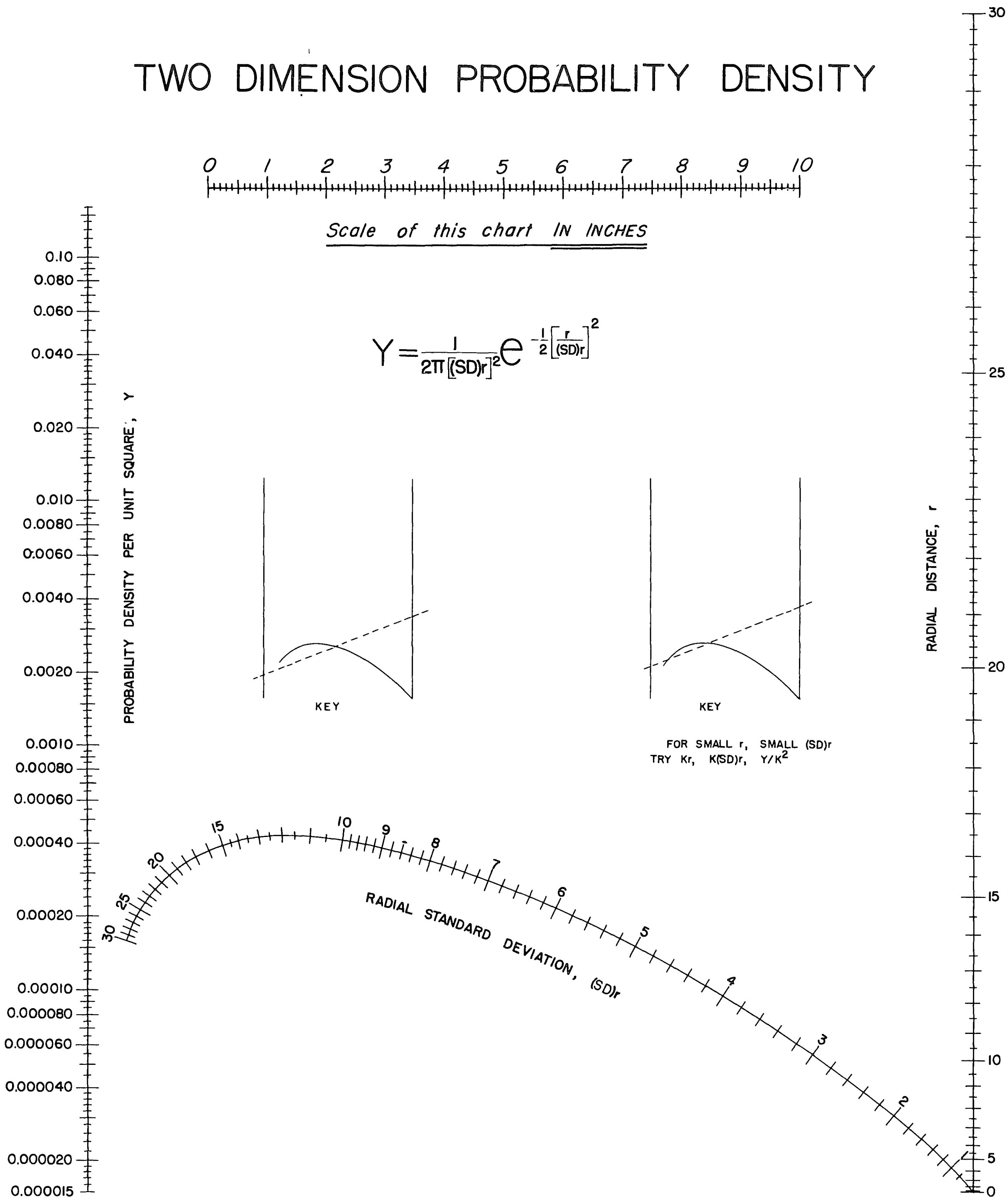
TRAJECTORY RANGE

Figure 1–13.



$$L = \frac{A^2 + B^2}{A + B}$$

Figure 1–14.



TWO DIMENSION PROBABILITY DENSITY
0 1 2 3 4 5 6 7 8 9 10
Scale of this chart IN INCHES
$Y = \frac{1}{2\pi[(SD)r]^2} e^{-\frac{1}{2}\left[\frac{r}{(SD)r}\right]^2}$
PROBABILITY DENSITY PER UNIT SQUARE, Y
0.10
0.080
0.060
0.040
0.020
0.010
0.0080
0.0060
0.0040
0.0020
0.0010
0.00080
0.00060
0.00040
0.00020
0.00010
0.000080
0.000060
0.000040
0.000020
0.000015
KEY
KEY
FOR SMALL r, SMALL (SD)r
TRY Kr, K(SD)r, Y/K²
RADIAL STANDARD DEVIATION, (SD)r
30 25 20 15 10 9 8 7 6 5 4 3 2
RADIAL DISTANCE, r
30
25
20
15
10
5
0

Figure 1–15.

D

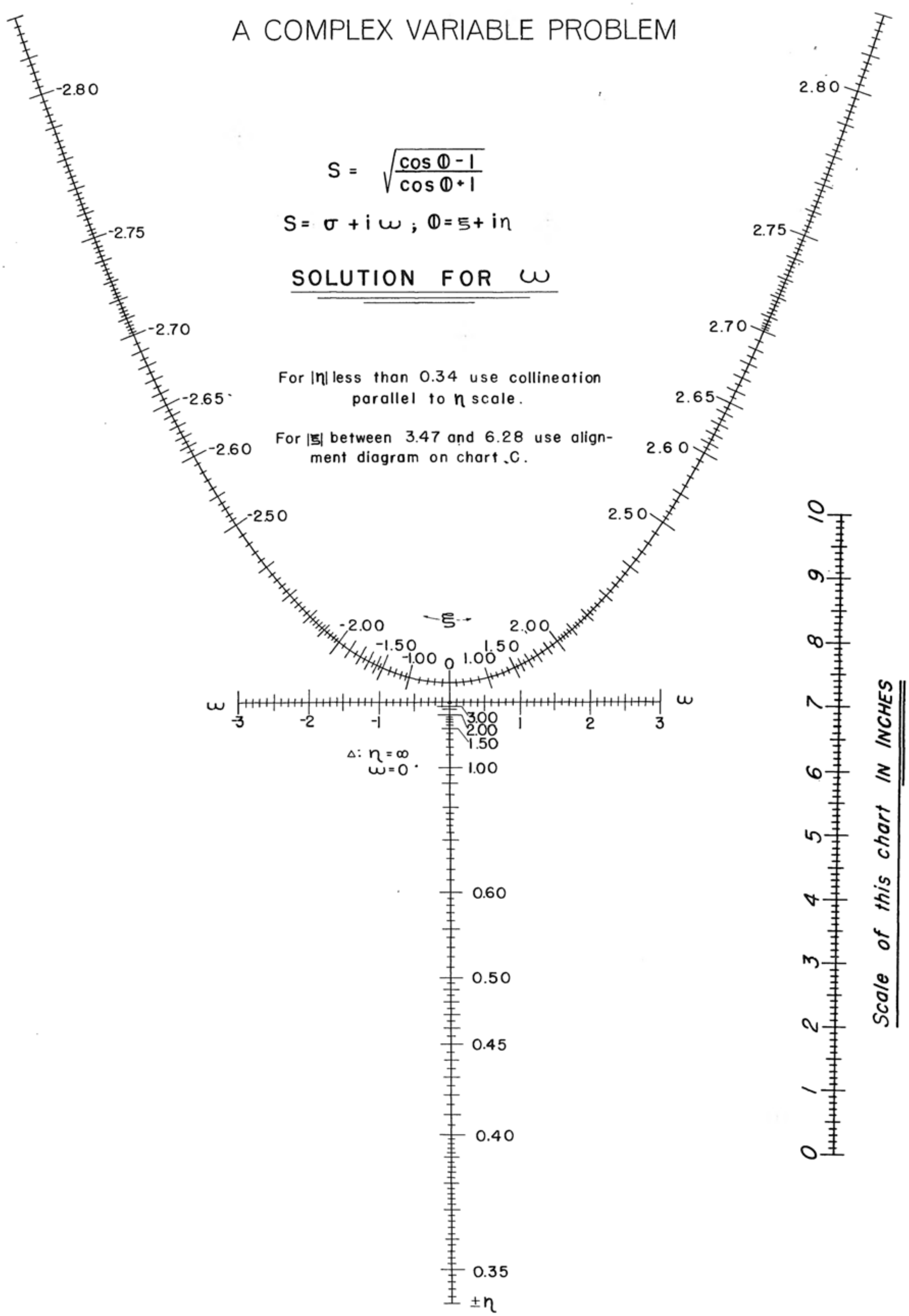

A COMPLEX VARIABLE PROBLEM
S = √((cos Φ - 1)/(cos Φ + 1))
S = σ + i ω ; Φ = ξ + iη
SOLUTION FOR ω
For |η| less than 0.34 use collineation parallel to η scale.
For |ξ| between 3.47 and 6.28 use alignment diagram on chart C.
-2.80
-2.75
-2.70
-2.65
-2.60
-2.50
-2.00
-1.50
-1.00
0
1.00
1.50
2.00
2.50
2.60
2.65
2.70
2.75
2.80
ξ
ω
-3
-2
-1
1
2
3
ω
3.00
2.00
1.50
1.00
Δ: η = ∞
ω = 0
0.60
0.50
0.45
0.40
0.35
±η
Scale of this chart IN INCHES
0
1
2
3
4
5
6
7
8
9
10

Figure 1–16.

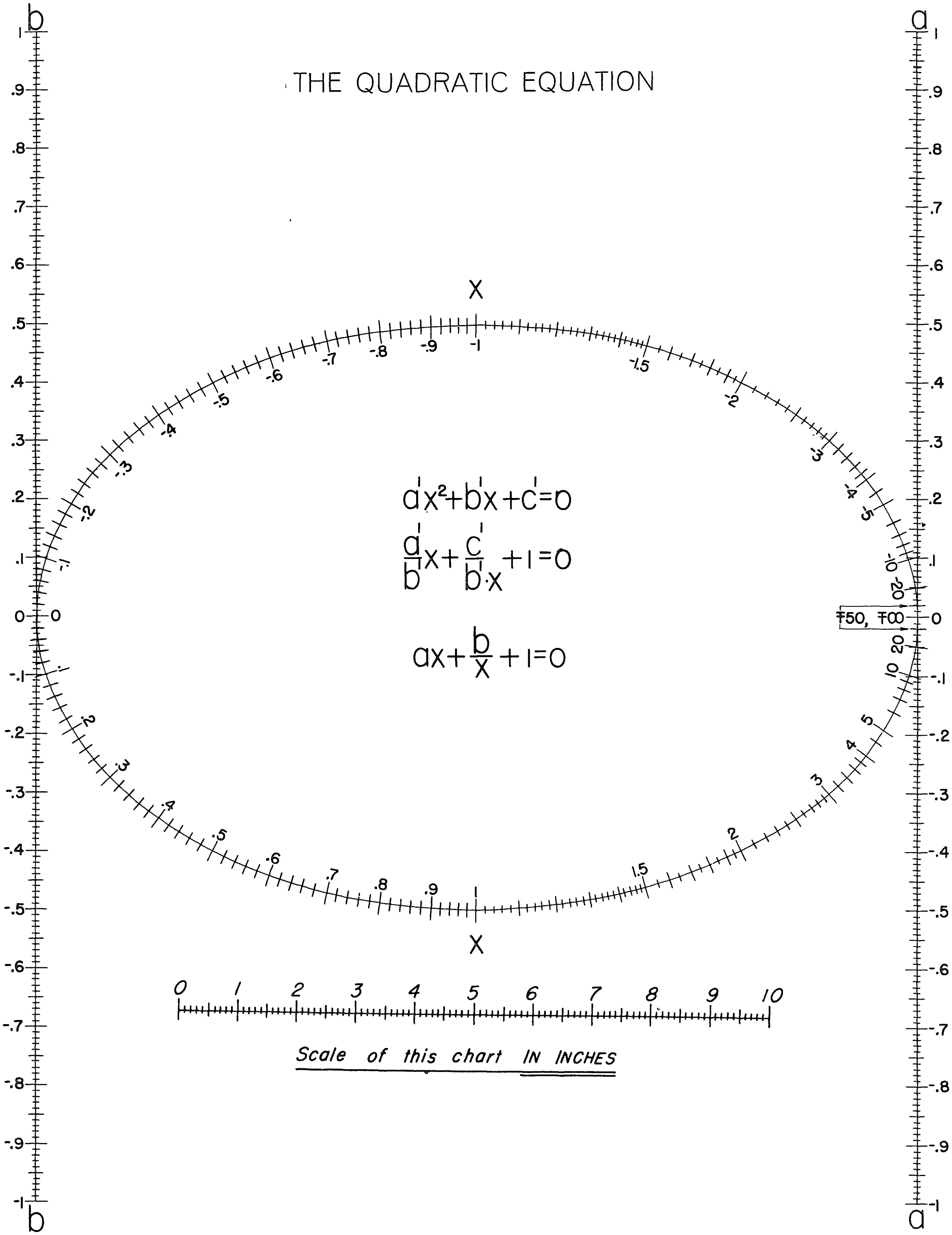

THE QUADRATIC EQUATION
$a'x^2+b'x+c'=0$
$\frac{a'}{b'}x+\frac{c'}{b'x}+1=0$
$ax+\frac{b}{x}+1=0$
Scale of this chart IN INCHES

Figure 1–17.

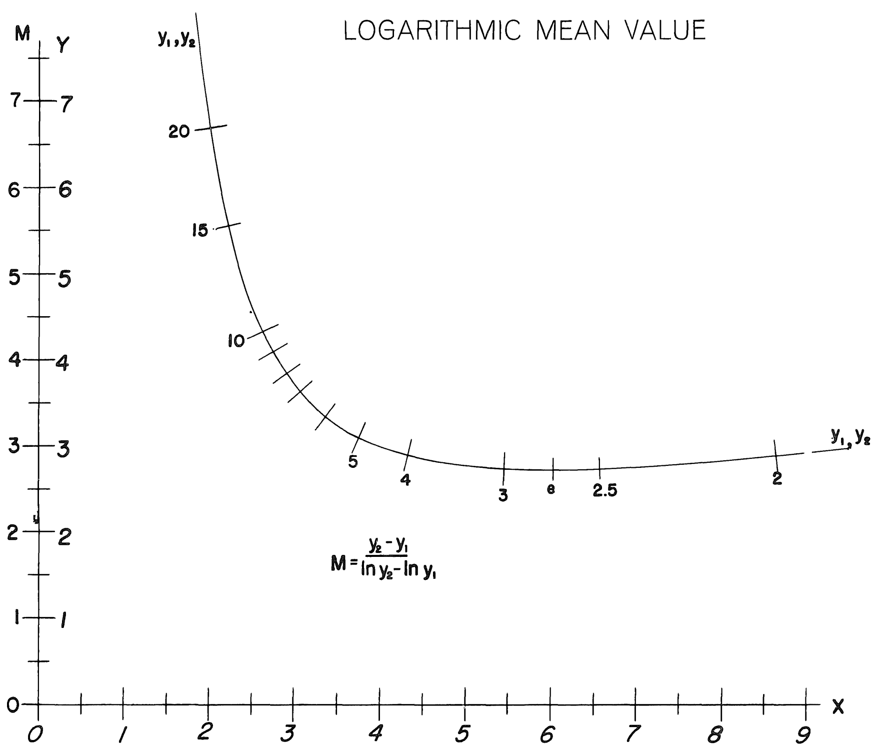

LOGARITHMIC MEAN VALUE
M
Y
y_1, y_2
20
15
10
5
4
3
e
2.5
2
y_1, y_2
$M = \frac{y_2 - y_1}{\ln y_2 - \ln y_1}$
0
1
2
3
4
5
6
7
8
9
X

Figure 1–18.

CHAPTER 2

EQUATIONS IN DETERMINANT FORM

2–1. *An Equation in the Form of Three Compatible Equations.* It is possible to write down three equations which express a single three-variable equation. These three equations are said to be compatible; that is, they do not conflict with each other, or with the original equation. Thus, if the original equation is

$$U + V = W \tag{2–1}$$

one can write:

$$\text{Let } A = U$$
$$B = V$$
$$\text{Then } A + B = W. \tag{2–2}$$

These three equations are compatible. They can be written in the form

$$A \cdot 1 + B \cdot 0 - U = 0$$
$$A \cdot 0 + B \cdot 1 - V = 0$$
$$A \cdot 1 + B \cdot 1 - W = 0. \tag{2–3}$$

(2–3) is identical with (2–1).

2–2. *An Equation Put in Determinant Form.* Since equations (2–3) are compatible, the determinant of their coefficient vanishes, or

$$\begin{vmatrix} 1 & 0 & -U \\ 0 & 1 & -V \\ 1 & 1 & -W \end{vmatrix} = 0. \tag{2–4}$$

Equation (2–4) is a relationship between U, V and W. It is consistent with the three equations (2–3) and (2–2) which were based upon (2–1). Hence, (2–4) and (2–1) must be identical. This can be proved by expanding (2–4) to prove it is indeed the relation

$$U + V = W.$$

Any equation F(U, V, W) = 0 can be put in determinant form, the simplest of which would be

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & F(U,V,W) \end{vmatrix} = 0 \tag{2–5}$$

but (2–4) has the advantage that it has distributed the variables of its equation into various places throughout the determinant.

Example 2–1. Place the equation $U \cdot V = W$ in determinant form in three different ways.

Method 1)

$$\text{Let } A = U$$
$$B = W$$
$$A \cdot 1 + B \cdot 0 - U = 0$$
$$A \cdot 0 + B \cdot 1 - W = 0$$
$$A \cdot V - B \cdot 1 + 0 = 0$$

Answer

$$\begin{vmatrix} 1 & 0 & -U \\ 0 & 1 & -W \\ V & -1 & 0 \end{vmatrix} = 0. \tag{2–6}$$

Check by expanding the determinant

$$U \cdot V = W$$

Method 2)

$$\text{Let } A = U$$
$$B = V$$
$$A \cdot 1 + B \cdot 0 - U = 0$$
$$A \cdot 0 + B \cdot 1 - V = 0$$
$$A \cdot V + B \cdot 0 - W = 0$$

Answer

$$\begin{vmatrix} 1 & 0 & -U \\ 0 & 1 & -V \\ V & 0 & -W \end{vmatrix} = 0. \tag{2–7}$$

Check by expanding the determinant

$$U \cdot V = W$$

Method 3)

$$\text{Let } A = U$$
$$B = V \text{ as in 2)}$$
$$A \cdot 1 + B \cdot 0 - U = 0$$
$$A \cdot 0 + B \cdot 1 - V = 0$$
$$A \cdot 0 + B \cdot U - W = 0$$

Answer

$$\begin{vmatrix} 1 & 0 & -U \\ 0 & 1 & -V \\ 0 & U & -W \end{vmatrix} = 0. \qquad (2\text{–}8)$$

Check by expanding the determinant

$$U \cdot V = W$$

All three of these ways are valid, but Method 1) will be shown to be better for our use than 2) or 3).

Example 2–2. Place the equation $x^2 + Ax + B = 0$ in determinant form. To avoid confusion, call the two unknowns here P and Q.

$$\text{Let } P = A$$

$$Q = B$$

$$P \cdot 1 + Q \cdot 0 - A = 0$$

$$P \cdot 0 + Q \cdot 1 - B = 0$$

$$P \cdot x + Q \cdot 1 + x^2 = 0$$

Answer

$$\begin{vmatrix} 1 & 0 & -A \\ 0 & 1 & -B \\ x & 1 & +x^2 \end{vmatrix} = 0. \qquad (2\text{–}9)$$

Check $x^2 + Ax + B = 0$

Conclusions:

a. Any equation can be put into determinant form.

b. This can be done in a number of different ways.

c. The expansion of the determinant then yields the equation.

d. The determinant is the equation in a special form.

Definition: Form ((A)). If an equation is in determinant form, it is said to be in Form ((A)).

2–3. *Disjoining Variables in a Determinant. Form* ((B)). It is often possible to make changes in the determinant according to the standard rules (see Appendix) until each of the three variables appears in only one row (or in only one column.)

Definition: Form ((B)). When each of the three variables of an equation in determinant form appears in only one row (or one column) the variables are said to be *disjoined*. The determinant equation is then said to have form ((B)). All equations in determinant form ((B)) are also in form ((A)), but those in form ((A)) are not in form ((B)) unless they are disjoined. In (2–4) the equation $U + V = W$ appeared in a determinant form

$$\begin{vmatrix} 1 & 0 & -U \\ 0 & 1 & -V \\ 1 & 1 & -W \end{vmatrix} = 0. \quad ((B)) \qquad (2\text{–}10)$$

Here disjunction had already occurred, so the determinant can be labeled ((B)).

In Example 2–1, Method 1), the equation

$$U \cdot V = W$$

appeared in the form (2–6).

$$\begin{vmatrix} 1 & 0 & -U \\ 0 & 1 & -W \\ V & -1 & 0 \end{vmatrix} = 0 \quad ((B)) \qquad (2\text{–}11)$$

which can be labeled ((B)). However, the *same equation* also appeared in the form, Method 2) (2–7)

$$\begin{vmatrix} 1 & 0 & -U \\ 0 & 1 & -V \\ V & 0 & -W \end{vmatrix} = 0 \quad ((A)) \qquad (2\text{–}12)$$

to which only the symbol ((A)) can be attached because the variables have not been disjoined since the bottom row contains both a V and a W. (2–12) can be put in form ((B)) by the following determinant changes:

$$0 = \begin{vmatrix} 1 & 0 & -U \\ 0 & 1 & -V \\ V & 0 & -W \end{vmatrix}$$

(a) Multiply column III by (-1)

(b) then multiply column II by $(-V)$ and add to column III

$$0 = \begin{vmatrix} 1 & 0 & U \\ 0 & 1 & 0 \\ V & 0 & W \end{vmatrix}$$

Add row II to row III

$$0 = \begin{vmatrix} 1 & 0 & U \\ 0 & 1 & 0 \\ V & 1 & W \end{vmatrix}$$

Multiply column II by $(-V)$ and add to column I.

$$0 = \begin{vmatrix} 1 & 0 & U \\ -V & 1 & 0 \\ 0 & 1 & W \end{vmatrix} \quad ((B)) \tag{2–13}$$

Conclusions: A determinant in form ((A)) *can sometimes be disjoined* and thus placed in form ((B)) by formal determinant changes.

Note: It is not always possible to change a determinant from form ((A)) to form ((B)). Necessary and sufficient conditions have been derived which can theoretically tell when an equation can and when it cannot be disjoined but they are not in useful form. The practitioner can only go through various manipulations such as appear above in the hope of effecting disjunction. Though success is possible, he may lack the skill to achieve it. Continued attempts at disjoining are always a gamble since the way may not be possible. After some experience, the student usually gets so that he can find any solution that exists in a reasonably short time.

RULE: As a first step toward placing a three-variable equation in disjoined form ((B)), one should look to see if there are any variables which appear only *once* in the equation. The substitutions should preferably be tried for those variables. In the example below, Methods 1) and 2) do not follow this rule, Method 3) does and gives the desired result quicker and easier.

Example 2–3. Place the following equation for two-dimensional probability density in form ((B)).

$$y = \frac{1}{2\pi S^2} e\{-r^2/2S^2\} \qquad \text{(exponential)}$$

Taking logs

$$\ln y + \ln 2\pi + \ln S^2 + r^2/2S^2 = 0$$

let

$$y' = \ln y + \ln 2\pi$$

$$2S^2y' + 4S^2 \ln S + r^2 = 0$$

Method 1) Let $A = S^2$

$$B = r^2$$

$$A \cdot 1 + B \cdot 0 - S^2 = 0$$

$$A \cdot 0 + B \cdot 1 - r^2 = 0$$

$$A \cdot 2y' + B \cdot 1 + 4S^2 \ln S = 0$$

$$0 = \begin{vmatrix} 1 & 0 & -S^2 \\ 0 & 1 & -r^2 \\ 2y' & 1 & 4S^2 \ln S \end{vmatrix}$$

(a) multiply column III by (-1)

(b) divide row II by r^2

$$0 = \begin{vmatrix} 1 & 0 & S^2 \\ 0 & 1/r^2 & 1 \\ 2y' & 1 & -4S^2 \ln S \end{vmatrix} \quad ((B))$$

now write rows as columns, or columns as rows

$$0 = \begin{vmatrix} 1 & 0 & 2y' \\ 0 & 1/r^2 & 1 \\ S^2 & 1 & -4S^2 \ln S \end{vmatrix} \quad ((B)) \tag{2–14}$$

Method 2) as before, let

$$A = S^2$$

$$B = r^2$$

$$A \cdot 1 + B \cdot 0 - S^2 = 0$$

$$A \cdot 0 + B \cdot 1 - r^2 = 0$$

$$A(2y' + 4 \ln S) + B \cdot 1 + 0 = 0$$

$$0 = \begin{vmatrix} 1 & 0 & -S^2 \\ 0 & 1 & -r^2 \\ 2y' + 4 \ln S & 1 & 0 \end{vmatrix}$$

(a) multiply row I by $4 \ln S$ and subtract from row III

(b) divide row II by r^2

$$0 = \begin{vmatrix} 1 & 0 & -S^2 \\ 0 & 1/r^2 & -1 \\ 2y' & 1 & 4S^2 \ln S \end{vmatrix} \quad ((B))$$

write rows for columns, columns for rows

$$0 = \begin{vmatrix} 1 & 0 & 2y' \\ 0 & 1/r^2 & -1 \\ -S^2 & -1 & 4S^2 \ln S \end{vmatrix} \quad ((B)) \tag{2–15}$$

Method 3) Let $A = y'$

$$B = r^2$$

$$A \cdot 1 + B \cdot 0 - y' = 0$$

$$A \cdot 0 + B \cdot 1 - r^2 = 0$$

$$A \cdot 2S^2 + B \cdot 1 + 4S^2 \ln S = 0$$

$$= \begin{vmatrix} 1 & 0 & -y' \\ 0 & 1 & -r^2 \\ 2S^2 & 1 & 4S^2 \ln S \end{vmatrix} \quad ((B)) \qquad (2\text{–}16)$$

In Methods 1) and 2), the disjoining occurred first by columns. In Method 3), disjunction occurred by rows and right away with no need for changes within the determinant. These three approaches show the effects of different substitutions and the value of trying different ones for A and B if the disjoining seems to be coming hard.

2–4. *Canonical Form of Variables in a Determinant. Form* ((C)).

Once disjunction of variables in an equation has been carried through and form ((B)) obtained, a further final form of the determinant can always easily be found. This is known as the canonical form, or form ((C)).

Definition. Form ((C)). If the disjunction of a determinant has been by rows, ((C)) is made to have a *column* of 1's usually placed on the right as in (1–8). If the disjunction of a determinant has been by *columns,* ((C)) is made to have a *row* of 1's usually placed along the bottom.

Example 2–4. The equation $U + V = W$ was placed in form ((B)) in (2–4). We now change it into canonical form ((C))

$$0 = \begin{vmatrix} 1 & 0 & -U \\ 0 & 1 & -V \\ 1 & 1 & -W \end{vmatrix}$$

(a) multiply column III by (-1)

(b) add column II to column I

$$0 = \begin{vmatrix} 1 & 0 & U \\ 1 & 1 & V \\ 2 & 1 & W \end{vmatrix}$$

(a) divide row III by 2

(b) shift column I to column III

$$\begin{vmatrix} O & U & 1 \\ 1 & V & 1 \\ 1/2 & W/2 & 1 \end{vmatrix} \quad ((C)) \qquad (2\text{–}17)$$

Following (2–12, 13) equation $U \cdot V = W$ can be placed in the canonical form ((C)) as follows:

$$\begin{vmatrix} 0 & U & 1 \\ 1 & W & 1 \\ \frac{1}{1-V} & 0 & 1 \end{vmatrix} = 0. \quad ((C)) \qquad (2\text{–}18)$$

Following (2–9), the equation $x^2 + Ax + B = 0$ can be put into canonical form ((C)) as follows:

$$\begin{vmatrix} 0 & A & 1 \\ 1 & B & 1 \\ \frac{1}{1+x} & \frac{-x^2}{1+x} & 1 \end{vmatrix} = 0. \quad ((C)) \qquad (2\text{–}19)$$

Following (2–16) the equation

$$y = \frac{1}{2\pi S^2} e\{-r/2S^2\} \quad \text{(exponential)}$$

can be put into canonical form ((C)) as follows:

$$\begin{vmatrix} 0 & y' & 1 \\ 1 & r^2 & 1 \\ \frac{1}{1+2S^2} & \frac{-4S^2 \ln S}{1+2S^2} & 1 \end{vmatrix} = 0. \quad ((C)) \qquad (2\text{–}20)$$

2–5. *Bringing New Quantities into the Determinant.* For the equation $U + V = W$, the substitution

$$A = uU$$

$$B = vV \qquad (2\text{–}21)$$

yields three equations:

$$A \cdot 1 + B \cdot 0 - uU = 0$$

$$A \cdot 0 + B \cdot 1 - vV = 0$$

$$A \cdot \frac{1}{u} + B \cdot \frac{1}{v} - W = 0$$

A canonical form ((C)) is now found into which we shall have brought several new quantities which can serve as adjustment factors:

$$0 = \begin{vmatrix} 1 & 0 & -uU \\ 0 & 1 & -vV \\ \frac{1}{u} & \frac{1}{v} & -W \end{vmatrix}$$

(a) multiply column III by (−1)
(b) add column II to column I
(c) then shift column I to column III

$$= \begin{vmatrix} 0 & uU & 1 \\ 1 & vV & 1 \\ \frac{1}{v} & W & \frac{u+v}{uv} \end{vmatrix}$$

(a) divide row III by $\frac{u+v}{u \cdot v}$

(b) multiply column I by G

$$= \begin{vmatrix} 0 & uU & 1 \\ G & vV & 1 \\ \frac{Gu}{u+v} & \frac{uvW}{u+v} & 1 \end{vmatrix} \qquad (2\text{–}22)$$

(2–22) will be found on expansion to yield the original equation U + V = W. It should then be compared with form ((C)) of (2–17) for the same equation. The new quantities give the interpretation of ((C)) much greater flexibility. G permits varying the width of the diagram. u and v are scale multipliers or scale factors and can be used to expand or contract their respective scales.

Note: Passing from form ((A)) to form ((B)) is usually simpler if no scale factors are present. It is better to put thru a "pilot plant" procedure for ((B)) with no scale factors present and then, if successful, to see if the change can still be made when scale factors are brought in.

2–6. *Finding Scale Factors for a Particular Chart. Breadth of the Chart.* The nomographer is usually interested in a certain range of each variable of a formula and in a nomogram of a certain physical size, usually rectangular in shape. This permits use of (1–18) to determine such quantities as u and v, as discussed in Section 1–6. Quantities like G controlling breadth can often be worked into the determinant toward the end whenever desirable.

Example 2–4. A pilot plant for Example 2–2 has already been put thru without scale factors. (2–9), (2–19). Scale factors are now desired, because variables A and B are both to range from −100 to 100 and the chart is to be 15 inches wide and 20 inches high. Following the pattern of Example 2–2, for the equation $x^2 + Ax + B = 0$

$$\text{Let } P = aA$$

$$Q = bB$$

$$P \cdot 1 + Q \cdot 0 - aA = 0$$

$$P \cdot 0 + Q \cdot 1 - bB = 0$$

$$P\frac{x}{a} + Q\frac{1}{b} + x^2 = 0$$

$$0 = \begin{vmatrix} 1 & 0 & -aA \\ 0 & 1 & -bB \\ x/a & 1/b & x^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & aA \\ 1 & 1 & bB \\ \frac{x}{a}+\frac{1}{b} & \frac{1}{b} & -x^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & aA \\ 1 & 1 & bB \\ 1 & \frac{a}{bx+a} & \frac{-abx^2}{bx+a} \end{vmatrix} .$$

Using the material from Section 1–6, and the equation (1–18)

$$20 = a(100-(-100))$$

$$a = 1/10$$

$$20 = b(100-(-100))$$

$$b = 1/10$$

$$0 = \begin{vmatrix} 0 & A/10 & 1 \\ 15 & B/10 & 1 \\ \frac{15}{x+1} & \frac{-x^2}{10(x+1)} & 1 \end{vmatrix}$$

PROBLEMS

Each of the following problems appeared in Chapter 1 under the same number.

1) Place each equation in canonical form ((C)) such that each term X_1, Y_1, X_2, Y_2, X_3, Y_3, of ((C))

$$\begin{vmatrix} X_1 & Y_1 & 1 \\ X_2 & Y_2 & 1 \\ X_3 & Y_3 & 1 \end{vmatrix} = 0$$

is the same as shown in the problems of corresponding number in Chapter 1.

2) By comparing with the problems in Chapter 1, read off the values of the scale factors and constants.

3) Using the scale lengths and ranges of the variables in the completed diagrams referred to in Chapter 1 for each problem, show that the values of scale factors agree with those of Chapter 1.

For example: In Problem 2–1 place the equation $U + V = W$ in the canonical form of Problem 1, Chapter 1:

$$\begin{vmatrix} 0 & uU & 1 \\ G & vV & 1 \\ \dfrac{uG}{u+v} & \dfrac{uvW}{u+v} & 1 \end{vmatrix} = 0$$

and by further comparison with that problem conclude that $G = 15$, $u = 10$, $v = 20$. On looking at completed Figure 1–8, it will be clear that these values of G, u and v *were* the ones used there since they make each variable's range fit its scale length and put it in proper position.

PROBLEM 2–1. $U + V = W$

PROBLEM 2–2. $x^2 + Ax + B = 0$

PROBLEM 2–3. $UV(U + V) = W(U + V)$

PROBLEM 2–4. $x^3 + Ax^2 + Bx + -0.4 = 0$

PROBLEM 2–5. $V = 52.8\left(\frac{1}{A}(1 - B/3V)\right)^{1/3}$

PROBLEM 2–6. $\frac{\sqrt{H}}{4} \cdot S - \frac{+H}{11.19} = W$

PROBLEM 2–7. $L = \frac{A^2 + B^2}{A + B}$

PROBLEM 2–8. $y = \frac{1}{2\pi S^2} \cdot e\left\{\frac{-r^2}{2S^2}\right\}$ (exponential)

PROBLEM 2–9. $W + V^2U + WU^2V^2 - U = 0$

PROBLEM 2–10. $a \cdot x + b/x + 1 = 0$

PROBLEM 2–11. $M = \frac{y_2 - y_1}{\ln y_2 - \ln y_1}$

CHAPTER 3

THE CANONICAL FORM FOR MORE THAN THREE VARIABLES

3–1. *Disjoining an Equation in Four Variables. Canonical Form ((C)).* Assume that it is possible to put an equation in four variables:

$$F(U, V, W, T) = 0 \tag{3–1}$$

in the form

$$\begin{vmatrix} U_1 & U_2 & 1 \\ V_1 & V_2 & 1 \\ (W,T)_1 & (W,T)_2 & 1 \end{vmatrix} = 0 \tag{3–2}$$

U_1, U_2 are functions of U only

V_1, V_2 are functions of V only

$(W,T)_1$, $(W,T)_2$ are functions of W and T only.

This would be done in the same way as described earlier for three variables. The equation in four variables is then said to be disjoined and in canonical form ((C)).

3–2. *Interpretation of a Disjoined Equation in Four Variables.* One writes the following equations, planning to interpret them as in Chapter 1 for three variables:

$$\left.\begin{array}{ll} X_1 = U_1; & Y_1 = U_2 \\ \multicolumn{2}{l}{\text{A parametric equation in U, yielding a curve graduated in U.}} \\ X_2 = V_1; & Y_2 = V_2 \\ \multicolumn{2}{l}{\text{A parametric equation in V, yielding a curve graduated in V.}} \\ X_3 = (W,T)_1; & Y_3 = (W,T)_2 \end{array}\right\} \tag{3–3}$$

To interpret X_3, Y_3, let T have the constant value $T = T_g$, then X_3 and Y_3 are functions of W only,

$$X_3 = (W,T_g)_1; \qquad Y_3 = (W,T_g)_2 \tag{3–4}$$

Figure 3–1 shows the three curves for U, V and W when $T = T_g$. Now repeat the process using a value of T differing just a little from T_g, namely T_h. Then

$$X_3 = (W,T_h)_1; \qquad Y_3 = (W,T_h)_2 \tag{3–5}$$

and again X_3 and Y_3 are functions only of W. The U, V and W curves when $T = T_h$ are shown on the same Figure 3–1, the U and V curves being the same as before. The scale stems bearing W values have been labelled for T_g and T_h. The new W graduations will not be far from the old ones because T_h differed very little from T_g. This process can be carried out many times, and with each new scale, each new W graduation moves a little way across the paper, tracing out a W curve with that value. Imagine many such values of T. Then all the T curves and all the W curves form a two-variable net, as shown in Figure 3–2. Actually the roles of T and W could have been interchanged. On holding W constant at value $W = W_k$, one would have X_3, Y_3 functions of T alone. The U and V curves and graduations would be the same as above; the third curve, graduated in T and labelled W_k, would turn out to be identical with the curve of the first net labelled W_k.

Such a chart consisting of U and V scales and a two-variable net of W and T curves aligns three points at values of U, of V, and the intersection of a W curve and a T curve such that these values of U, V, W and T satisfy the equations (3–1) (3–2). It is a common, effective form of chart.

Suggestion: As a first step toward placing a multiple variable chart in disjoined form ((C)), if there are any variables which appear only once in that equation, the substitutions should preferably be made for those variables.

If two variables play symmetric roles in an equation, the substitutions should be tried for either those variables or others.

The first of these rules is used in Examples 3–1, 3–2; the second in Example 3–3.

Example 3–1. Figure 3–3 represents the equation

$$X^3 + AX^2 + BX + C = 0 \tag{3–6}$$

nomographically. A, B, C range from −100 to +100, X is dependent. The available chart space is 15 inches wide and 20 inches high.

$$\left.\begin{array}{r} \text{Let } \phi = aA \\ \theta = bB \end{array}\right\} \tag{3–7}$$

$$\phi \cdot 1 + \theta \cdot 0 - aA = 0 \tag{3–8}$$

$$\phi \cdot 0 + \theta \cdot 1 - bB = 0$$

$$\phi \cdot \frac{X^2}{a} + \theta \cdot \frac{X}{b} + X^3 + C = 0 \qquad (3\text{–}8)$$

These are three linear, compatible equations in ϕ and θ and hence the determinant of these coefficients vanishes.

$$\begin{vmatrix} 1 & 0 & -aA \\ 0 & 1 & -bB \\ \frac{X^2}{a} & \frac{X}{b} & X^3 + C \end{vmatrix} = 0 \quad ((B)) \qquad (3\text{–}9)$$

The disjunction of variables has already occurred here since the first row contains only functions of A, the second only functions of B, the third only functions of X and C.

One now proceeds toward a canonical form ((C)) for four variables as follows:

$$\begin{vmatrix} 1 & 0 & aA \\ 1 & 1 & bB \\ \left(\frac{X^2}{a} + \frac{X}{b}\right) & \frac{X}{b} & -(X^3 + C) \end{vmatrix} = 0 \qquad (3\text{–}10)$$

$$\begin{vmatrix} 1 & 0 & aB \\ 1 & 1 & bB \\ 1 & \frac{X}{b}\frac{ab}{bX^2 + aX} & \frac{-ab(X^3 + C)}{bX^2 + aX} \end{vmatrix} = 0, \text{ form } ((C)) \qquad (3\text{–}11)$$

$$\begin{vmatrix} 0 & aA & 1 \\ G & bB & 1 \\ \frac{GX}{X + \frac{b}{a}X^2} & \frac{-b(X^3 + C)}{X + \frac{b}{a}X^2} & 1 \end{vmatrix} = 0, \text{ form } ((C)) \qquad (3\text{–}12)$$

Applying (1–18) to evaluate scale factors, one has

$a(100-(-100)) = 20$: $\quad a = 1/10$

$b(100-(-100)) = 20$; $\quad b = 1/10$

and making the width 15,

$$G = 15$$

$$\begin{vmatrix} 0 & A/10 & 1 \\ 15 & B/10 & 1 \\ \frac{15X}{X + X^2} & \frac{-(X^3 + C)}{10(X + X^2)} & 1 \end{vmatrix} \text{0, final form ((C))} \qquad (3\text{–}13)$$

Vertical, uniform scales in A and B, 15 inches apart, are the result. Variables X and C yield an X, C net, which at first glance poses formidable plotting problems, but a moment's inspection provides an easy way out:

In the equations

$$X_3 = \frac{15X}{X + X^2}\,; \quad Y_3 = \frac{-(X^3 + C)}{10(X + X^2)}. \qquad (3\text{–}14)$$

One observes:

1) when X is kept constant and C varies, only Y_3 is affected. Hence, "X = constant" curves are vertical lines on this net.

2) when X is kept constant, Y_3 varies uniformly with C. Hence, on a vertical line for given X, location of one high and one low value of C permits rapid determination, by *uniform subdivision* of the interval, for those values of C lying in between. This does not imply that the C curves are straight lines—they are wavy.

Example 3–2. The substitution (3–7) above was a natural start toward ending up the uniform scales in A and B on the sides of the diagram and this turned out to be the case. It is sometimes possible to know in advance which variables it would be good to have in these positions. Such a case would arise, as shown later, when it is clear that their diagram will have to be joined to another one by means of such a scale along the side. The diagrams resulting from starting with variables A and C, and B and C now follow, on the left and right down the page. Figures 3–4 and 3–5.

Let $\phi = aA$

$\theta = cC$

$$\left.\begin{aligned} \phi \cdot 1 + \theta \cdot 0 - aA &= 0 \\ \phi \cdot 0 + \theta \cdot 1 - cC &= 0 \\ \phi \frac{x^2}{a} + \theta \frac{1}{c} + x^3 + Bx &= 0 \end{aligned}\right\} \qquad (3\text{–}15)$$

Let $\phi = cC$

$\theta = bB$

$$\left.\begin{aligned} \phi \cdot 1 + \theta \cdot 0 - cC &= 0 \\ \phi \cdot 0 + \theta \cdot 0 - bB &= 0 \\ \phi \cdot \frac{1}{c} + \theta \cdot \frac{x}{b} + x^3 + Ax^2 &= 0 \end{aligned}\right\} \qquad (3\text{–}16)$$

$$\begin{vmatrix} 1 & 0 & -aA \\ 0 & 1 & -cC \\ \frac{x^2}{a} & \frac{1}{c} & x^3 + Bx \end{vmatrix} = 0 \qquad (3\text{–}17) \qquad \begin{vmatrix} 1 & 0 & -cC \\ 0 & 1 & -bB \\ \frac{1}{c} & \frac{x}{b} & x^3 + Ax^2 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 0 & aA \\ 1 & 1 & cC \\ \frac{cx^2 + a}{ac} & \frac{1}{c} & -(x^3 + Bx) \end{vmatrix} = 0 \qquad (3\text{–}18) \qquad \begin{vmatrix} 1 & 0 & cC \\ 1 & 1 & bB \\ \frac{b + cx}{bc} & \frac{x}{b} & -(x^3 + Ax^2) \end{vmatrix} = 0$$

$$\begin{vmatrix} 0 & aA & 1 \\ G & cC & 1 \\ \frac{G}{\frac{cx^2}{a} + 1} & \frac{-c(x^3 + 3x)}{\frac{cx^2}{a} + 1} & 1 \end{vmatrix} = 0 \qquad (3\text{–}19) \qquad \begin{vmatrix} 0 & cC & 1 \\ G & bB & 1 \\ \frac{Gx}{x + \frac{b}{c}} & \frac{-b(x^3 + Ax^2)}{x + \frac{b}{c}} & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 0 & A/10 & 1 \\ 15 & C/10 & 1 \\ \frac{15}{x^2 + 1} & \frac{-(x^3 + Bx)}{10(x^2 + 1)} & 1 \end{vmatrix} = 0 \qquad (3\text{–}20) \qquad \begin{vmatrix} 0 & C/10 & 1 \\ 15 & B/10 & 1 \\ \frac{15x}{x + 1} & \frac{-(x^3 + Ax^2)}{10(x + 1)} & 1 \end{vmatrix} = 0$$

3–3. *Scope of the General Case.* The fullest use of the method just developed would be given by the form

$$\begin{vmatrix} (R, S)_1 & (R, S)_2 & 1 \\ (T, U)_1 & (T, U)_2 & 1 \\ (V, W)_1 & (V, W)_2 & 1 \end{vmatrix} = 0 \qquad \text{where } (R, S)_1 \text{ is a function of R and S only, etc.} \qquad (3\text{–}21)$$

which would have come from an equation in six variables,

$$F(R, S, T, U, V, W) = 0 \qquad (3\text{–}22)$$

and would be interpreted as three nets, namely those of R and S, T and U, V and W. Equations in six variables adaptable to this form are not common. One classical case often referred to is that of the stress in the walls of a thick hollow cylinder with closed ends under internal pressure, external pressure or both. It has a twin equation where the cylinder has open ends. These formulas are called Clavarino's and Birnie's respectively, and they state:

$$p = \frac{r_1^2 \cdot w_1 - r_2^2 \cdot w_2 + 4r_1^2 r_2^2 (w_1 - w_2)/r^2}{3(r_2^2 - r_1^2)} \qquad (3\text{–}23)$$

$$p = \frac{2r_1^2 \cdot w_1 - 2r_2^2 \cdot w_2 + 4r_1^2 r_2^2 (w_1 - w_2)/r^2}{3(r_2^2 - r_1^2)} \qquad (3\text{–}24)$$

Example 3–3. Figure 3–6. Prepare alignment diagrams for Clavarino's and Birnie's equations. In Clavarino's equation, (3–23) p enters simply and w_1 and w_2 next simply. Because of the somewhat symmetric roles of the latter, one might try the substitution

$$\text{let } A = w_1$$

$$\text{let } B = w_2 \qquad (3\text{–}25)$$

$$A \cdot 1 \quad + \quad B \cdot 0 \quad - \quad w_1 \quad = 0$$

$$A \cdot 0 \quad + \quad B \cdot 1 \quad - \quad w_2 \quad = 0$$

$$A\left(r_1^2 + \frac{4r_1^2 r_2^2}{r^2}\right) + B\left(-r_2^2 - \frac{4r_1^2 r_2^2}{r^2}\right) + 3p(r_1^2 - r_2^2) = 0 \tag{3–26}$$

$$\begin{vmatrix} 1 & 0 & -w_1 \\ 0 & 1 & -w_2 \\ r_1^2 + \frac{4r_1^2 r_2^2}{r^2} & -r_2^2 - \frac{4r_1^2 r_2^2}{r^2} & 3p(r_1^2 - r_2^2) \end{vmatrix} = 0 \tag{3–27}$$

(a) add column II to column I
(b) multiply column III by (−1).

$$\begin{vmatrix} 1 & 0 & w_1 \\ 1 & 1 & w_2 \\ r_1^2 - r_2^2 & -r_2^2 - \frac{4r_1^2 r_2^2}{r^2} & -3p(r_1^2 - r_2^2) \end{vmatrix} = 0 \tag{3–28}$$

(a) divide row III by $r_1^2 - r_2^2$.
(b) *then* multiply column II by $r_1^2 - r_2^2$.

$$= \begin{vmatrix} 1 & 0 & w_1 \\ 1 & r_1^2 - r_2^2 & w_2 \\ 1 & -r_2^2 - \frac{4r_1^2 r_2^2}{r^2} & -3p \end{vmatrix} \tag{3–29}$$

Multiply column I by r_2^2 and add to column II

$$= \begin{vmatrix} 1 & r_2^2 & w_1 \\ 1 & r_1^2 & w_2 \\ 1 & -\frac{4r_1^2 r_2^2}{r^2} & -3p \end{vmatrix} \tag{3–30}$$

(a) Interchange column I and column III
(b) Multiply column II by $10/r_1^2 r_2^2$

$$\begin{vmatrix} w_1 & 1/r_1^2 \cdot 10 & 1 \\ w_2 & 1/r_2^2 \cdot 10 & 1 \\ -3p & -4/r^2 \cdot 10 & 1 \end{vmatrix} = 0 \tag{3–31}$$

The twin determinant, for equation (3–24), reads by analogy

$$\begin{vmatrix} 2w_1 & 2/r_1^2 \cdot 10 & 1 \\ 2w_2 & 2/r_2^2 \cdot 10 & 1 \\ -3p & -4/r^2 \cdot 10 & 1 \end{vmatrix} = 0 \tag{3–32}$$

Figure (3–6) shows how both of these diagrams can be condensed onto a single diagram. To avoid having the nets overlap, the x-coordinate is reversed for the second problem, thereby reversing the direction of reading p. A y-coordinate scale multiplier of 10 has been introduced in the last form of each determinant. The x-coordinate scales have been kept natural size.

Equations (3–23), (3–24) are also good illustrations of "homogeneous equations," for if p, r, r_1, r_2, w_1, w_2 are solution values, then kp, kr, kr_1, kr_2, kw_1, kw_2 are also a set of solution values. Here k can take on any value but is the same for a particular use. This fortunate fact implies that a diagram made for a limited range of the variables will be usable for a greater range. A favorite value for k is 10, but others such as 2 or 3 are also sometimes practical. Equations which are not completely homogeneous may be "semi-homogeneous" as in Problem 1–8. Equations in five variables, interpretable as two nets and a scale, are met reasonably often, while those with a single net, like Figures 3–3, –4, and –5 are quite common.

3–4. *Collapsed Net in a Diagram. Remedies.* Occasionally an equation in more than three variables turns up in a canonical form, where an expected net has "collapsed" onto an axis or some other line or curve, as where F(U, V, R, S) = 0 takes on the form:

$$\begin{vmatrix} U_1 & U_2 & 1 \\ V_1 & V_2 & 1 \\ (R, S)_1 & 0 & 1 \end{vmatrix} = 0 \qquad (3\text{–}33)$$

Here, what should be an R, S net lies only along the X-axis. Not being spread out, a pair of values, R, S do not fix a point through which an alignment can be drawn. On the other hand, one has the relationship

$$X_3 = (R, S)_1 \qquad (3\text{–}34)$$

which is a three-variable equation for which an alignment diagram can perhaps be made. If this diagram can be arranged to have a linear scale for the variable X_3, this scale can be placed in coincidence with the X-axis. A collineation through a value of U and of V, and one through a value of R and of S will then cross the X-axis at the same point, X3, if U, V, R and S satisfy the equation F(U, V, R, S) = 0. If values of three of these variables are known, that of the fourth can be found in this way. (Problems 3–3, 3–4.)

PROBLEMS

PROBLEM 3–1. Given the equation

$$P = \frac{b \cdot h}{H - h},$$

use the substitution

$$A = b,$$
$$B = H$$

to derive the canonical form

$$\begin{vmatrix} 0 & b & 1 \\ G & H & 1 \\ \frac{GP}{P-h} & \frac{Ph}{P-h} & 1 \end{vmatrix} = 0. \qquad (3\text{–}35)$$

In the bottom row of this determinant, eliminate first P (leaving h constant) and then h (leaving P constant) to show that both families of the net are straight lines. Make a quick sketch of the diagram and show that it works.

PROBLEM 3–2. Given the equation

$$P = \frac{b \cdot h}{H - h}$$

use the substitution

$$A = P$$
$$B = b$$

to derive the canonical form

$$\begin{vmatrix} 0 & P & 1 \\ H & +b & 1 \\ h & 0 & 1 \end{vmatrix} = 0. \qquad (3\text{–}36)$$

Make a quick sketch of the diagram and show that it works.

PROBLEM 3–3. Given the same equation

$$P = \frac{b \cdot h}{H - h}$$

use the substitution

$$A = P$$
$$B = b$$

to derive the canonical form

$$\begin{vmatrix} 0 & P & 1 \\ G & b & 1 \\ \frac{Gh}{2h-H} & 0 & 1 \end{vmatrix} = 0 \qquad (3\text{–}37)$$

Here the h, H net has collapsed onto the X-axis. However, the expression

$$X_3 = \frac{Gh}{2h - H} \qquad (3\text{–}38)$$

can be placed in the canonical form below. Derive the form

$$\begin{vmatrix} X_3 & 0 & 1 \\ \frac{G}{2} & \frac{H}{2} & 1 \\ 0 & h & 1 \end{vmatrix} = 0 \qquad (3\text{–}39)$$

Here X_3 is interpreted as an x-coordinate. Hence, a diagram with a collapsed net can be replaced by two diagrams (a compound diagram) using two alignments which cross the X-axis at the value X_3. Make a quick sketch of the compound diagram and show that it works.

PROBLEM 3–4. Given the same equation

$$P = \frac{b \cdot h}{H - h}$$

use the substitution

$$A = h$$

$$B = H$$

to derive the canonical form

$$\begin{vmatrix} 0 & h & 1 \\ G & H & 1 \\ \frac{-GP}{b} & 0 & 1 \end{vmatrix} = 0. \qquad (3\text{–}40)$$

Here the P, b net has collapsed onto the X-axis. However, the expression

$$X_3 = \frac{-GP}{b}$$

can be thought of as ordinary multiplication

$$b \cdot X_3 = -GP.$$

The collapsed net diagram of (3–40) can be replaced by a compound diagram using two collineations, (one for each of its component charts) which cross the X-axis at the value $X_3 = -GP/b$. Make a quick sketch of the diagram and show that it works.

PROBLEM 3–5. Given the same equation

$$P = \frac{b \cdot h}{H - h}$$

derive a canonical form for it different from any of the above four forms. Use any substitution you wish. The substitutions 1) $A = P, B = H$; 2) $A = P, B = h$; 3) $A = b, B = h$, have not been used above up to now.

PROBLEM 3–6. The equation from textiles

$$1.03\,N - C \cdot I + 0.911S(C - 9.43) = 0$$

can be placed in a variety of canonical forms. Derive the three different canonical forms for this equation needed to yield three charts (two of which are shown complete in Figure 3–7). Introduce scale multipliers in each case and evaluate them to yield these charts. Expand each canonical form to prove that it does represent the original equation. In each case, derive the equations of any families of straight lines in the net.

PROBLEM 3–7. Figure 3–8. The law of cosines can be placed in desirable nomographic form:

$$A^2 = B^2 + C^2 - 2BC \cdot \cos\alpha \qquad (3\text{–}41)$$

$$\begin{vmatrix} 15 & A^2/20 & 1 \\ 0 & 10 + 10\cos\alpha & 1 \\ \frac{150}{10 + BC/10} & \frac{[1/2](B^2 + C^2) + BC}{10 + BC/10} & 1 \end{vmatrix} = 0 \qquad (3\text{–}42)$$

Derive (3–42) from (3–41). It will be necessary to introduce scale factors and give them the values needed for this result. Show that this was the form used to obtain the completed diagram of Figure 3–8. *Note:* The equations for the net in the bottom row of (3–42) are symmetric in B and C, causing the B and C curves to coincide. This does not interfere with the use of the net unless B and C happen to have the same value. Even then the coordinates of a point for each such pair of equal values of B and C are definite, for such a point will lie at the point of tangency of the overlapping pair with the envelope of the family and have been marked there on the chart.

PROBLEM 3–8. Figure 3–9. The law for design factors for close wound coils can be written

$$L = \frac{0.2A^2N^2}{3A + 9DN} \qquad (3\text{–}43)$$

A chart for this diagram appears in Figure 3–9. A smaller chart for a limited portion appears to the upper left. Derive the canonical forms required to provide these diagrams. Evaluate all scale factors.

PROBLEM 3–9. In cycloidal cam theory, one meets the equation

$$2M \tan \mu = \tan (M\alpha) + (M - L)\, \alpha(\csc^2 M\alpha)$$

Bearing in mind the rule about substitutions quoted in Section 3–2, derive the canonical form

$$\begin{vmatrix} 1 & \tan \mu & 1 \\ 0 & L & 1 \\ \dfrac{2M}{2M + \alpha \cdot \csc^2 M\alpha} & \dfrac{\tan M\alpha + M\alpha \cdot \csc^2 M\alpha}{2M + \alpha \cdot \csc^2 M\alpha} & 1 \end{vmatrix} = 0. \qquad (3\text{–}44)$$

Plot enough points to check the diagram. See also Problems 4–4, 4–5.

PROBLEM 3–10. Figure 3–10. The solution of the second order secular equation can be represented graphically by the nomogram of Figure 3–10. Derive this diagram, especially the two forms shown there using respectively the vertical and inclined lettering.

This is an interesting chart. It is clear that the range of the chart can be changed by multiplying all the variables by the same factor. It is also clear that

$$\begin{vmatrix} A_1 - \lambda & R \\ R & A_2 - \lambda \end{vmatrix} = \begin{vmatrix} 10 - A_1 - (10 - \lambda) & -R \\ -R & 10 - A_2 - (10 - \lambda) \end{vmatrix} = 0 \qquad (3\text{–}45)$$

This justifies the vertical numerals if the inclined ones have already been derived, or conversely. The user should verify that either set of numerals will give both answers although one of the answers will lie each time in the congested corner of the net and be hard to read. The diagram based on the other set of values (vertical or inclined, respectively) gets around this difficulty and was derived for this reason.

PROBLEM 3–11. Figure 3–11. An alignment diagram for speed at sea level is given in Figure 3–11. Derive this figure and check several values.

PROBLEM 3–12. The design of reinforced concrete columns uses a formula for which a diagram has been made in Figure 3–12. Derive this diagram in the form shown there.

PROBLEM 3–13. The astronomical triangle formula appears in Figure 3–13. Derive the diagram shown there and check it with numerous measurements.

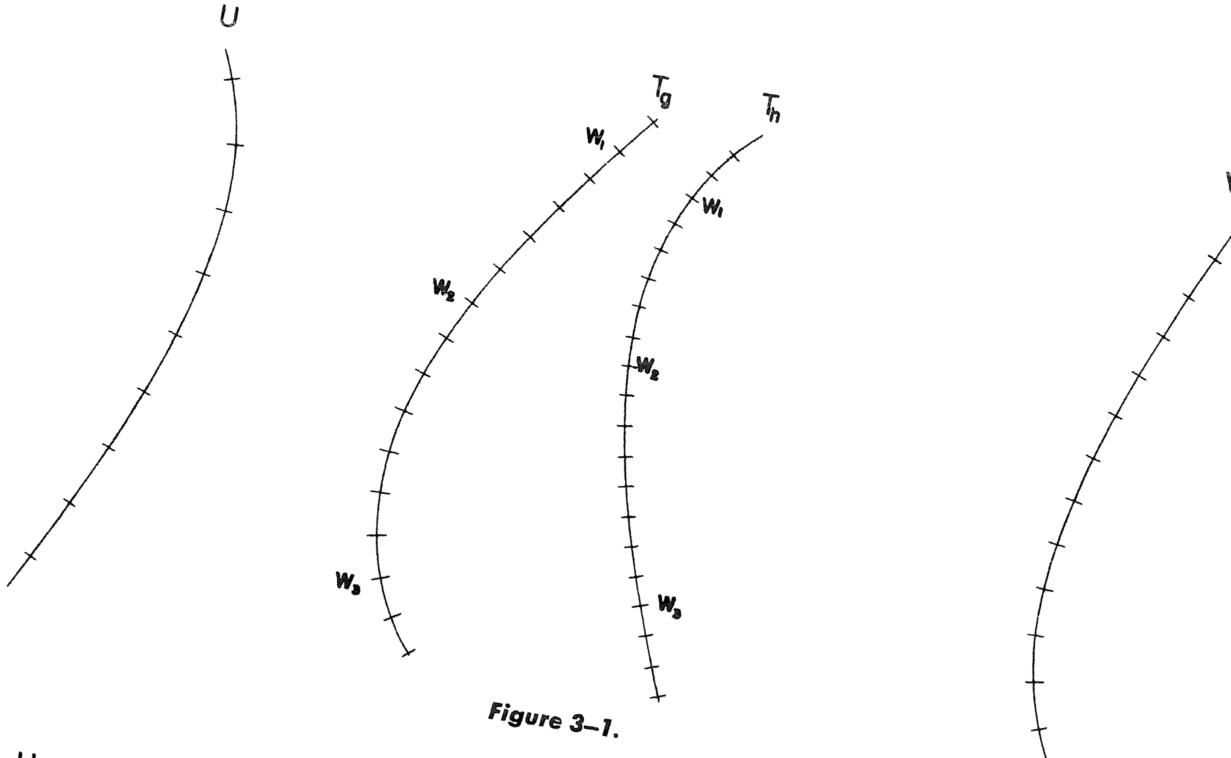


Figure 3–1.

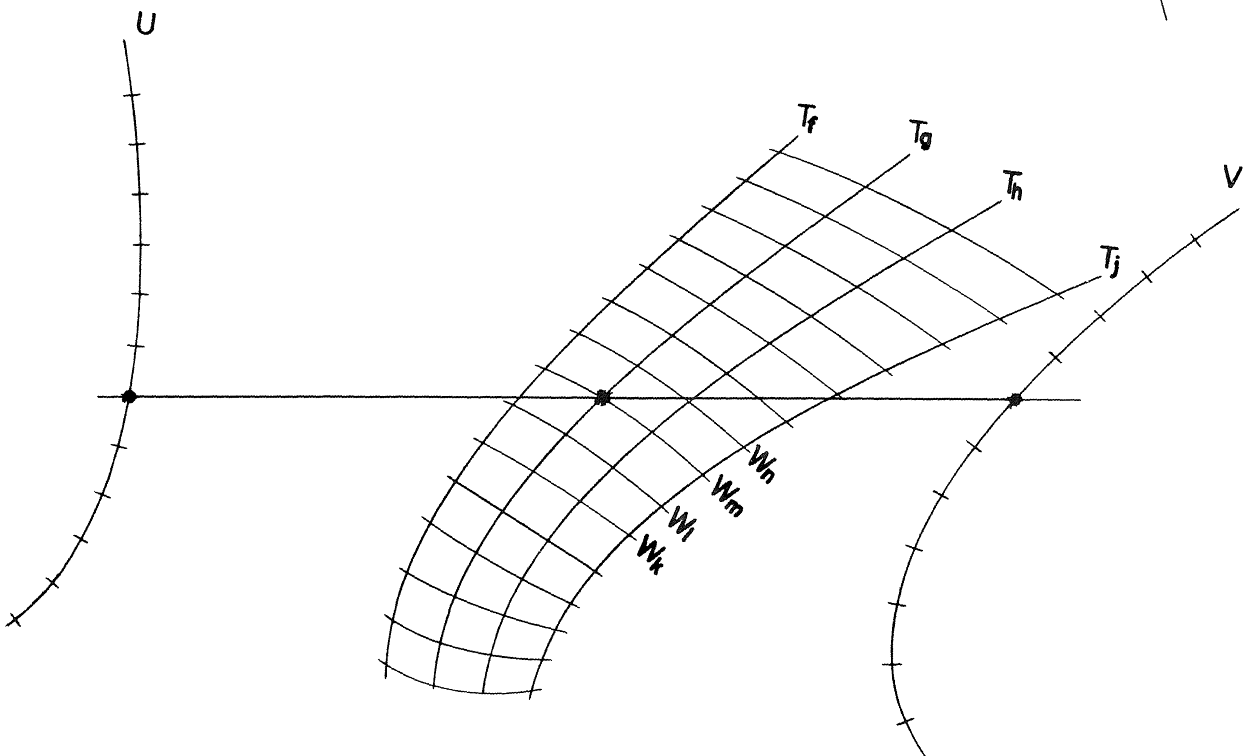
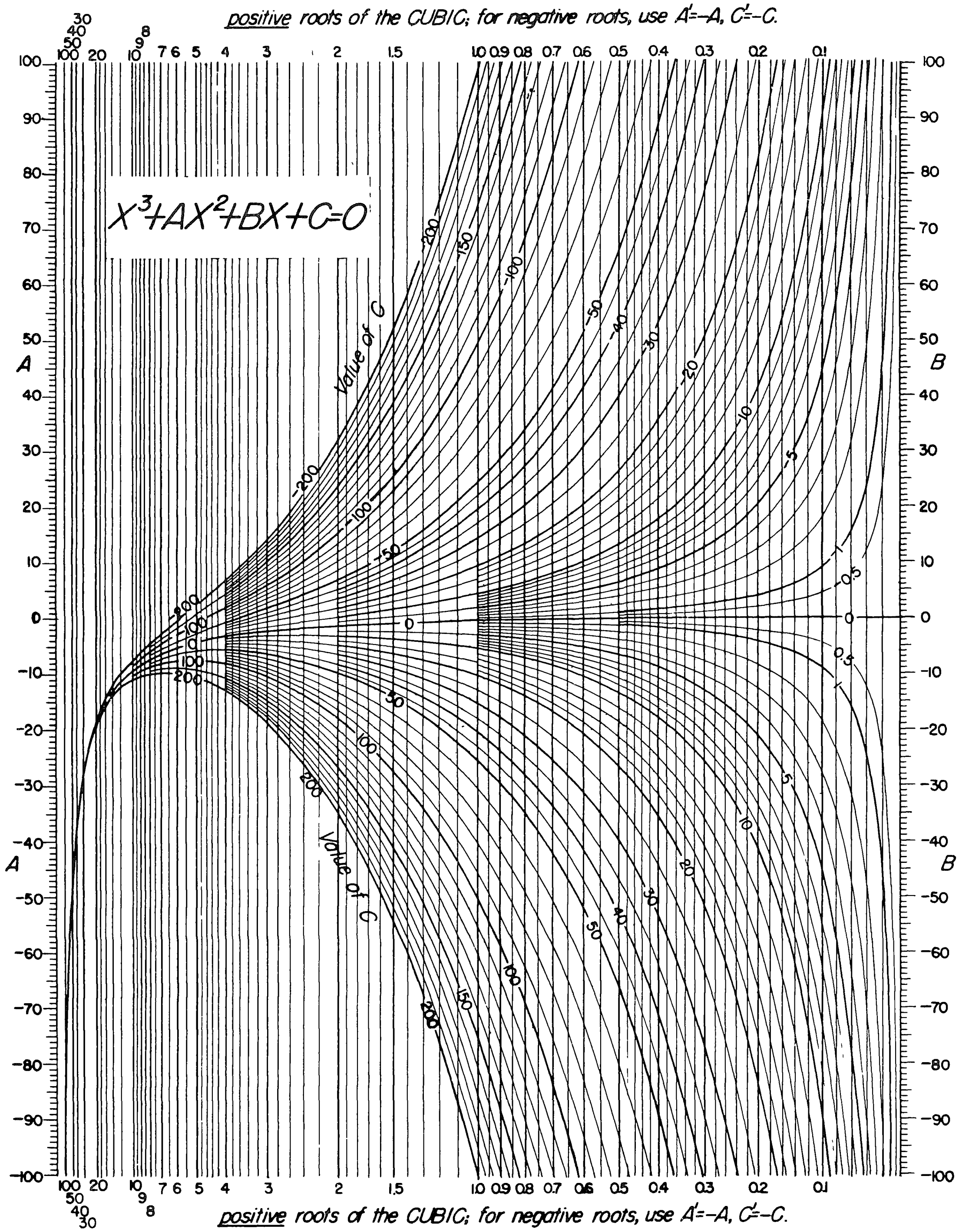


Figure 3–2.



THE ROOTS OF THE CUBIC EQUATION

Figure 3–3.

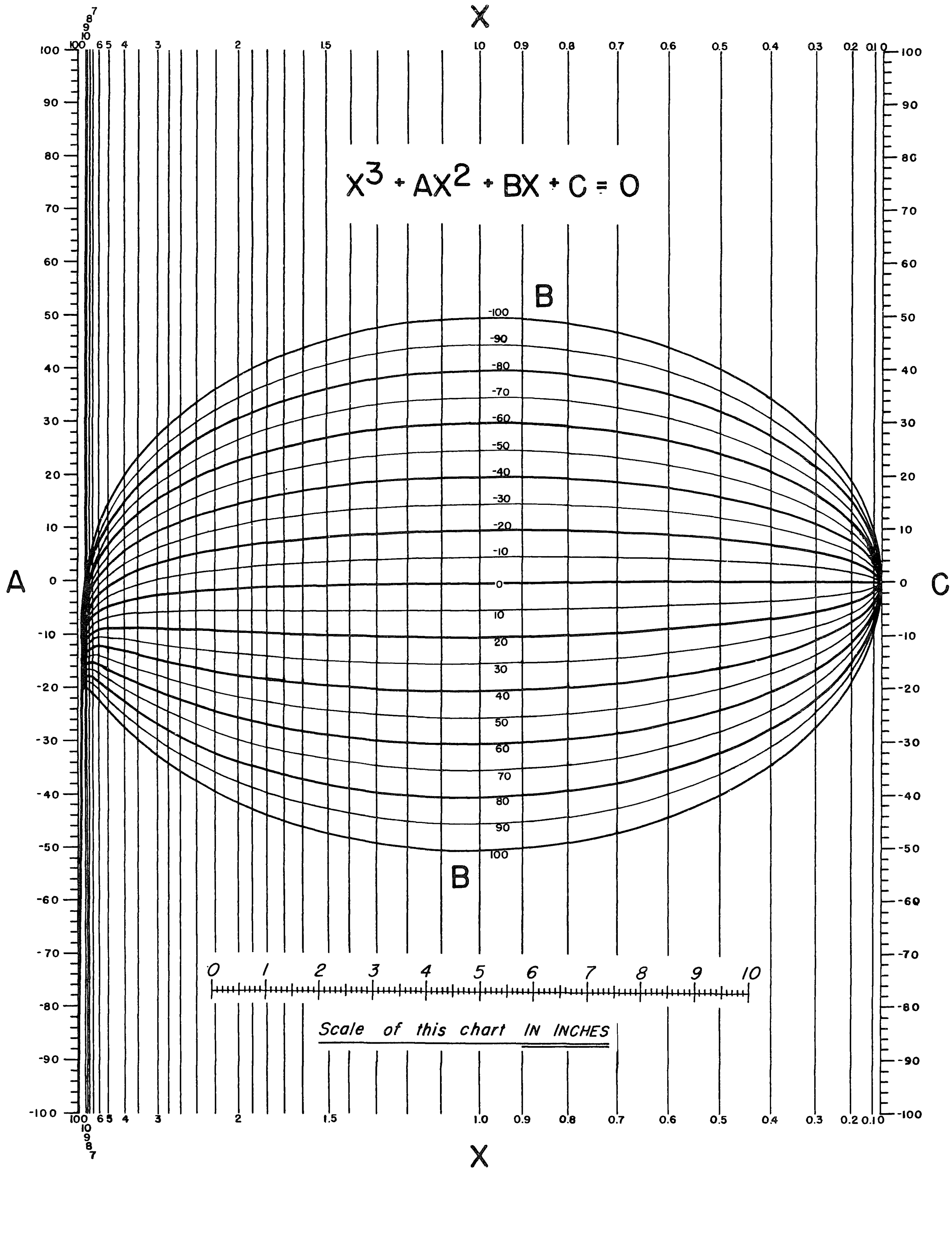


Figure 3–4.

$$X^3 + AX^2 + BX + C = 0$$

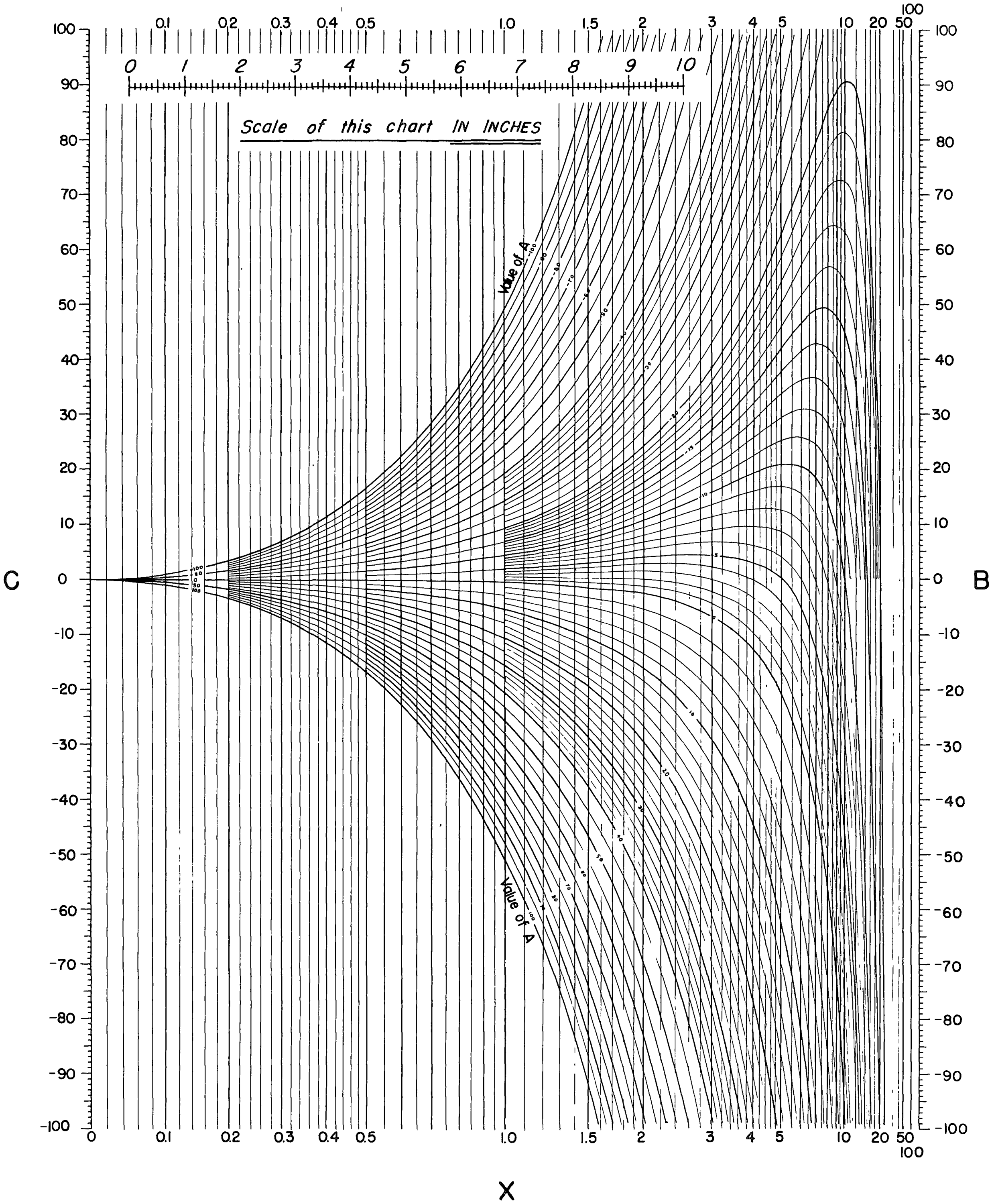
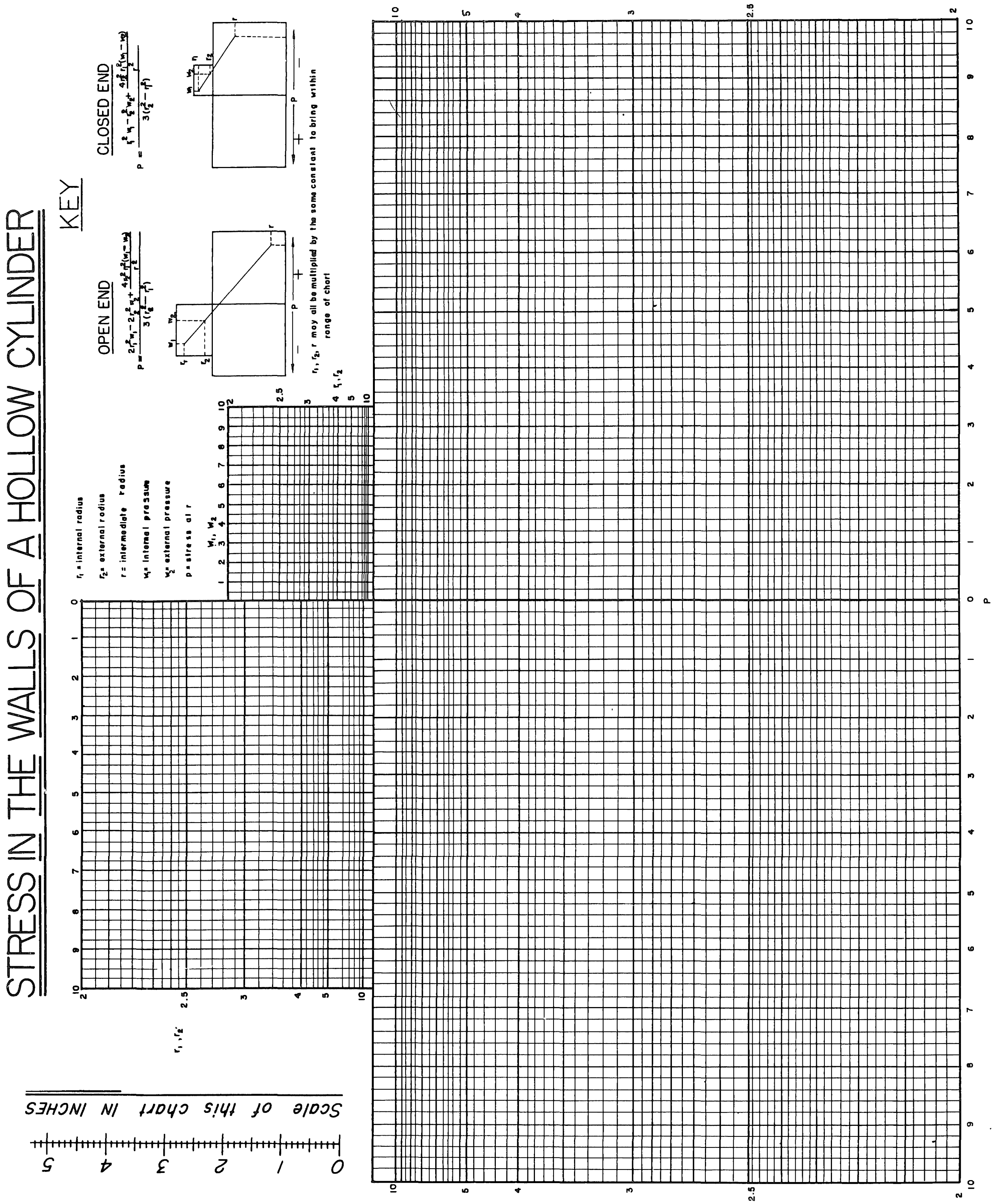
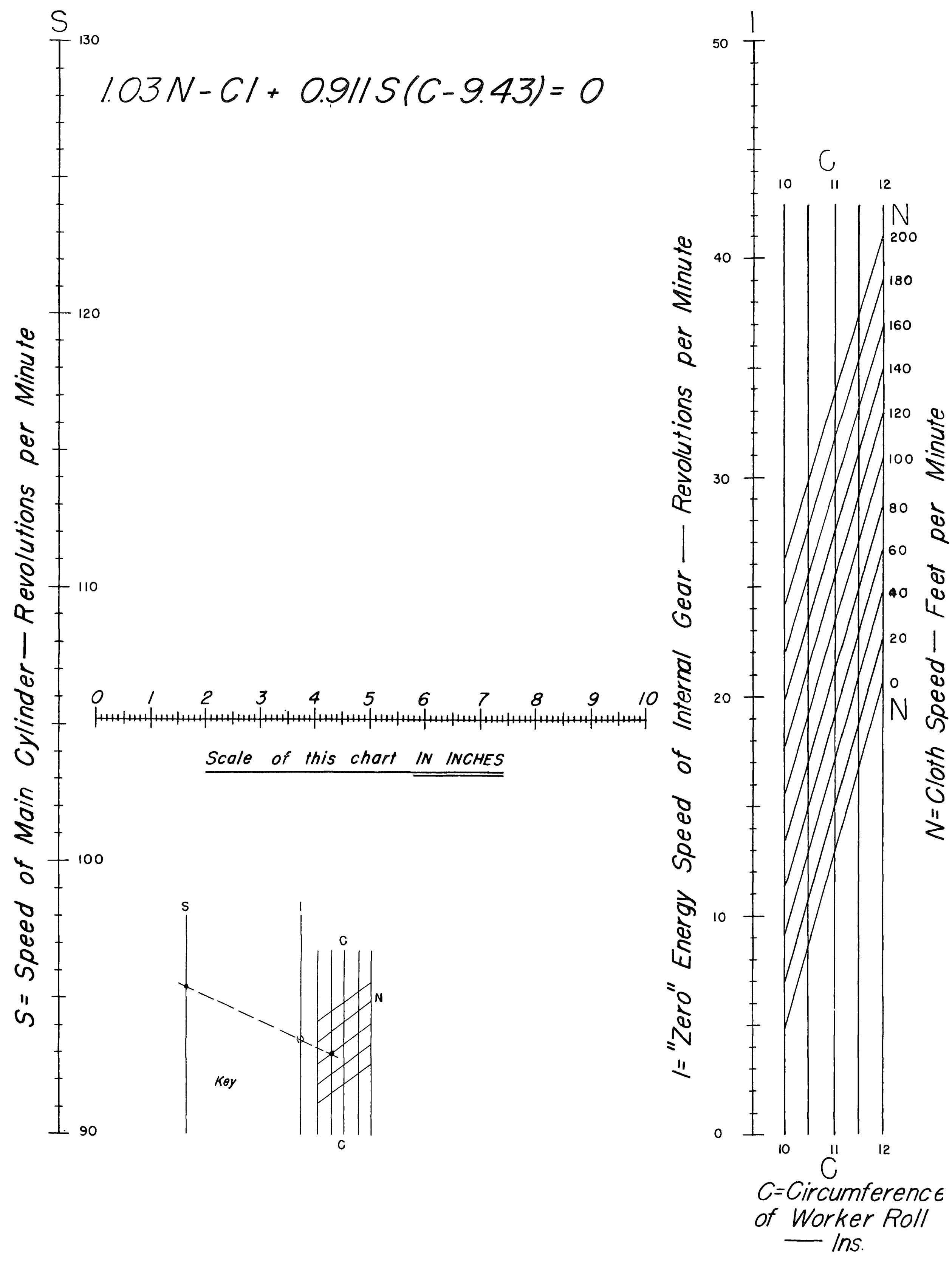


Figure 3–5.



STRESS IN THE WALLS OF A HOLLOW CYLINDER
KEY
OPEN END
CLOSED END
r1 = internal radius
r2 = external radius
r = intermediate radius
w1 = internal pressure
w2 = external pressure
p = stress at r
r1, r2, r may all be multiplied by the same constant to bring within range of chart
Scale of this chart IN INCHES

Figure 3–6.



1.03N - CI + 0.911S(C-9.43) = 0
S
130
120
110
100
90
S = Speed of Main Cylinder — Revolutions per Minute
0 1 2 3 4 5 6 7 8 9 10
Scale of this chart IN INCHES
S
I
C
N
Key
C
I
50
40
30
20
10
0
I = "Zero" Energy Speed of Internal Gear — Revolutions per Minute
C
10 11 12
N
200
180
160
140
120
100
80
60
40
20
0
N
N = Cloth Speed — Feet per Minute
10 11 12
C
C = Circumference of Worker Roll — Ins.

Figure 3–7a.

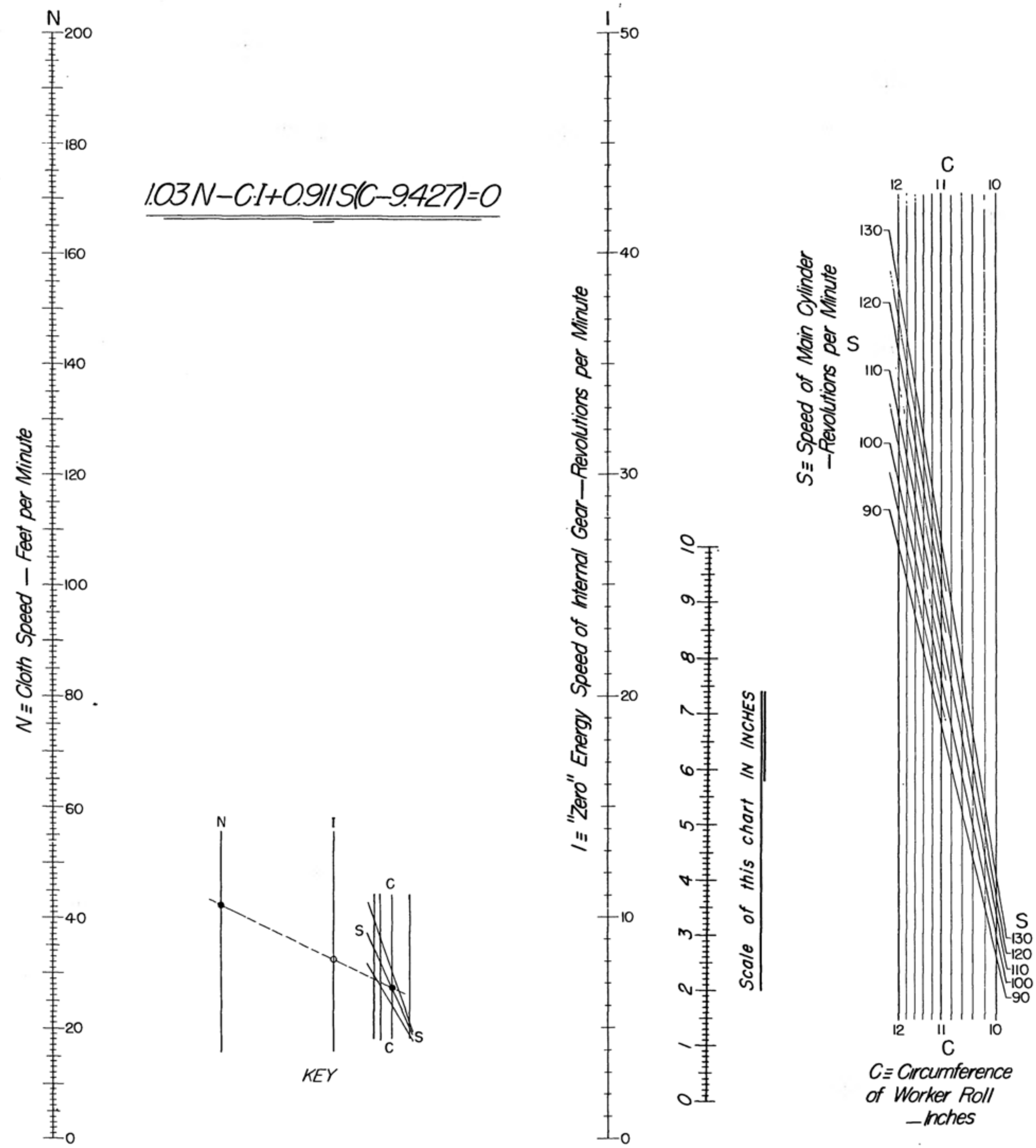
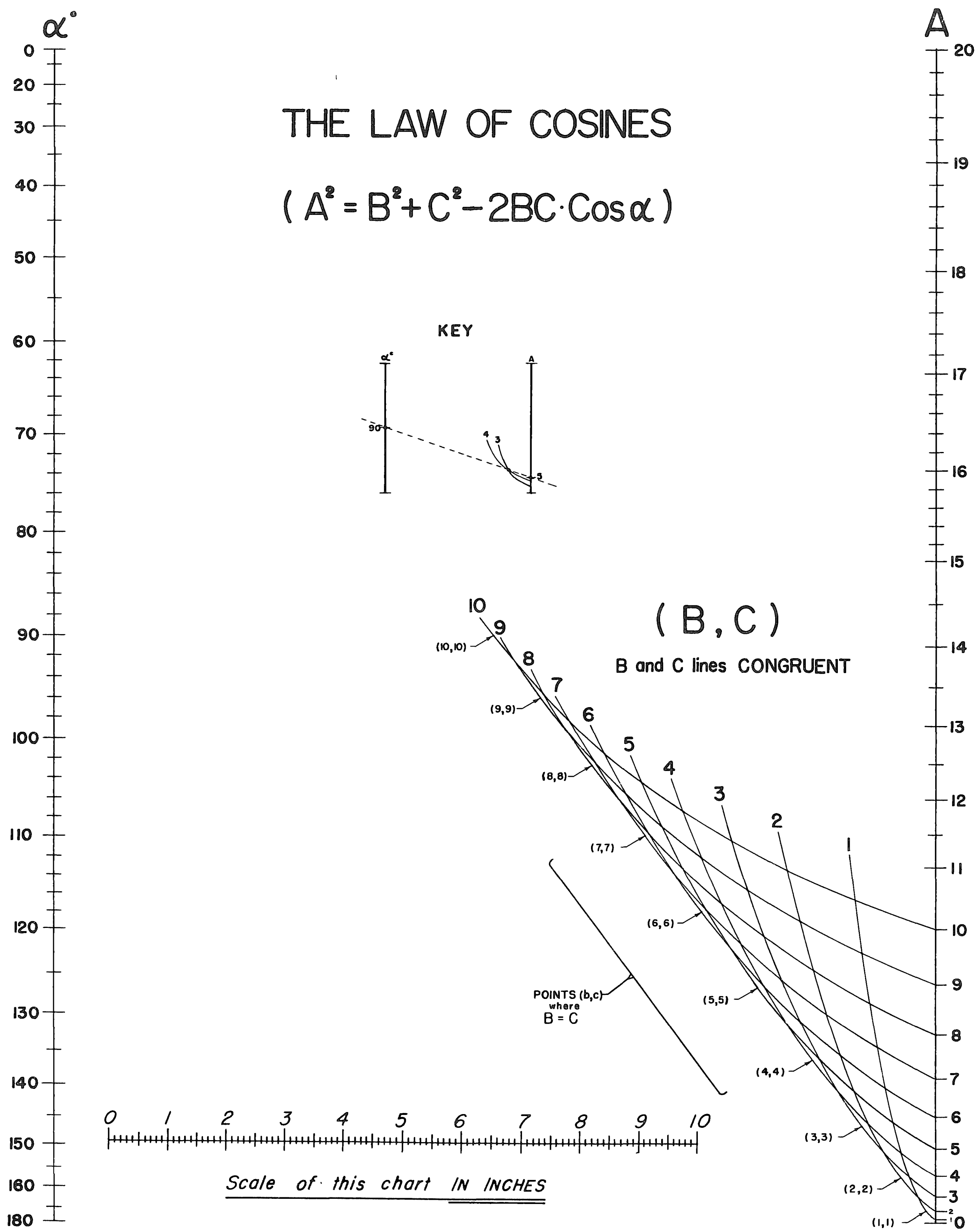

1.03N−CI+0.911S(C−9.427)=0
N ≡ Cloth Speed — Feet per Minute
I ≡ "Zero" Energy Speed of Internal Gear—Revolutions per Minute
S ≡ Speed of Main Cylinder —Revolutions per Minute
C ≡ Circumference of Worker Roll —Inches
Scale of this chart IN INCHES
KEY

Figure 3–7b.



THE LAW OF COSINES
(A² = B² + C² − 2BC·Cos α)
α°
A
KEY
(B, C)
B and C lines CONGRUENT
POINTS (b,c) where B = C
(10,10)
(9,9)
(8,8)
(7,7)
(6,6)
(5,5)
(4,4)
(3,3)
(2,2)
(1,1)
0 1 2 3 4 5 6 7 8 9 10
Scale of this chart IN INCHES

Figure 3–8.

DESIGN FACTORS FOR CLOSE WOUND COILS

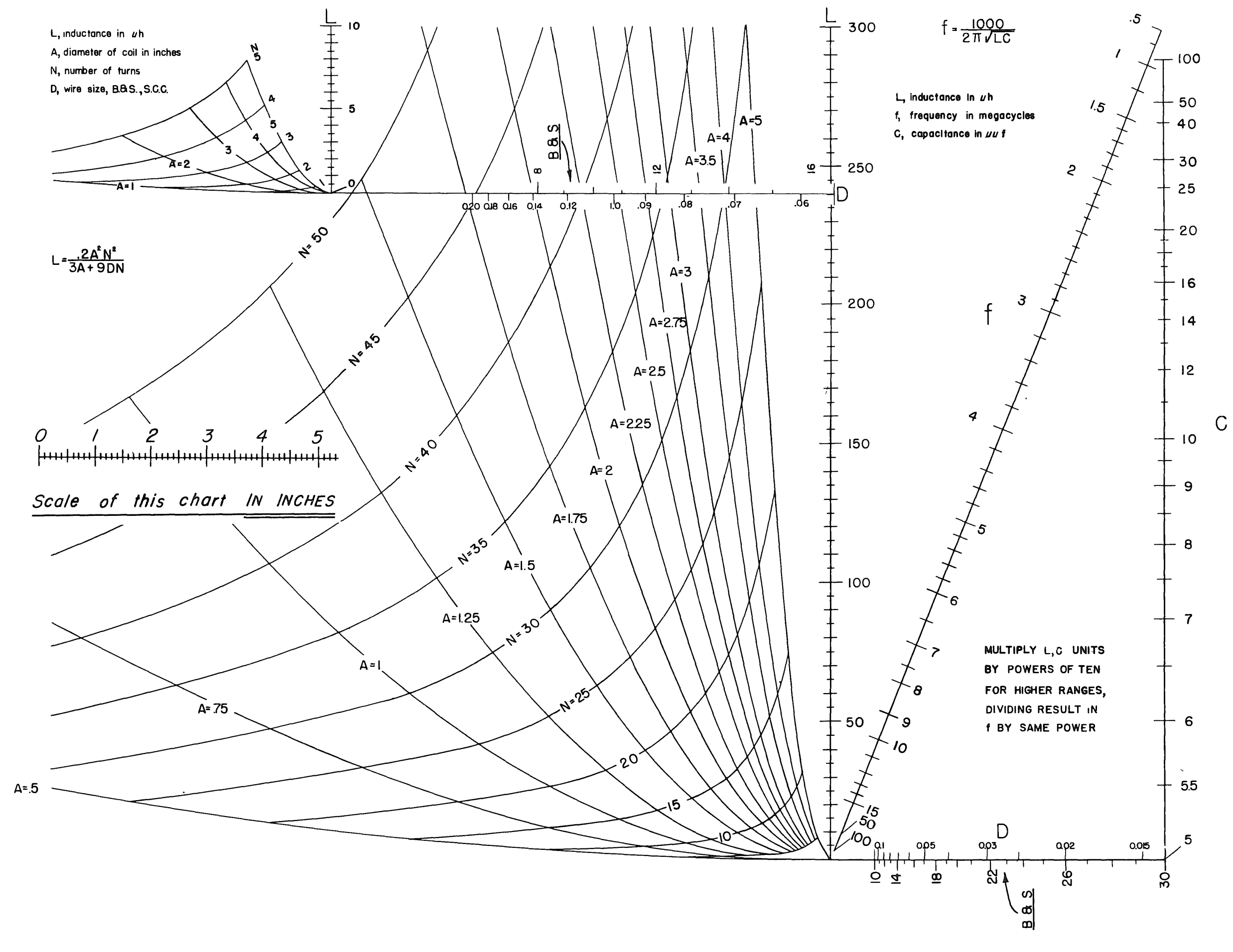
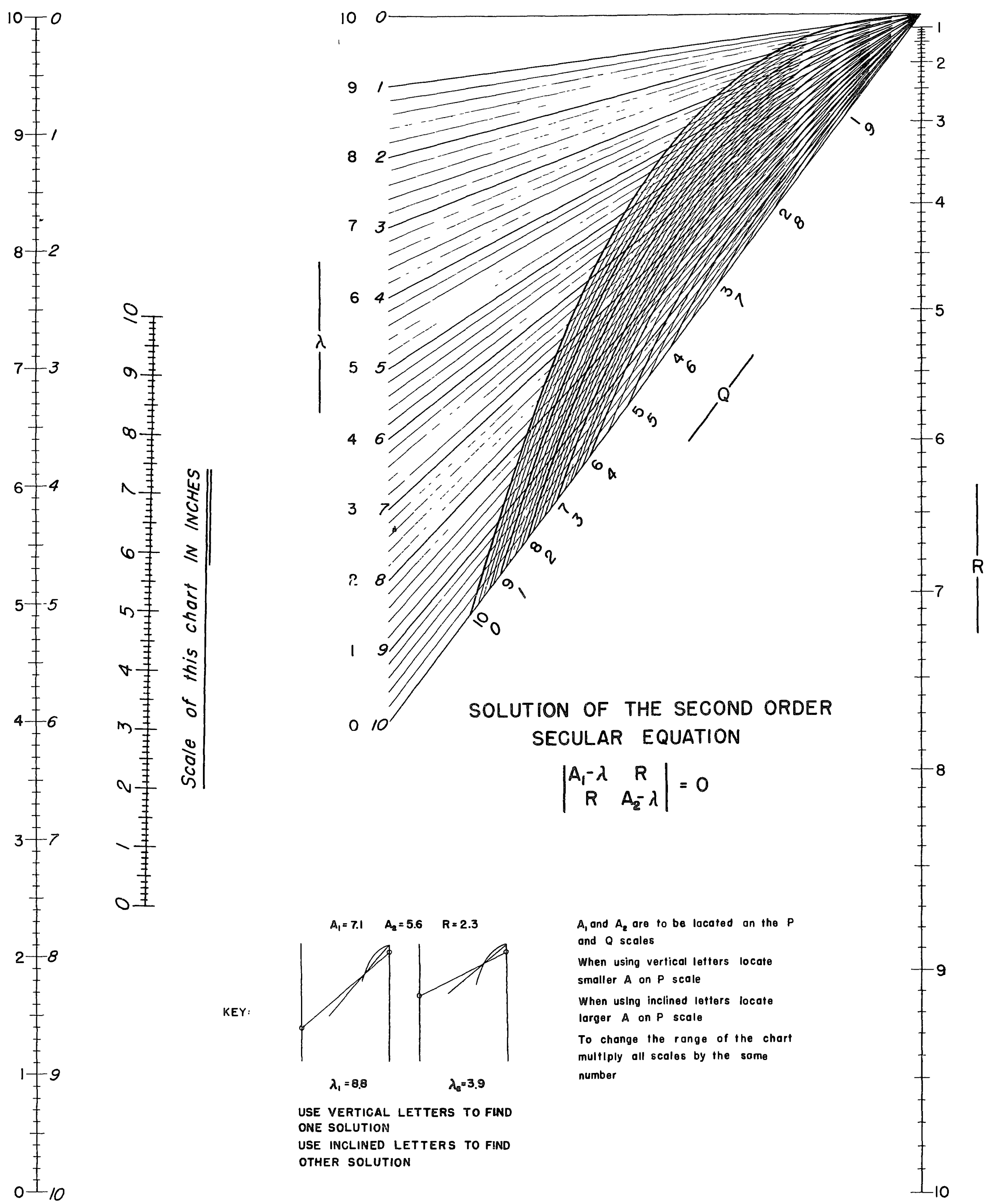
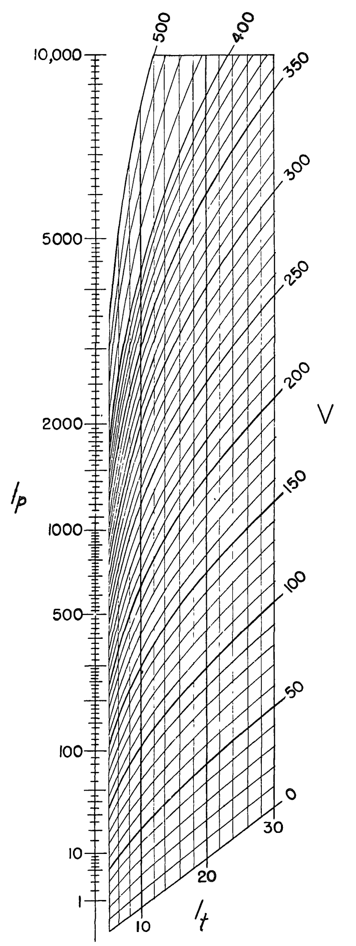


Figure 3–9.



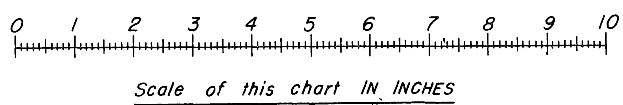
SOLUTION OF THE SECOND ORDER SECULAR EQUATION
$\begin{vmatrix} A_1-\lambda & R \\ R & A_2-\lambda \end{vmatrix} = 0$
λ
Q
R
Scale of this chart IN INCHES
KEY:
$A_1 = 7.1$ $A_2 = 5.6$ $R = 2.3$
$\lambda_1 = 8.8$ $\lambda_2 = 3.9$
USE VERTICAL LETTERS TO FIND ONE SOLUTION
USE INCLINED LETTERS TO FIND OTHER SOLUTION
A_1 and A_2 are to be located on the P and Q scales
When using vertical letters locate smaller A on P scale
When using inclined letters locate larger A on P scale
To change the range of the chart multiply all scales by the same number

Figure 3–10.



ALIGNMENT DIAGRAM FOR OSWALD'S
MAXIMUM SPEED AT SEA LEVEL

$$V_{max_{SL}} = 52.8\left(l_p/l_t\right)^{\frac{1}{3}} - .11\, l_s l_t$$



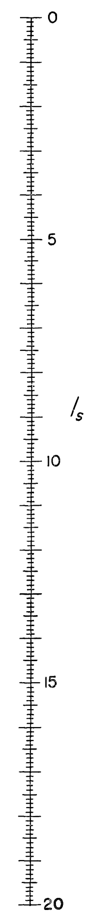
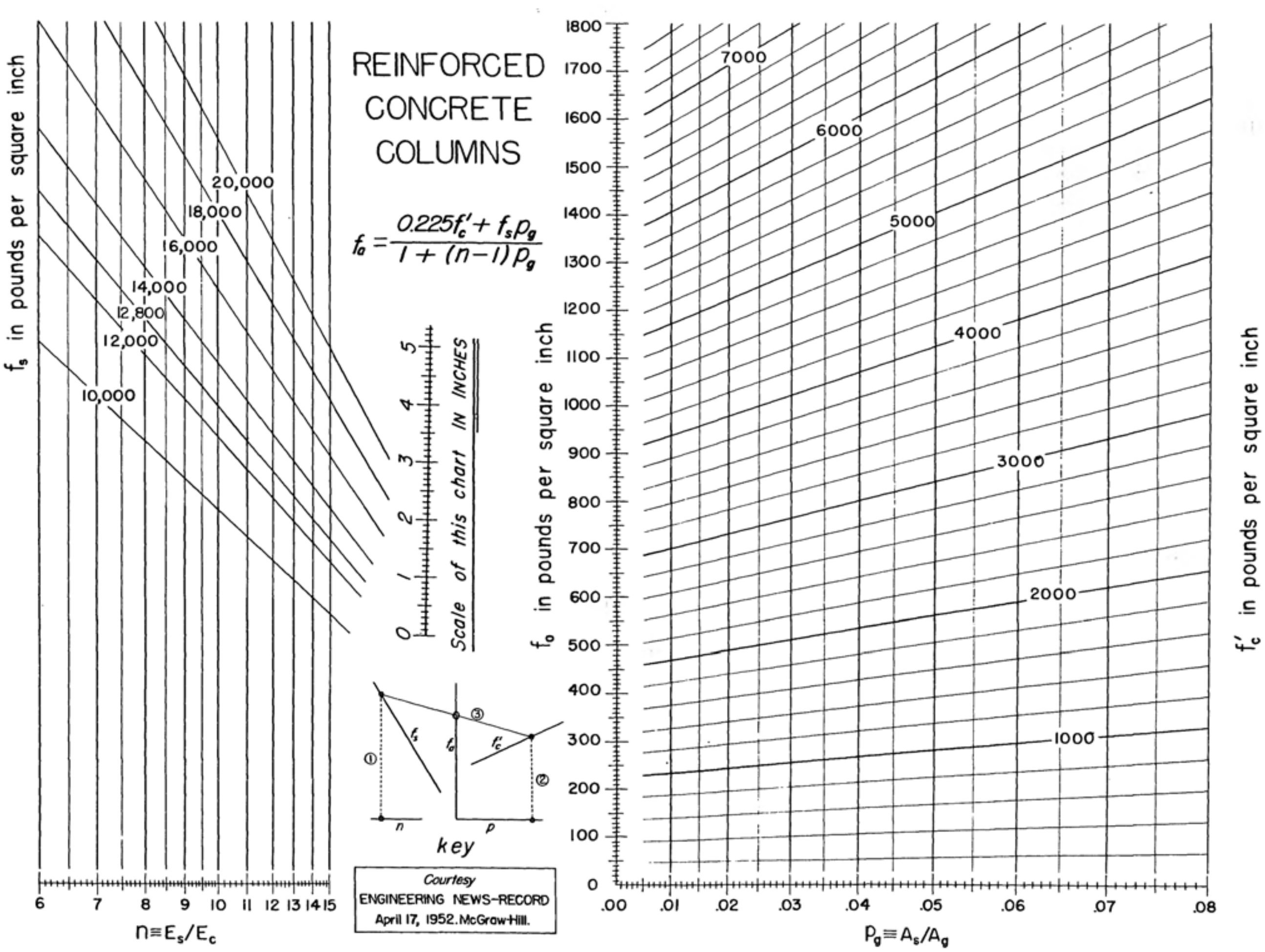


Figure 3–11.



REINFORCED CONCRETE COLUMNS
$f_a = \frac{0.225f_c' + f_s p_g}{1 + (n-1) p_g}$
Scale of this chart IN INCHES
0 1 2 3 4 5
f_s in pounds per square inch
20,000
18,000
16,000
14,000
12,800
12,000
10,000
6 7 8 9 10 11 12 13 14 15
$n \equiv E_s/E_c$
key
f_s
f_a
f_c'
n
p
f_a in pounds per square inch
1800 1700 1600 1500 1400 1300 1200 1100 1000 900 800 700 600 500 400 300 200 100 0
7000
6000
5000
4000
3000
2000
1000
.00 .01 .02 .03 .04 .05 .06 .07 .08
$p_g \equiv A_s/A_g$
f_c' in pounds per square inch
Courtesy
ENGINEERING NEWS-RECORD
April 17, 1952. McGraw-Hill.

Figure 3–12.

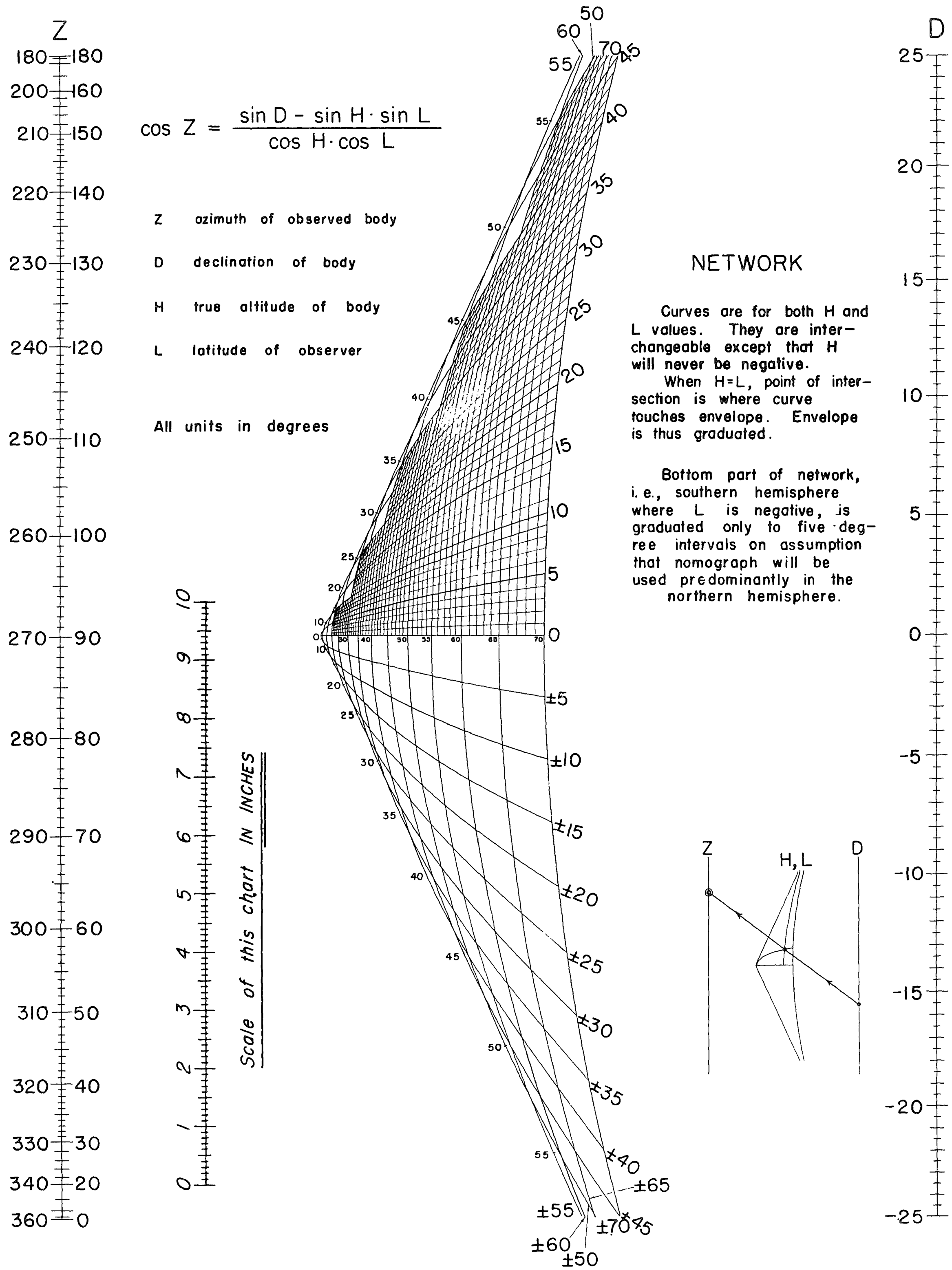

Z
D
cos Z = (sin D − sin H · sin L) / (cos H · cos L)
Z azimuth of observed body
D declination of body
H true altitude of body
L latitude of observer
All units in degrees
NETWORK
Curves are for both H and L values. They are interchangeable except that H will never be negative.
When H=L, point of intersection is where curve touches envelope. Envelope is thus graduated.
Bottom part of network, i. e., southern hemisphere where L is negative, is graduated only to five degree intervals on assumption that nomograph will be used predominantly in the northern hemisphere.
Scale of this chart IN INCHES
H, L

Figure 3–13.

PART II

Chapters 4 through 10 show what can be done with the approach outlined in Part I, when ingenuity and extrapolation of ideas are practiced. They also develop the underlying theory of central projection and the general projective transformation into effective operator form. The useful parallelisms in duality of point and line are naturally placed here.

CHAPTER 4

THE IMPERFECT CANONICAL FORM ((C)) • INGENUITY • NETWORK CHARTS DEPENDENT AND INDEPENDENT VARIABLES • COMPOUND DIAGRAMS SUBSTITUTIONS • ELEMENTARY DIAGRAMS

4–1. *Imperfect Canonical Form.* ((C)). A nomographer may be unsuccessful in attempting to place an equation in canonical form ((C)). However, he may arrive at a stage that is not perfect but lends itself to a practical interpretation.

Example 4–1. (See also Problem 4–10). Figure 4–1. The equation

$$10(W + V - 2U) + (W + V)(U - V) = 0 \tag{4–1}$$

can be placed in the form

$$\begin{vmatrix} 5 & U & 1 \\ 10 & V & 1 \\ \frac{W+V}{2} & W & 1 \end{vmatrix} = 0 \tag{4–2}$$

Figure 4–1 is a direct interpretation of this canonical form. The V scale is vertical, linear and placed at $X_2 = 10$. Its scale factor is unity. The U scale is vertical, linear, placed at $X_1 = 5$, with a scale factor of unity.

A W scale of unit scale factor appears at the extreme left, fixing the height of the coordinate Y_3. To find the x-coordinate of the W point, $X_3 = \frac{W+V}{2}$. Hence W and V are treated by a diagram for addition (Chapter 1) whose sum scale is halfway between the W and V scales and with a scale factor 1/2. When W is joined to V, the join cuts this central stem at a height $X_3 = \frac{W+V}{2}$. This ordinate is then brought to the 45° ray. This fixes the abscissa value X_3 and with the original ordinate W, the third point of the collineation. If U is the dependent variable, the steps for finding it are as numbered on the figure.

Example 4–2. Figure 4–2.
Represent in alignment diagram form the equation.

$$y^2 = \frac{x^2 + \frac{2}{K-1}}{\frac{2K\,x^2}{K-1} - 1} \tag{4–3}$$

$$2Kx^2y^2 - (K-1)y^2 - (K-1)x^2 - 2 = 0 \tag{4–4}$$

Let $A = x^2$

$B = y^2$

$$A \cdot 1 + B \cdot 0 - x^2 = 0$$

$$A \cdot 0 + B \cdot 1 - y^2 = 0$$

$$A(-1) + B\left\{\frac{2Kx^2}{K-1} - 1\right\} - \frac{2}{K-1} = 0$$

$$0 = \begin{vmatrix} 1 & 0 & -x^2 \\ 0 & 1 & -y^2 \\ -1 & \left\{\frac{2K\,x^2}{K-1} - 1\right\} & \frac{-2}{K-1} \end{vmatrix}$$

(a) multiply column III by (-1)

(b) multiply column I by $\frac{2K\,x^2}{K-1}$ and add to column II. (4–5)

$$0 = \begin{vmatrix} 1 & \frac{2K\,x^2}{K-1} & x^2 \\ 0 & 1 & y^2 \\ -1 & -1 & \frac{2}{K-1} \end{vmatrix}$$

(a) divide row I by x^2

(b) multiply column I by x^2

(c) multiply row III by $K - 1$. (4–6)

$$0 = \begin{vmatrix} 1 & \frac{2K}{K-1} & 1 \\ 0 & 1 & y^2 \\ -x^2(K-1) & -(K-1) & 2 \end{vmatrix}$$

write rows for columns, columns for rows (4–7)

$$0 = \begin{vmatrix} 1 & 0 & -x^2(K-1) \\ \frac{2K}{K-1} & 1 & -(K-1) \\ 1 & y^2 & 2 \end{vmatrix}$$

(a) move column I to column III

(b) divide column II by $-(K-1)$ (4–8)

$$0 = \begin{vmatrix} 0 & x^2 & 1 \\ 1 & 1 & \frac{2K}{K-1} \\ y^2 & \frac{-2}{K-1} & 1 \end{vmatrix}$$

(a) multiply row II by $\frac{K-1}{2K}$

(b) multiply column I by G (4–9)

$$0 = \begin{vmatrix} 0 & x^2 & 1 \\ \frac{(K-1)}{2K} G & \frac{K-1}{2K} & 1 \\ y^2 G & \frac{-2}{K-1} & 1 \end{vmatrix} = 0 \qquad (4\text{–}10)$$

Several other forms of arranging the three variables of this equation are also possible, depending on what the dependent variable is.

In (4–10) X_2, Y_2 define a ray through the origin which can be graduated in K. The only difficult point of the collineation, $X_3 = y^2G$; $Y_3 = \frac{-2}{K-1}$. If $X_2 = \frac{(K-1)}{2K} G$ is now correlated with Y_3 and if one eliminates the parameter K, one derives the curve given below, where X_2, Y_2, X_3, Y_3 refer to form shown in equation (1–4)

$$X_2 = \frac{G}{2 - Y_3} \qquad (4\text{–}11)$$

This is the rectangular hyperbola shown in Figure 4–2 with asymptotes $X = 0$ and $Y = 2$. As shown in the key, entry in K supplies an abscissa value $X_2 = \frac{K-1}{2K} G$ which, through the curve, yields an ordinate value $Y_3 = \frac{-2}{K-1}$. With $X_3 = y^2G$, this fixes the third point of the collineation.

4–2. *Ingenuity in Nomography.* Section 4–1 has just shown how ingenuity can help the nomographer. The distinction between a seemingly obvious and an ingenious scheme will vary with the individual depending upon his knowledge of the subject and his experience with it. An awareness of a wide variety of possible developments will give him flexibility. See Problem 4–2 and especially Problem 4–19, Figure 4–33, for an ingenious treatment of Example 4–2. A broad base of information is helpful in conceiving such possibilities. Properties of conics, elements and processes of projective geometry, procedures in algebra and calculus, can contribute to facility with nomography.

4–3. *Network Charts.* A network chart is a device for showing the behavior between three variables, $F(U, V, W) = 0$. Two of the variables, say U and V, are plotted (usually, though not necessarily) as Cartesian coordinates. If the third is held constant at some value $W = W_k$, a relationship between U and V results which can be plotted and labelled $W = W_k$. W is sometimes called the "family variable" or "family parameter" because differing values of it give rise to a whole family of such W curves on the U, V grid. Schemes like changing the family parameter from W to U or to V may change the nature of the curves of the family drastically.

Example 4–3. Show various forms of network chart for the equation $U \cdot V = W$.

1. Letting U and V be uniform scales at right angles (conventional Cartesian scales) then for each $W = W_k =$ constant, one has the relation $U \cdot V = W_k$, which yields a family of rectangular hyperbolas. Figure 4–3.

2. Letting U and W be plotted uniformly on Cartesian axes and V be family parameter, the ratio of W to U remains constant for constant V and the family of V curves appear as straight lines through the origin. Figure 4–4.

2a. The roles of U and V can be interchanged here with V plotted along the prime axis and constant U lines through the origin. Figure 4–5.

3. Taking logs of $U \cdot V = W$, $\log U + \log V = \log W$. Plotting $X = \log U$, $Y = \log V$, then constant W yields $X + Y = \log W_k =$ constant. Figure 4–6.

4. Returning to 1 and plotting a reciprocal scale in U, $U' = 1/U$, rather than a uniform scale of U on the prime axis, one has $V/U' = W_k =$ constant, yielding straight lines through the origin on the W-family. Figure 4–7.

Other forms of a network chart appear in the Appendix. When an equation cannot be put into practicable alignment diagram form it can always appear in network chart form. Empirical behavior of variables is usually recorded graphically in network chart form. The development of an alignment diagram for such empirical behavior must then always come from this graphical plot—a technique discussed later. Having the "family variable" graph as a set of straight lines is an important special case which will be discussed later.

4–4. *Dependent and Independent Variables.* This subject is being treated more frequently in texts for it must be understood for good results. In the chart for the equation U + V = W, Figure 4–8, a value of W can always be found for each pair of assigned U, V values, that is, the chart is NOT well adapted to finding V when U and W are given, because the line of collineation could cut the V scale at a value outside the chart (dotted line). For the same reason, it is NOT well adapted to finding U when V and W are given. In other words, it is best adapted to the situation where values of U and V are chosen freely or *independently* and W is then found as a result—W being *dependent* on the choice of U and V. In practice, in any problem it should be possible to know which variable is regarded as dependent. If this is not clearly stated in words, the ranges assigned the variables will frequently indicate which is the *dependent* or *answer* variable. In contrast with the above, in Figure 4–9 it is now V which will always yield an answer for any chosen values of U and W. V is the dependent variable, U and W are independent variables. The ranges of U and W will be observed to be the same in both figures. The difference in the ranges of the two V scales shows the difference in the diagrams.

The answer scale for the dependent variable should apparently lie *in the area between* the scales of the independent variable for the most effective use of chart space. This broad principle should be kept in mind and violated only for good reasons. This will almost always improve the efficiency of a chart. Techniques by which a dependent variable scale can be *made* to fall *within* the space between other scales are treated later and are part of this subject. In diagrams where the dependent variable scale must, for good reason, lie outside the scales of the independent variables, it must be extensive enough to "receive" any ray determined by values of the independent variables. *Figure 4–10* shows the situation where V is dependent variable and lies *outside* the U, W, scales. Compare the poor use of space here to the better use in *Figure 4–9*.

4–5. *Compound Diagrams.* It may be impossible to place an equation in four or more variables in determinant form needing only one alignment. Sometimes the best way seems to be to use two or more alignments by hitching two or more alignment diagrams in tandem, or by using one or more network charts in tandem with alignment diagrams. Problems 3–3 and 3–4 have already used this approach.

Example 4–4. Figure 4–11. Make a diagram for the equation U · V = R + S. Writing W = U · V and W = R + S, we have two diagrams whose alignment and network types are familiar to us. Two alignment diagrams, or an alignment diagram and network chart can be joined in tandem provided their W-scales are identical. This requires that the scale factors and placement of the two W scales be identical. Assuming V is dependent variable, then U, R and S are independent. Hence in using values of R and S, W must first be regarded as dependent. Once its value has been learned, it is then considered an independent variable along with U for the purpose of determining the dependent variable answer in V.
Frequently an equation in several variables is best treated, or can only be treated, by a succession of elementary or advanced diagrams operating in series in this way.

Example 4–5. The relationships of stereoscopic drawing are given in an equation which can be expressed by means of three alignment diagrams—two are N-shaped charts and the other, three concurrent lines. The final diagram is shown in Figure 4–12. The effective joining of diagrams is one of the chief skills of the professional nomographer. Facility with all the forms each component equation could take is needed to yield the optimum pattern of the combined diagrams. The best design is based upon simplicity and ease of operation as well as accuracy of solution values. It is always hoped that the chart can be designed so that no variables will have to be scaled more than once. The natural goal for each chart is "single entry".

There are many relationships from geometry which can help to join together alignment diagrams, network charts and combinations of them. As noted earlier the ingenuity of the nomographer has to fit the pieces together. What needs to be done is often quite clear and he shows his ability by hitting upon the simplest combination of devices to do it.

Example 4–6. Figure 4–13. A cost equation for peat excavation appears on Figure 4–13. It can be figured by hand but with frequent errors. The nomogram is for checking purposes. Here only the variable h has required double entry. It has been necessary, however, to combine the use of five N-shaped diagrams for multiplication with an alignment diagram and a reference curve. In many instances, a chart would not be worthwhile if it had to be used *in such a complicated way,* but for *checking,* this one was worthwhile.

4–6. *The Principle of Substitutions.* It is often possible to use an alignment diagram for a simple equation as the basis of a diagram for a complicated equation. If an equation in P, Q, and R can be put in the form,

$$f(P) + g(Q) = h(R) \tag{4–12}$$

then, regardless of how involved these functions are, one can make the substitutions:

$$\text{let } U = f(P) \quad V = g(Q) \quad W = h(R) \tag{4–13}$$

or,

$$U + V = W \tag{4–14}$$

The limits on variables P, Q and R are assumed to be known so that those on U, V and W can be computed. Imagine that a chart in U, V and W for these limits has been made. It would be possible to regraduate it in P, Q and R by means of (4–13). As soon as the scale equations for U, V and W are known, it would be possible to substitute (4–13) into these scale equations and obtain directly scale equations in P, Q and R which would enable the scales to be graduated in these variables directly without intermediate plotting of U, V or W. This is the principle of substitutions. It can be applied best where an *elementary* equation, such as (4–14) can be seen by inspection to be the result of substitutions. The technique is particularly effective where an equation has a good many variables in it and can be simplified by such substitutions into more easily treated combinations of variables.

Example 4–7. Figure 4–14. Here the substitutions shown below have permitted the final determinant forms:

Original equation:

$$X = \left[\frac{Z + 1 - Y_o}{Y_o - 1}\right]^2 [\log_e Y_o - (1 - 1/Y_o)]$$

Substitution:

$$u = Y_o - 1;\ V = \sqrt{\log_e (u + 1) - u/(u + 1)}$$

Final forms:

$$\text{for } Y_o < 1 \quad \begin{vmatrix} 0 & X & 1 \\ 20 & 20 - Z/5 & 1 \\ \dfrac{100}{5 - u/V} & \dfrac{100 - u}{5 - u/V} & 1 \end{vmatrix} \qquad \begin{vmatrix} 0 & X & 1 \\ 20 & 20 - Z/5 & 1 \\ \dfrac{100}{5 + u/V} & \dfrac{100 - u}{5 + u/V} & 1 \end{vmatrix} \quad \text{for } Y_o > 1$$

The principle is also useful sometimes to permit quick identification of cognate types of diagrams such as were found in various canonical forms ((C)) coming from the same form ((A)) in Chapter 1. Consider the equation $U' + V' = W'$ in the form

$$W' - V' = U'$$

$$\text{Let } U = W' \quad V = -V' \quad W = U' \tag{4–15}$$

Then $U + V = W$. The charts for $U + V = W$ and the corresponding one in the original, primed variables appear in Figure 4–15(a). When the equation $U' + V' = W'$ is given in the form

$$U' - W' = -V' \tag{4–16}$$

one writes

$$U = U' \quad V = -W' \quad W = -V' \tag{4–17}$$

and the charts in U, V and W, U′, V′ and W′ appear as in Figure 4–15(b). These diagrams are the same variations discussed in Section 4–4 under Dependent and Independent Variables. In the standard diagram for U + V = W, the W scale appears between the U and V scales. If in the equation U′ + V′ = W′, W′ is dependent, then the standard diagram form can be used with W′ between U′ + V′. If, however, U′ is dependent, or V′ dependent, then procedures (4–15), (4–17) will place the dependent variable in the interior position between the other two scales. Similar arrangements can be made where U, V, W are more complicated functions of P, Q and R as in (4–12), (4–13).

In Chapters 1 and 2, different substitutions frequently led to different forms ((A)) and hence of the final determinant form ((C)). Also, the same form ((A)) could be made to yield several different forms of ((C)) by changing the determinant differently. Earlier in this chapter, principles of dependent and independent variables showed the need for these different forms ((C)). Now they are shown to be related through substitutions. In Chapter 6 the cause of these extensive interrelationships will be shown to be that these various charts are frequently linked by central projection. See *Elements of Nomography*, R. D. Douglass and D. P. Adams, McGraw-Hill (1947), Chapters X, XI.

4–7. *Elementary Diagrams.* See *Elements of Nomography* for a thorough treatment of this subject. Our application of determinants to nomography has used the most elementary equations and diagrams possible and all explanations of these diagrams thus far have been made on this basis. Elementary diagrams can also be based on plane geometry and trigonometry and can sometimes be thought of most effectively in these simpler terms. The diagrams, their derivations, scale equations and characteristics are now listed briefly.

The Three-Parallel-Line Chart. U + V = W. U · V = W. Figure 4–16.

$$\frac{S_V - S_U}{S_W - S_U} = \frac{a+b}{a}; \; S_U = uU; \; S_V = vV, \; S_W = wW$$

$$vV - uU = -\frac{a+b}{a}uU + \frac{a+b}{a}wW$$

$$\frac{b}{a}uU + vV = wU\frac{(a+b)}{a} + wV\frac{(a+b)}{a}$$

$$u = \frac{w(a+b)}{b} \qquad v = \frac{w(a+b)}{a}$$

$$w = \frac{u \cdot v}{u+v}; \; \frac{u}{v} = \frac{a}{b} \tag{4–18}$$

U, V, W scales are *uniform* for U + V = W

U, V, W scales are logarithmic for U · V = W.

Example 4–8. Figure 4–17 is a combination of three nomograms of the form (d) above, which is a very useful special case of (a). For day-of-the-week purposes, any elapsed number of days can be divided by seven and represented by the remainder (a number from 0 to 6) two days being represented by the same number if they are an even number of weeks apart. Scales I, III, V and VII record in this way respectively elapsed days from the beginning of the month to the given day, from the beginning of the year to the beginning of the month, from the beginning of the calendar to the beginning of the century, and from the beginning of the century to the beginning of the year. The three combined charts add all these, the inner scale being central in each case and twice as densely spaced as the outer two, as in Figure 4–16(d). Each non-leapyear advances by one position because 365 = 7 · 52 + 1 and is represented by one (1). The change of calendar in 1582 omitting ten days appears in column V; leapyears are compensated for at the beginning of the year in column VII and this being premature for January and February, requires double entry for those months in column III.

N-, Z-, or H-Chart. U · V = W.

$$\frac{S_U}{S_W} = \frac{K - S_V}{S_V} = \frac{K}{S_V} - 1;$$

$$S_U = uU; \; S_W = wW. \qquad \text{(See Figure 4–18)}$$

$$\frac{uU}{wW} + 1 = \frac{K}{S_V}$$

$$\frac{KV}{V + \frac{u}{w}} = S_V; \qquad S'_V = \frac{K}{\frac{w}{u}V + 1} \tag{4–19}$$

U and W scales are uniform. Midpoint of V scale is u/w (or −u/w, below).

Hexagonal Chart. 1/U + 1/V = 1/W.

$$\frac{DB}{OC} = \frac{OA + OD}{OA}$$

$$\frac{OB}{OC} = \frac{OB}{OA} + 1 \qquad \text{(See Figure 4–19(a))}$$

$$1/OA + 1/OB = 1/OC$$

$$OA = gU; \; OB = gV; \; OC = gW$$

$$1/U + 1/V = 1/W \tag{4–20}$$

$$S_U = uU;\ S_V = vV \qquad \text{(See Figure 4–19(b))}$$

$$S_W = \sqrt{u^2 + v^2} = W$$

$$\tan\theta = \frac{v}{u} \qquad (4\text{–}21)$$

All scales everywhere are uniform.

Circular Chart. Parabolic Chart. U · V = W.

$$\frac{\sin(90 - \theta)}{S_W} = \frac{\sin(90 - \phi + \theta)}{a \cdot \cos\phi}$$

$$S_W = a/(1 + \tan\theta\tan\phi)$$

Let: $U = \tan\theta;\ V = \tan\phi$

Then: $S_W = a/(1 + UV) = a/(1 + W)$

Let: $mU' = \tan\theta;\ nV' = \tan\phi$

Then: $S_W = a/(1 + mnW')$

$$U \cdot V = W$$

$$U' \cdot V' = W' \qquad (4\text{–}22)$$

$$\frac{W - U^2}{V^2 - U^2} = \frac{U}{V + U}$$

$$\frac{W - U^2}{V - U} = U \qquad U \neq -V$$

$$U \cdot V = W \qquad (4\text{–}23)$$

(See Figure 4–19(c))

PROBLEMS

PROBLEM 4–1. Example 4–1 can be done in the following way. In imperfect canonical form:

$$\begin{vmatrix} 5 - U/2 & U & 1 \\ 10 - V/2 & V & 1 \\ V/2 & W & 1 \end{vmatrix} = 0 \qquad (4\text{–}24)$$

Expand this determinant to check that it represents (4–1). Draw a key to show a simple way of using this imperfect canonical form. Sketch the diagram and show that it works.

PROBLEM 4–2. In Example 4–2, Figure 4–2, the canonical form had three functions of k in it so that the chart had an x-scale, a k-scale and a y,k-scale but was arranged to need entry in k only once. Place this same equation in a canonical form shown below having three functions of x in it so that the chart has a k-scale, an x-scale and a y, x-scale and can be used by entering only once in x. Sketch this chart and show that it works.

$$\begin{vmatrix} \frac{2}{1 - k} & 0 & 1 \\ 2 - 1/x^2 & 1/x^2 & 1 \\ x^2 & y^2 & 1 \end{vmatrix} = 0$$

Compare and contrast it with Figure 4–2.

PROBLEM 4–3. Place the equation

$$U \log U + VW - VU^2 - UW = 0$$

in the canonical form

$$\begin{vmatrix} U & U^2 & 1 \\ V & \log U & 1 \\ 0 & W & 1 \end{vmatrix} = 0$$

Sketch a diagram that will permit solution of this equation using this canonical form. Include several major calibrations of each variable and at least one alignment for checking. Draw a key which explains clearly how the diagram is used.

PROBLEM 4–4. Figure 4–20. The equation for the angle α of a cycloidal cam is given by

$$2M \tan\mu = \tan M\alpha + (M - L)\,\alpha\,[\csc^2 M\alpha],$$

α in radians.

In this figure, $X_1 = L/\tan\mu$, $Y_1 = 0$; $X_2 = 0$; $Y_2 = L$. Derive the canonical form that has these quantities in it this way with M and α entering in an M, α net. Plot enough points to check the diagram. See also Problems 3–9 and 4–5.

PROBLEM 4–5. Figure 4–21. The equation from Problem 4–3 can be put in the canonical form

$$\begin{vmatrix} \frac{1}{1 - \alpha\csc^2 M\alpha} & \frac{\tan M\alpha}{1 - \alpha\csc^2 M\alpha} & 1 \\ 0 & M - L & 1 \\ -1 & 2M\tan\mu & 1 \end{vmatrix} = 0.$$

Here an elementary diagram for multiplication for $Y_3 = 2M\tan\mu$, and for subtraction for $Y_2 = M - L$ can be arranged to use the same M scale and yet to have Y_2 and Y_3 scales fall in the right places at $X_2 = 0$ and $X_3 = -1$ respectively.

1) Arrange these elementary diagrams this way.

2) Sketch the entire diagram including a number of values.

3) Show two alignments working successfully.

4) Compare the form of this diagram with that of the preceding problem and describe clearly the comparative merits of the two as practical nomograms.

5) Compare the form of this diagram with those of Problems 3–9 and 4–4.

PROBLEM 4–6. 1) In each of the problems for Chapters 1, 2 and 3, where the chart for the problem is supplied, identify the dependent variable.

2) Where a net is present, note that either of the two variables of the new net may be dependent or they may be taken as a dependent pair.

PROBLEM 4–7. In Problem 3–6 note that different forms of the diagram imply different variables as dependent variable. Note the changes in the form of the rest of the diagram such as the direction and relative placement of vertical scales. These will be referred to later and used as the means of making one variable or another dependent. Can you reach any conclusions about dependent variables and direction of vertical scales?

PROBLEM 4–8. Combinations of Elementary Diagrams.

The inductance of a single-layer air core solenoid is given by the formula shown in Figure 4–22. Derive the canonical forms, with scale-factors, of the equations that give the form of the diagram shown for this equation.

PROBLEM 4–9. The equation for Weick's coefficient in the relation between speed and power can be expressed by means of two three-parallel-line diagrams shown in Figure 4–18. Derive this diagram.

PROBLEM 4–10. The equation for the gain of an amplifier can be expressed by two alignment diagrams compounded as in Figure 4–24. Derive this diagram.

PROBLEM 4–11. Return on investment is governed by an equation which can be represented by two alignment diagrams with a common index scale. Show that Figure 4–25 expresses this relationship accurately by deriving the scale equations for this diagram.

PROBLEM 4–12. In wood beam design, the load, stress, and beam dimensions are related as shown by the formula in Figure 4–26. Show that the combined diagrams drawn there accurately give the behavior of the variables. Note that W and L have been expressed in feet and should be changed to inches for checking purposes.

PROBLEM 4–13. The equation for upper half-power frequency in an R-C Coupled Amplifier appears in the chart of Figure 4–27. The key shows that this diagram has been compounded from three elementary diagrams — two with three concurrent scales and an N-shaped diagram. Derive the scale equations for these diagrams and show that they work.

PROBLEM 4–14. The relationship between Cartesian and polar coordinates can be expressed by two diagrams which can be so combined, as in Figure 4–28, that a single collineation will change from one pair of coordinates to the other. Derive this chart.

PROBLEM 4–15. In treating the circular nomogram for the equation $U \cdot V = W$, Example 1–7, the upper half of the circle bearing the positive U scale was given parametrically by the equations

$$X = \frac{a}{1 + U^2}; \qquad Y = \frac{aU}{1 + U^2}.$$

Then $$Y/X = U.$$

A line from the origin with slope $Y/X = \tan\theta$ will cut the U-scale on the circle at the value $\tan\theta = U$. If a chord from the zero end of the W scale to this point of the U scale has length aG and makes an angle ϕ with the W scale, then $\phi = \cos^{-1}G$, $U = \tan\theta = \cot\phi = \cot\cos^{-1}G$. The equation $S = r\cot\cos^{-1}(\cos U - d\#)$ relates the settings of the Buerger Precession Camera as indicated in Figure 4–29. Derive the scale equations for this diagram. See *Review of Scientific Instruments*, Volume 20, March 1949, pages 150-160. Derive the scales of Figure 4–29.

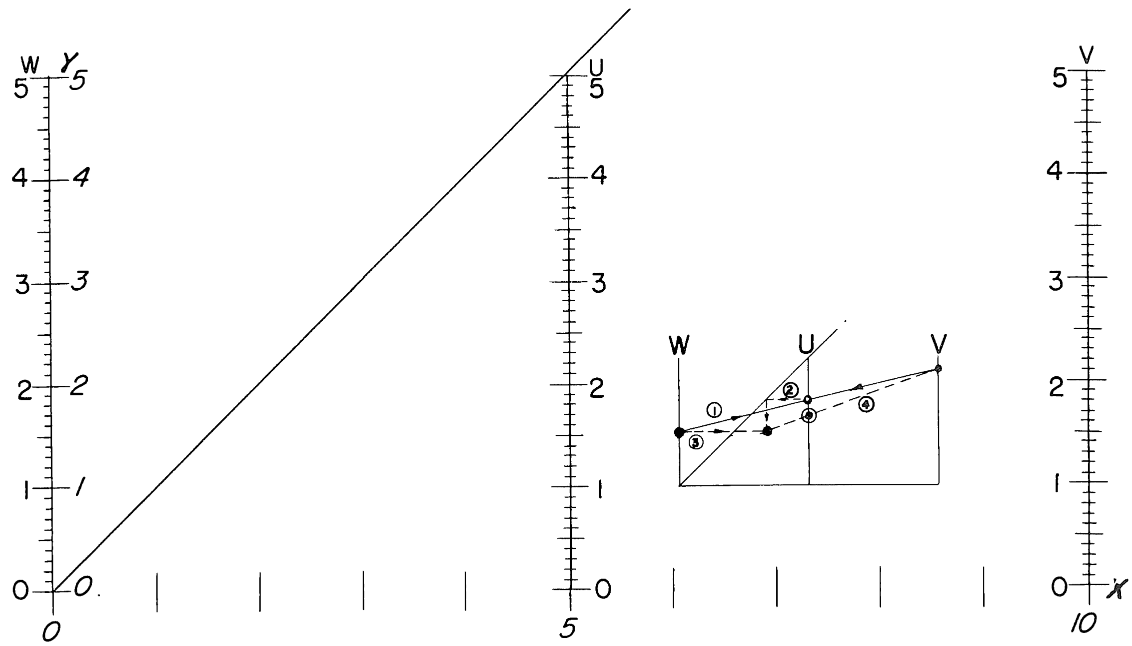
PROBLEM 4–16. The determination of stress in small wires under tension uses the equations shown on the chart of Figure 4–30. This diagram needs two N-diagrams and an alignment diagram with a net. The latter is somewhat unusual in that one of the scales is reduced to a point. Derive the scale equations for this diagram and verify its operation.

PROBLEM 4–17. The dependence of the length of a wound roll upon the thickness of the material and the inner and outer radii of the roll is expressed by the equation in Figure 4–31. If the equation is put in a form to yield the S and T scales shown there, it will be found that an r_o, r_c net results which, for the scale factors present, lies for all purposes right along the X-axis. It is impossible to separate the curves there sufficiently to use the net in a practical manner. Hence an artificial network chart has been made by assigning even levels to r_c values and plotting r_o curves on them so that the x values of intersection would agree with those of the actual net. Verify the chart.

If r_c, r_o, and t are multiplied by a factor k, then S is multiplied by this factor. This makes it possible to interpret the given ranges of the variables three different times, as the vertical numbers, the inclined numbers and the primed numbers. In view of the fact that t turned out to be a reciprocal scale, this is a great help.

PROBLEM 4–18. The Lamé-Maxwell Equation of Equilibrium appears on Figure 4–32. Its solution in that figure uses a three-parallel-line chart, a fixed point, a network chart and an N diagram. Derive the equations for this diagram and show that it works.

PROBLEM 4–19. Figure 4–33 (not drawn at time of publication). Problem 4–3, it turns out, can result in a diagram requiring only one collineation. This is not done thru a canonical form, but rather by arranging two diagrams to share the same collineation. Identify the component parts of these diagrams and derive the equations for them.



$10(W+V-2U)+(W+V)(U-V)=0$

Figure 4–1.

ONE-DIMENSIONAL NORMAL-SHOCK FUNCTION

$$M_Y^2 = (M_X^2 + \frac{2}{K-1}) / (\frac{2K}{K-1} M_X^2 - 1)$$

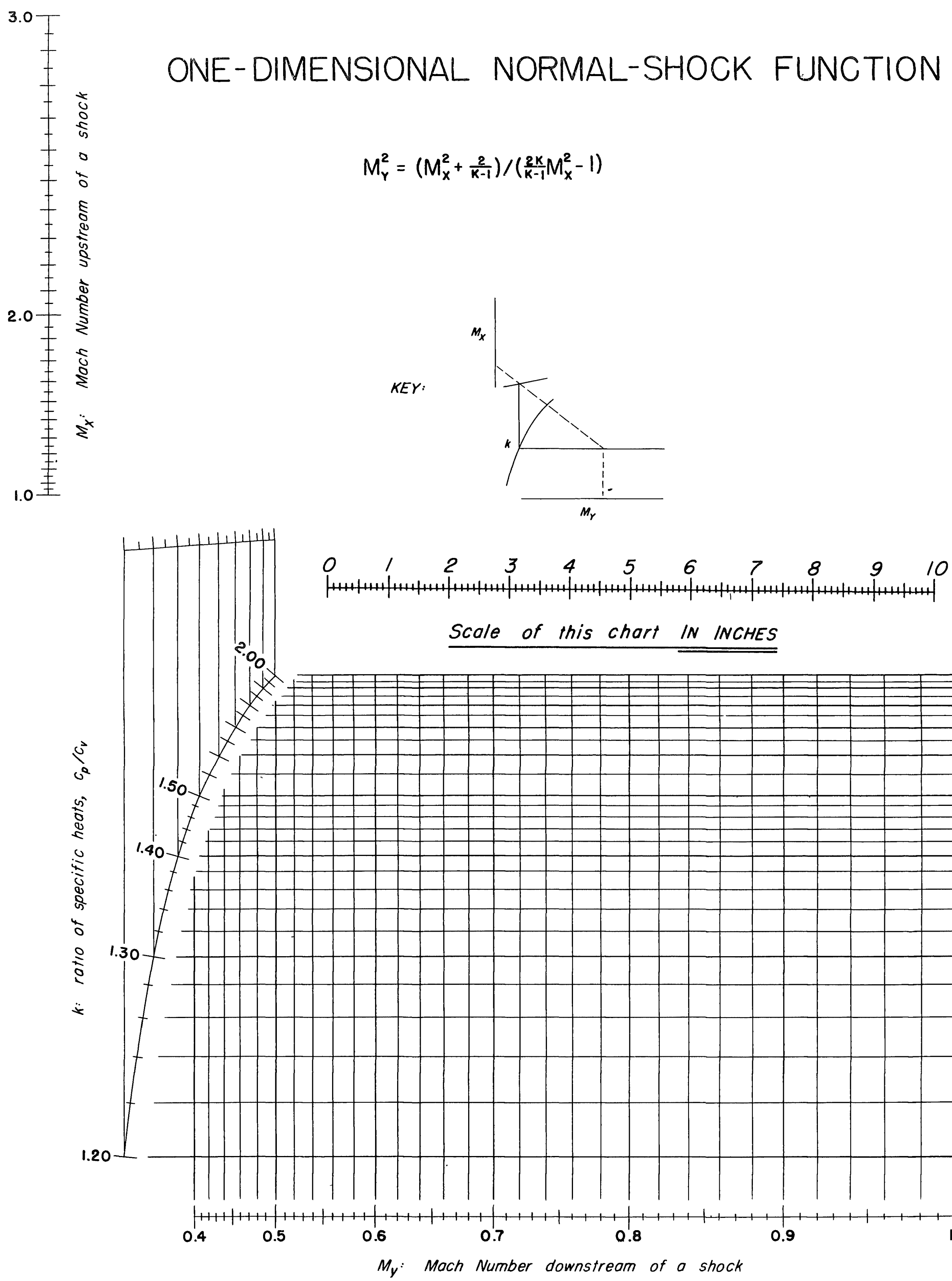


Figure 4–2.

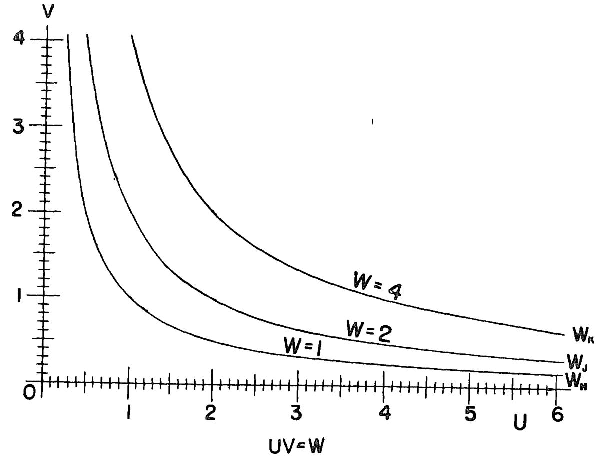


Figure 4–3.

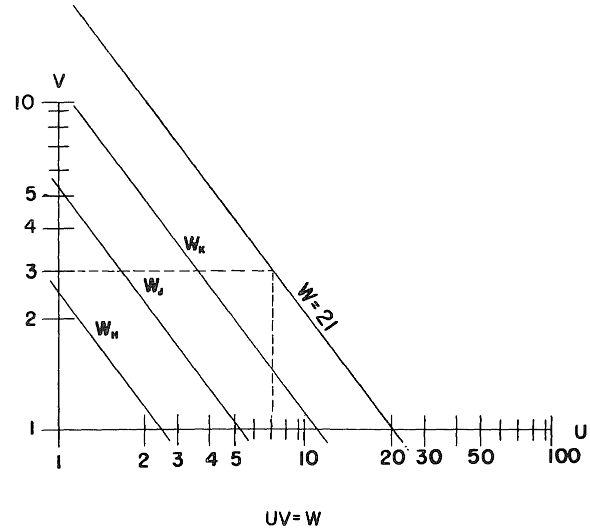


Figure 4–6

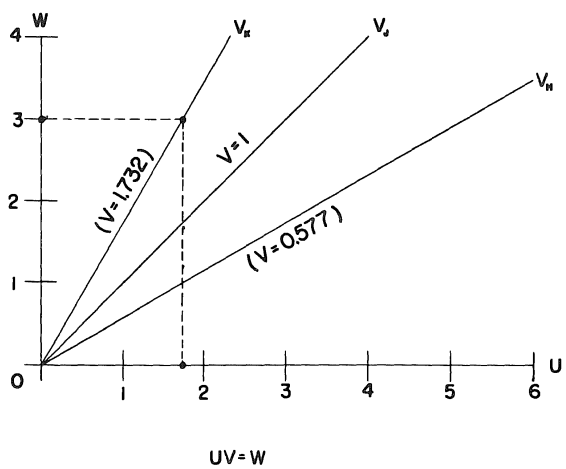


Figure 4–4.

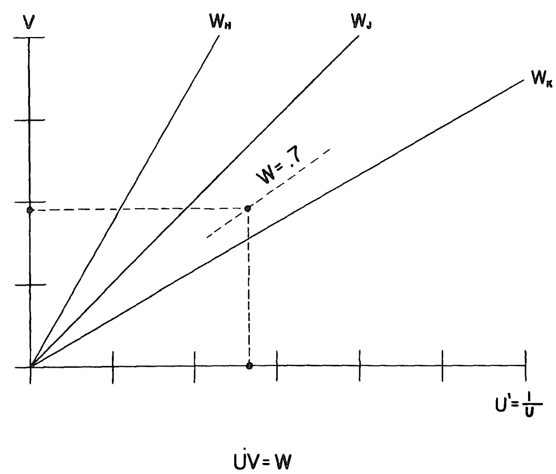


Figure 4–7.

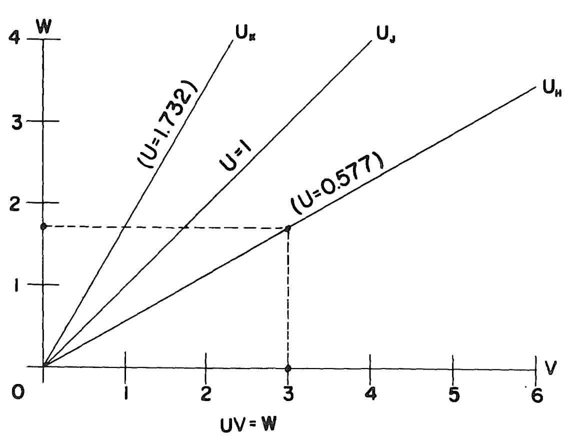


Figure 4–5.

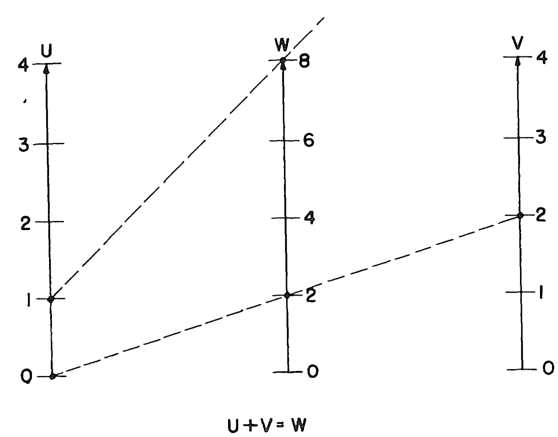


Figure 4–8.

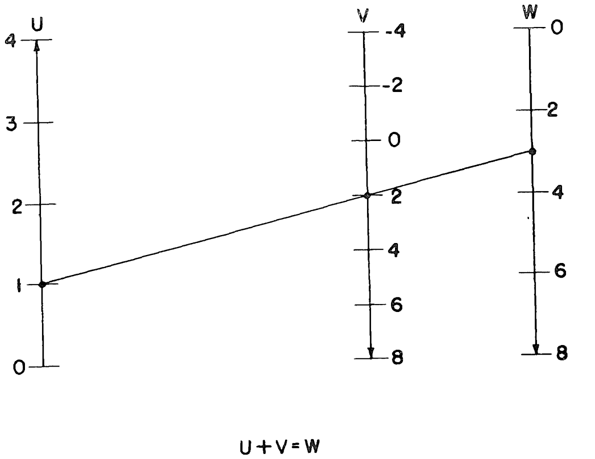


Figure 4–9.

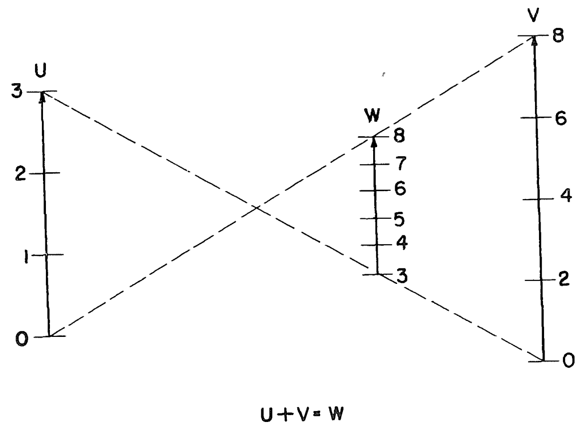


Figure 4–10.

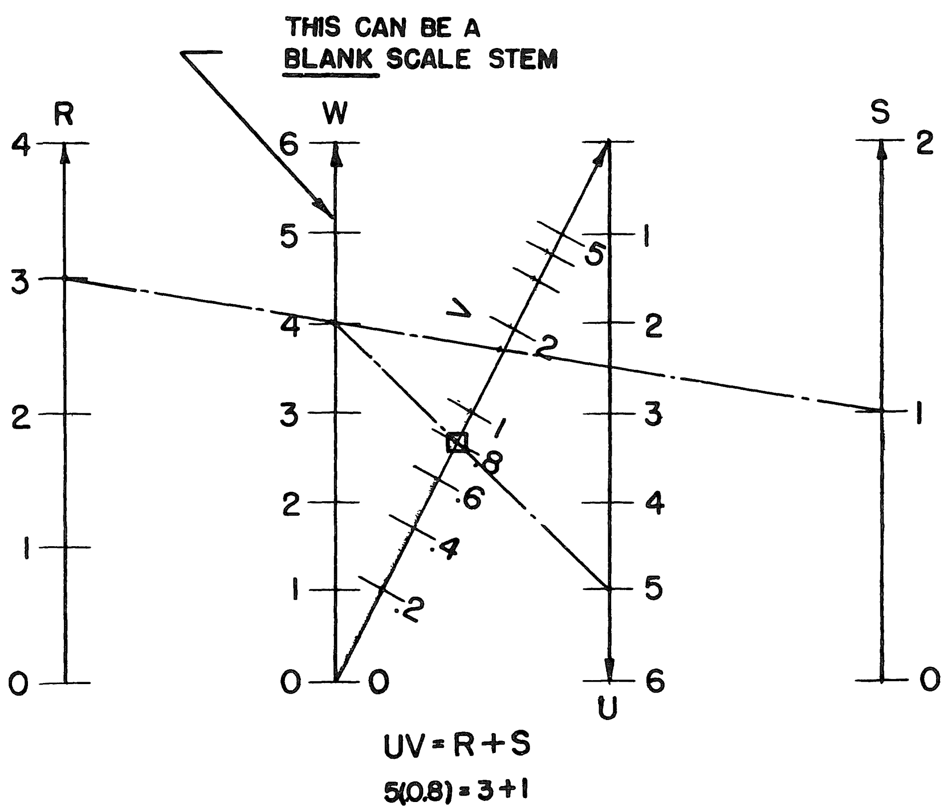


Figure 4–11.

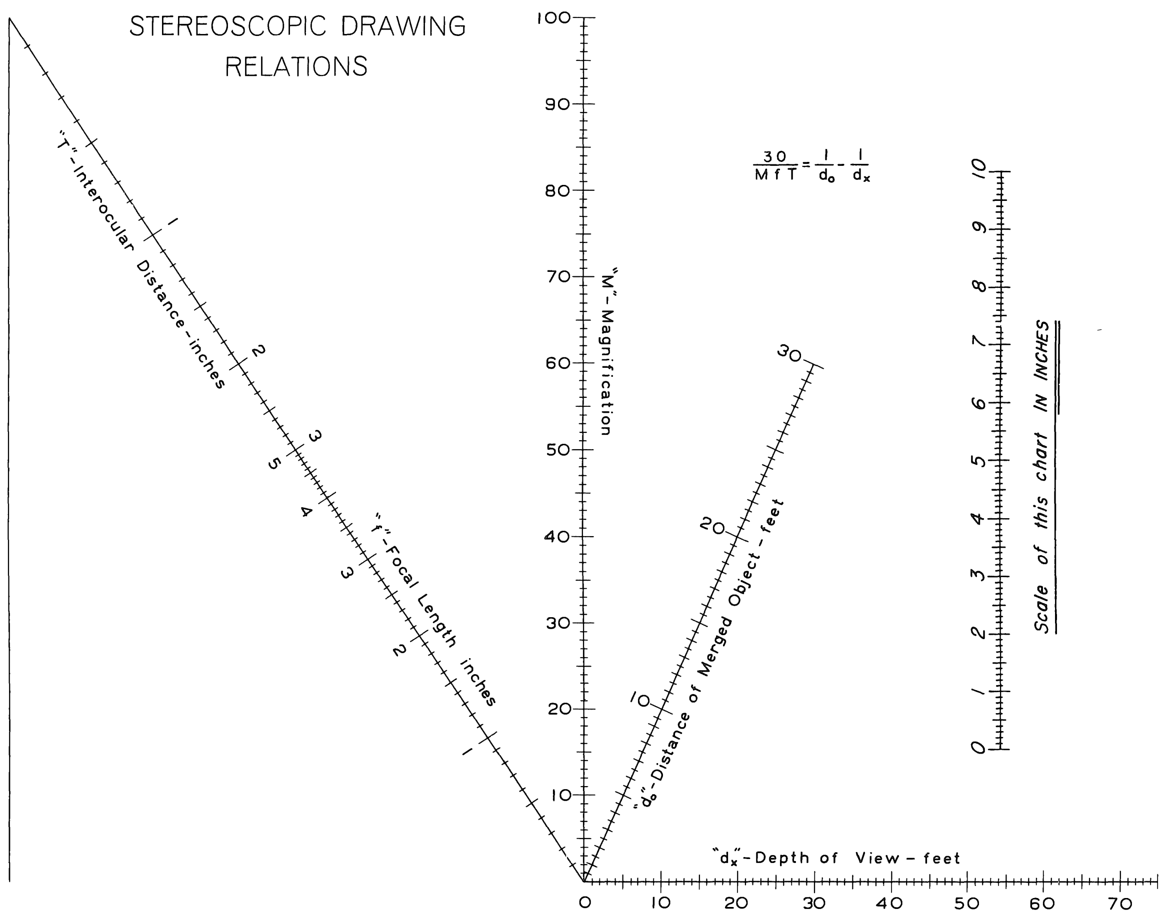

STEREOSCOPIC DRAWING RELATIONS
"T"-Interocular Distance-inches
"f"-Focal Length inches
"M"-Magnification
"d₀"-Distance of Merged Object-feet
"dx"-Depth of View-feet
30/MfT = 1/d₀ - 1/dx
Scale of this chart IN INCHES

Figure 4–12.

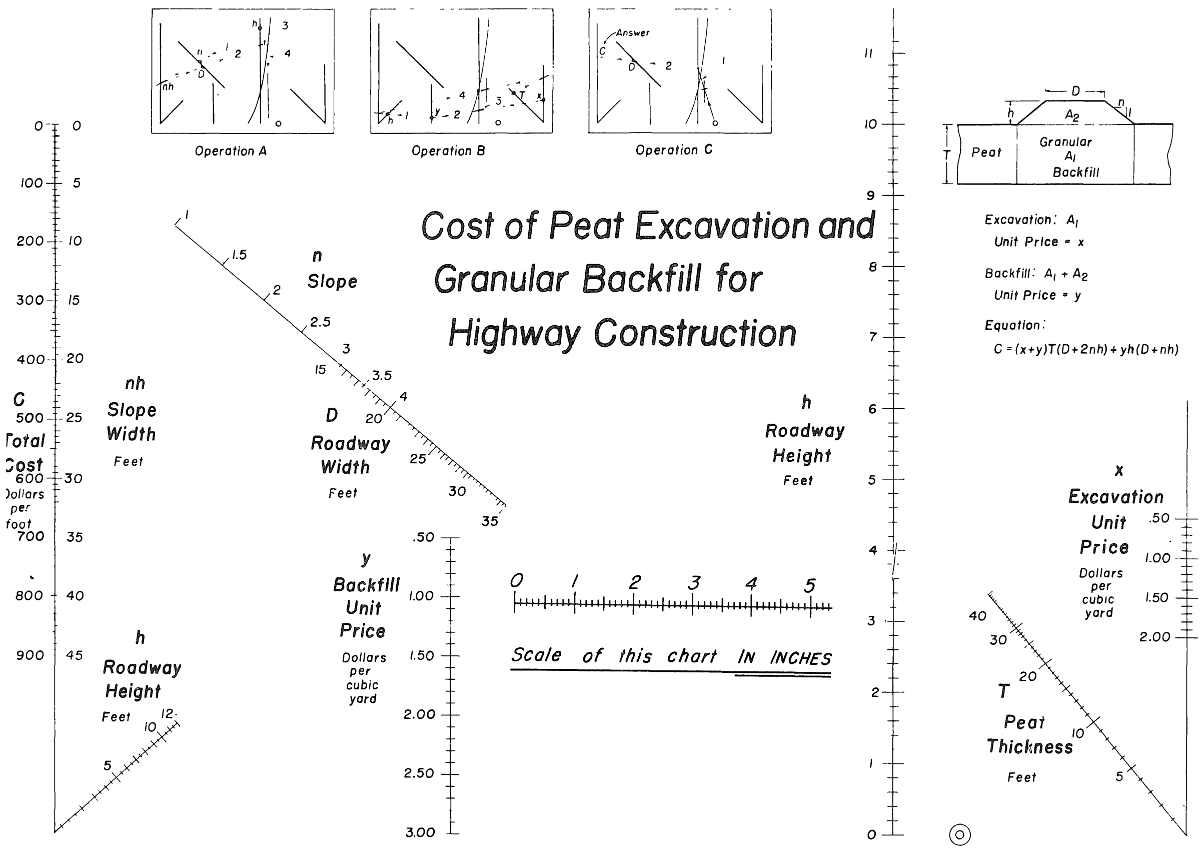

Cost of Peat Excavation and
Granular Backfill for
Highway Construction
Operation A
Operation B
Operation C
Answer
C
Total
Cost
Dollars
per
foot
nh
Slope
Width
Feet
n
Slope
D
Roadway
Width
Feet
h
Roadway
Height
Feet
y
Backfill
Unit
Price
Dollars
per
cubic
yard
Scale of this chart IN INCHES
x
Excavation
Unit
Price
Dollars
per
cubic
yard
T
Peat
Thickness
Feet
Peat
Granular
A1
Backfill
A2
Excavation: A1
Unit Price = x
Backfill: A1 + A2
Unit Price = y
Equation:
C = (x+y)T(D+2nh) + yh(D+nh)

Figure 4—13.

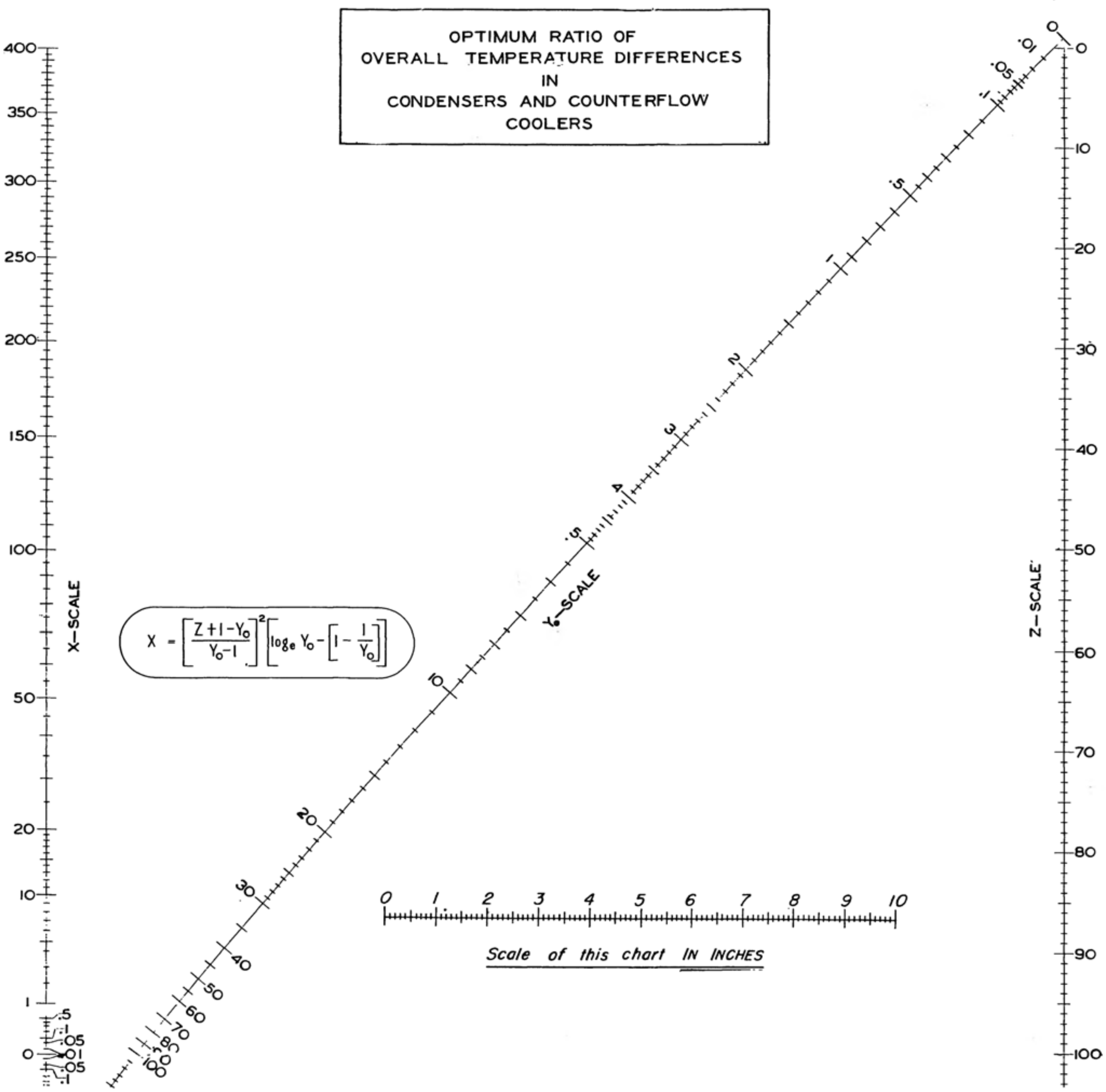

OPTIMUM RATIO OF
OVERALL TEMPERATURE DIFFERENCES
IN
CONDENSERS AND COUNTERFLOW
COOLERS
X-SCALE
Y-SCALE
Z-SCALE
$X = \left[\frac{Z+1-Y_o}{Y_o-1}\right]^2 \left[\log_e Y_o - \left[1 - \frac{1}{Y_o}\right]\right]$
Scale of this chart IN INCHES

Figure 4–14.

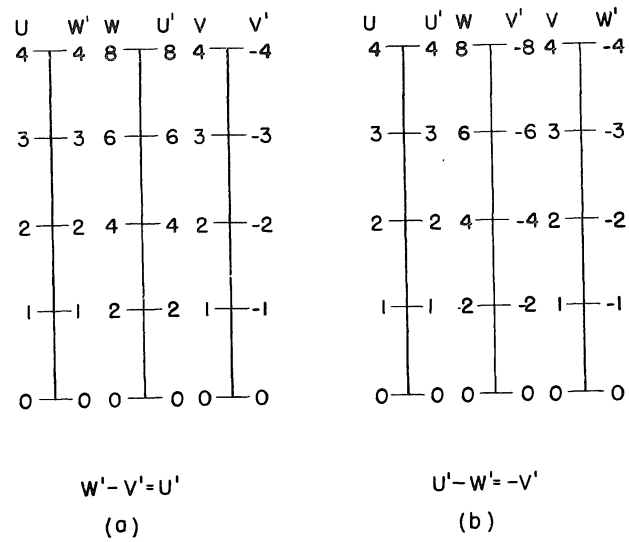


Figure 4–15.

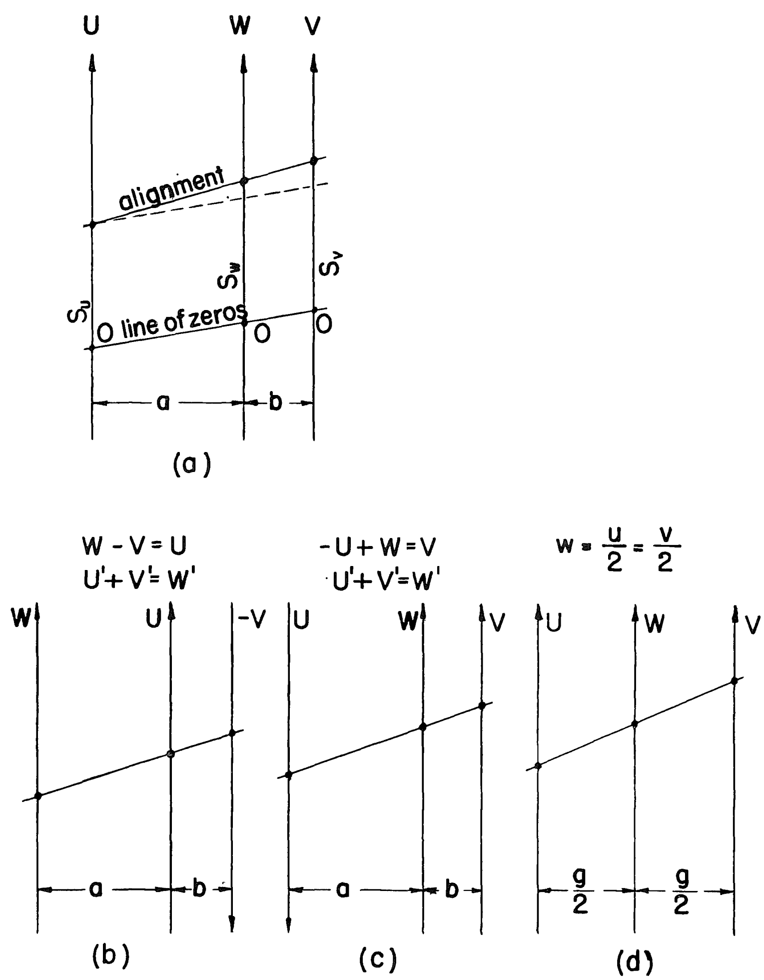
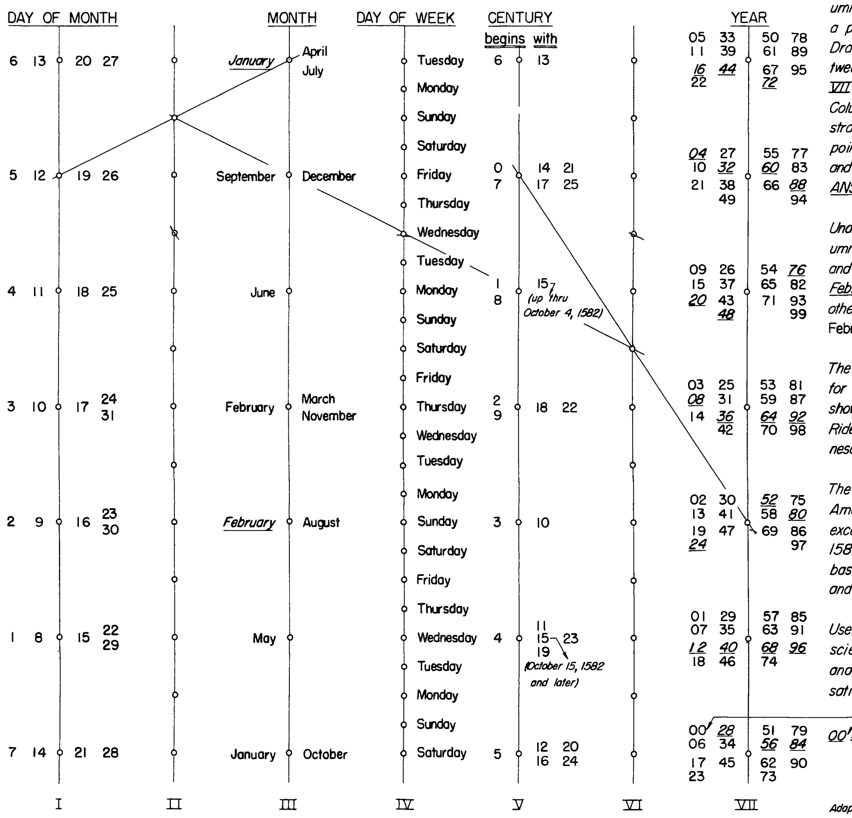


Figure 4–16.

THE DAY-OF-THE-WEEK FOR ANY DATE OF HISTORY BACK TO THE BIRTH OF CHRIST—*by CREPIN*



TO USE: Draw a straight line between Columns I and III, obtaining a point on Column II. Draw a straight line between Columns V and VII, obtaining a point on Column VI. Draw a straight line between the points of Columns II and VI, obtaining the ANSWER in Column IV.

•

Underlined years in Column VII are Leap Years and require January and February in Column III, otherwise use January and February.

•

The chart has been used for April 19, 1775 and shows that Paul Revere's Ride occurred on a Wednesday.

• • •

The chart represents American chronology except for the period 1582 to 1752. It is based upon the Julian and Gregorian Calendars.

• • •

Useful in banking, law, science, family records and after-dinner conversation.

• • •

00' ≡ for centuries 1–16, 20, 24.

Adapted by Douglas P. Adams

Figure 4–17.

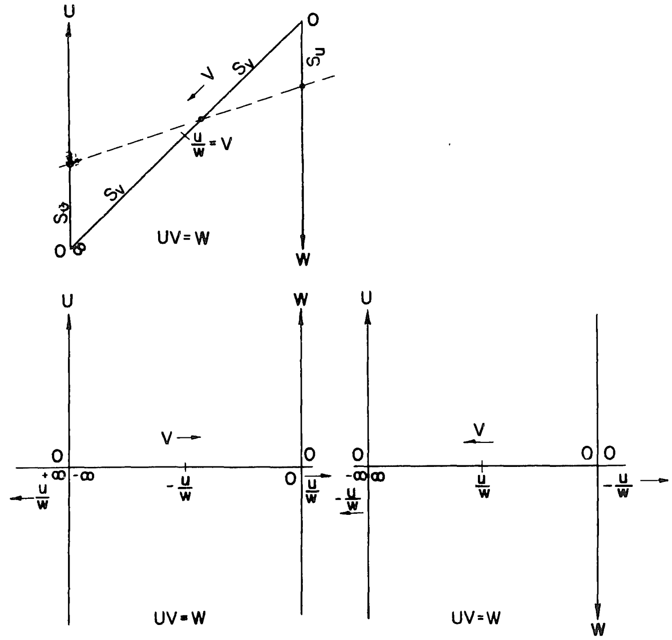


Figure 4–18.

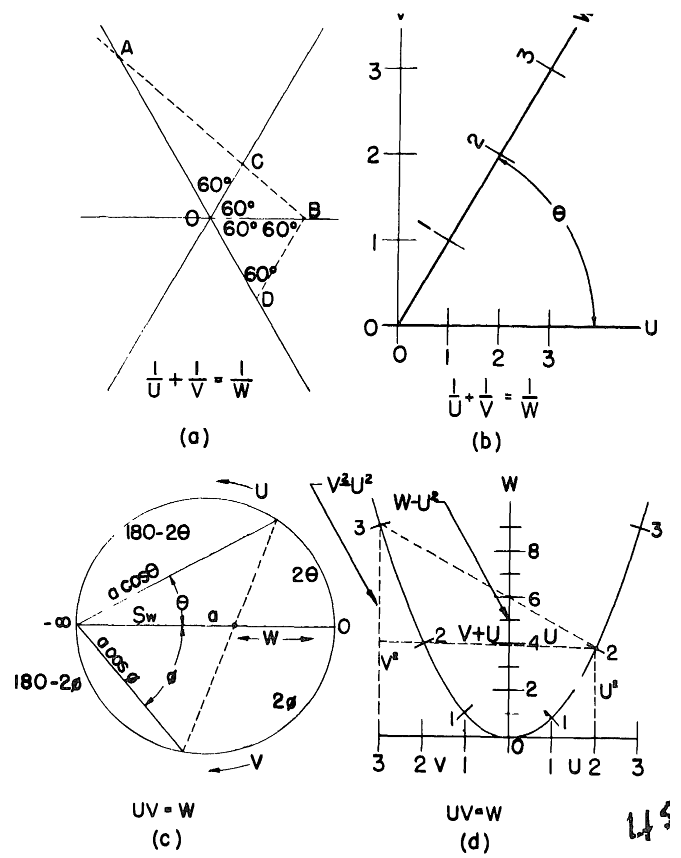
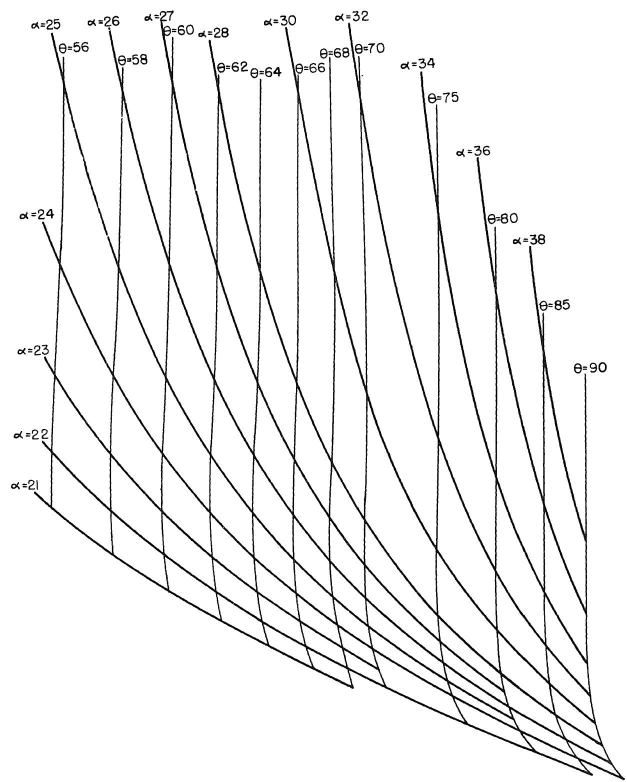
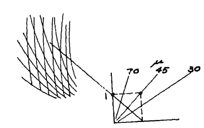


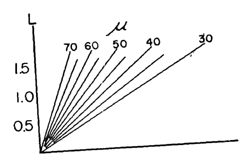
Figure 4–19.

NOMOGRAM FOR DETERMINATION OF α FOR CYCLOIDAL CAM





LET L=1 $\mu = 45°$ $\theta = 62°$ $\alpha = 27°$



CAM EQUATION:

$$K \tan \mu = \frac{2}{\sin K\alpha} - \frac{L\alpha}{(1-\cos K\alpha)} + \frac{K\alpha - \sin K\alpha}{(1-\cos K\alpha)}$$

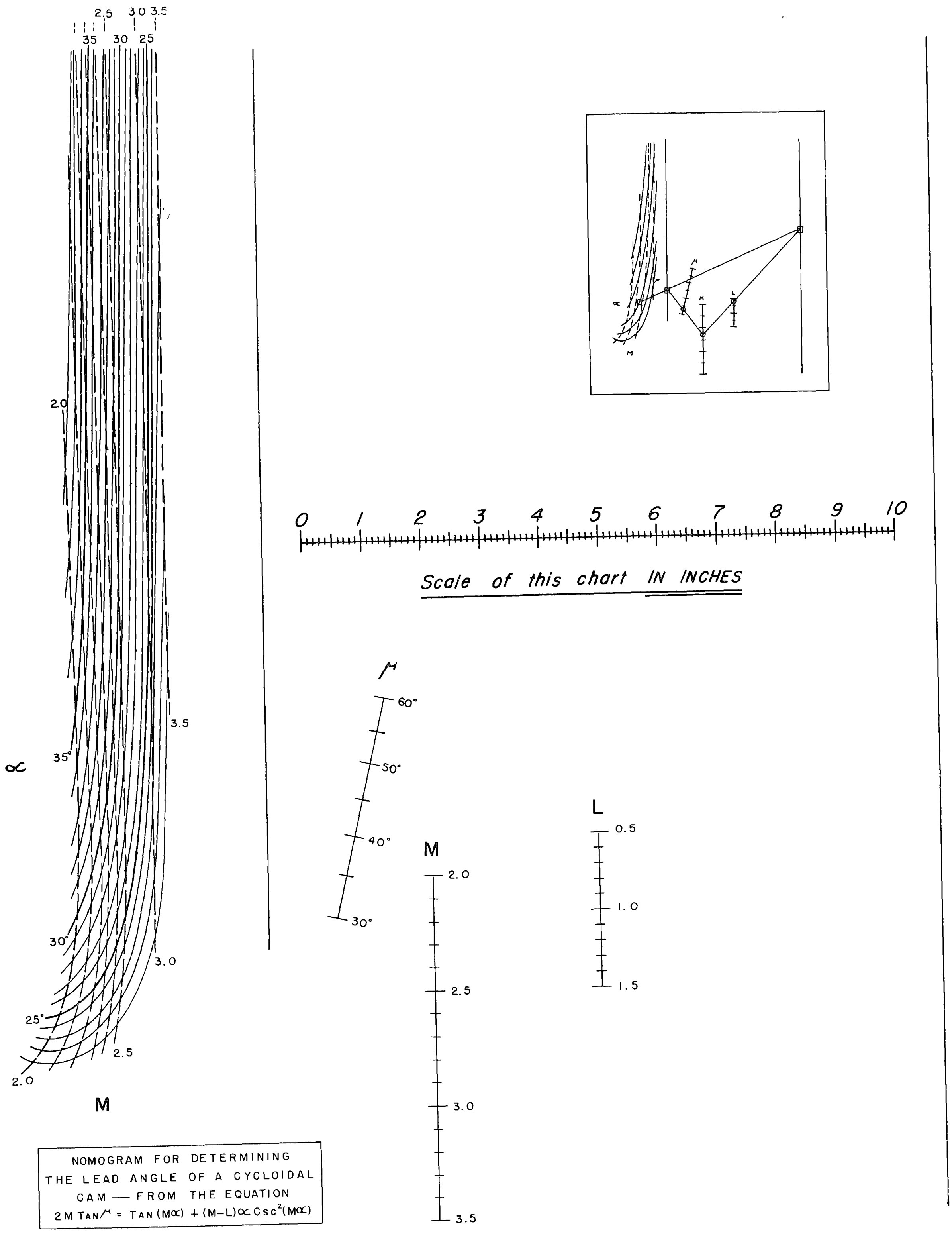
where $K = \frac{360}{\theta}$

Transformed To:

$$2M \tan \mu = \tan M\alpha + (M-L)\alpha \csc^2 M\alpha$$

where $M = \frac{180}{\theta}$

Figure 4–20.



2.5 30 3.5
35 30 25
2.0
3.5
α
35°
30°
3.0
25°
2.5
2.0
M
0 1 2 3 4 5 6 7 8 9 10
Scale of this chart IN INCHES
μ
60°
50°
40°
30°
M
2.0
2.5
3.0
3.5
L
0.5
1.0
1.5
NOMOGRAM FOR DETERMINING
THE LEAD ANGLE OF A CYCLOIDAL
CAM — FROM THE EQUATION
2M Tan μ = Tan (Mα) + (M−L)α Csc²(Mα)

Figure 4–21.

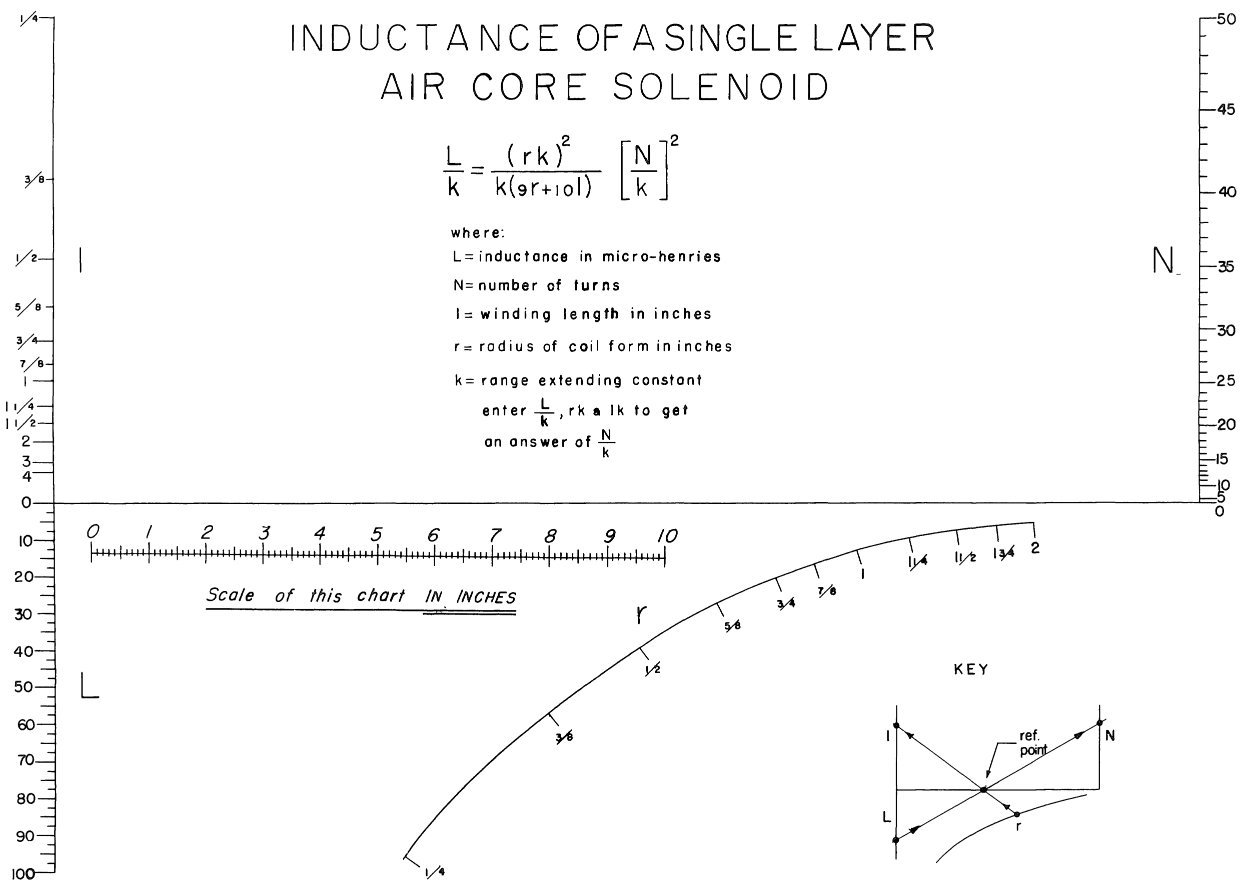
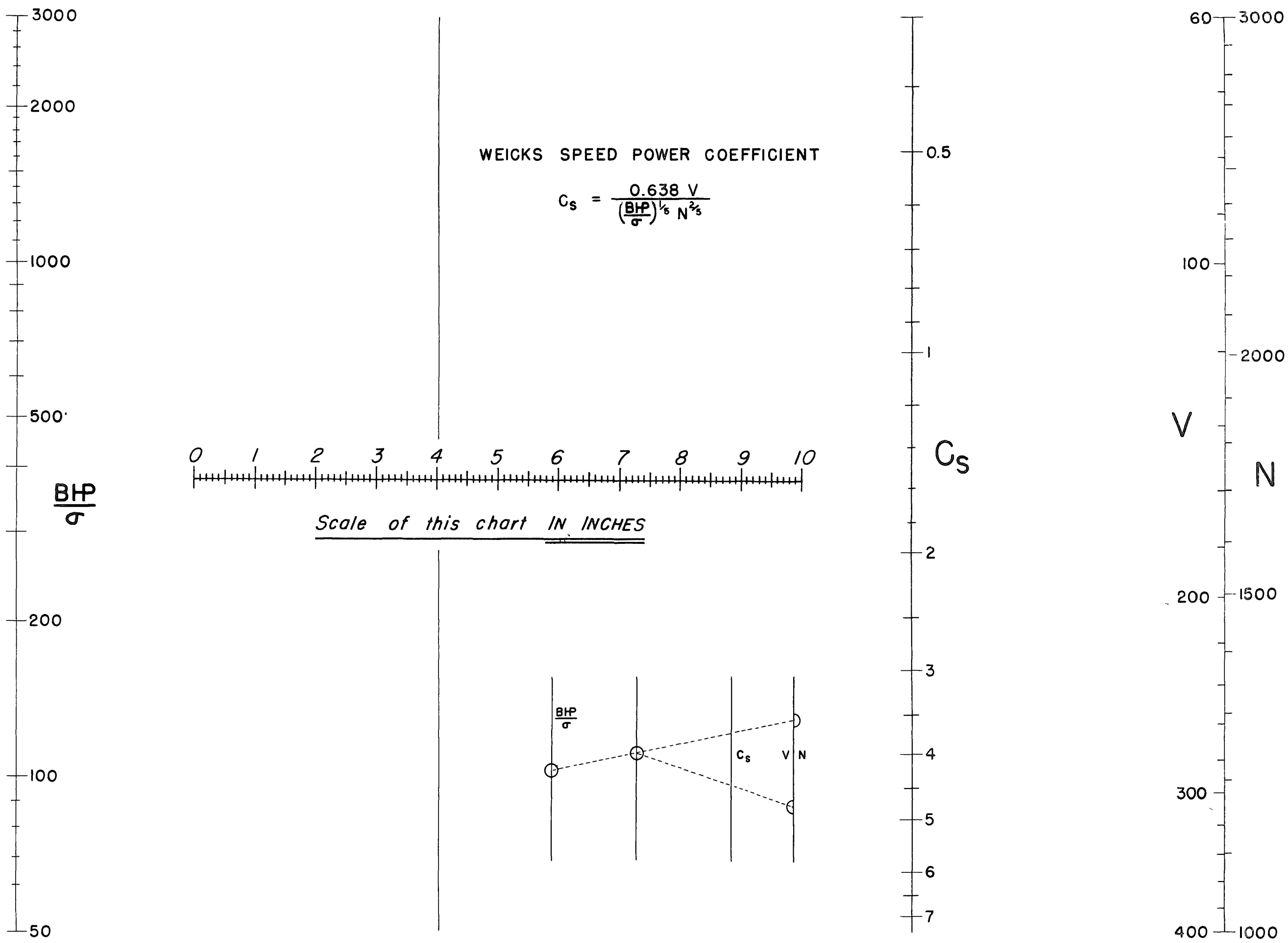

INDUCTANCE OF A SINGLE LAYER AIR CORE SOLENOID
$$\frac{L}{k} = \frac{(rk)^2}{k(9r+10l)} \left[\frac{N}{k}\right]^2$$
where:
L = inductance in micro-henries
N = number of turns
l = winding length in inches
r = radius of coil form in inches
k = range extending constant
enter $\frac{L}{k}$, rk & lk to get
an answer of $\frac{N}{k}$
l
N
L
r
Scale of this chart IN INCHES
KEY
ref. point

Figure 4–22.



WEICKS SPEED POWER COEFFICIENT
$C_s = \frac{0.638\ V}{\left(\frac{BHP}{\sigma}\right)^{1/5} N^{2/5}}$
$\frac{BHP}{\sigma}$
3000
2000
1000
500
200
100
50
0 1 2 3 4 5 6 7 8 9 10
Scale of this chart IN INCHES
C_s
0.5
1
2
3
4
5
6
7
V
60
100
200
300
400
N
3000
2000
1500
1000
$\frac{BHP}{\sigma}$
C_s
V
N

Figure 4–23.

THE GAIN OF AN AMPLIFIER

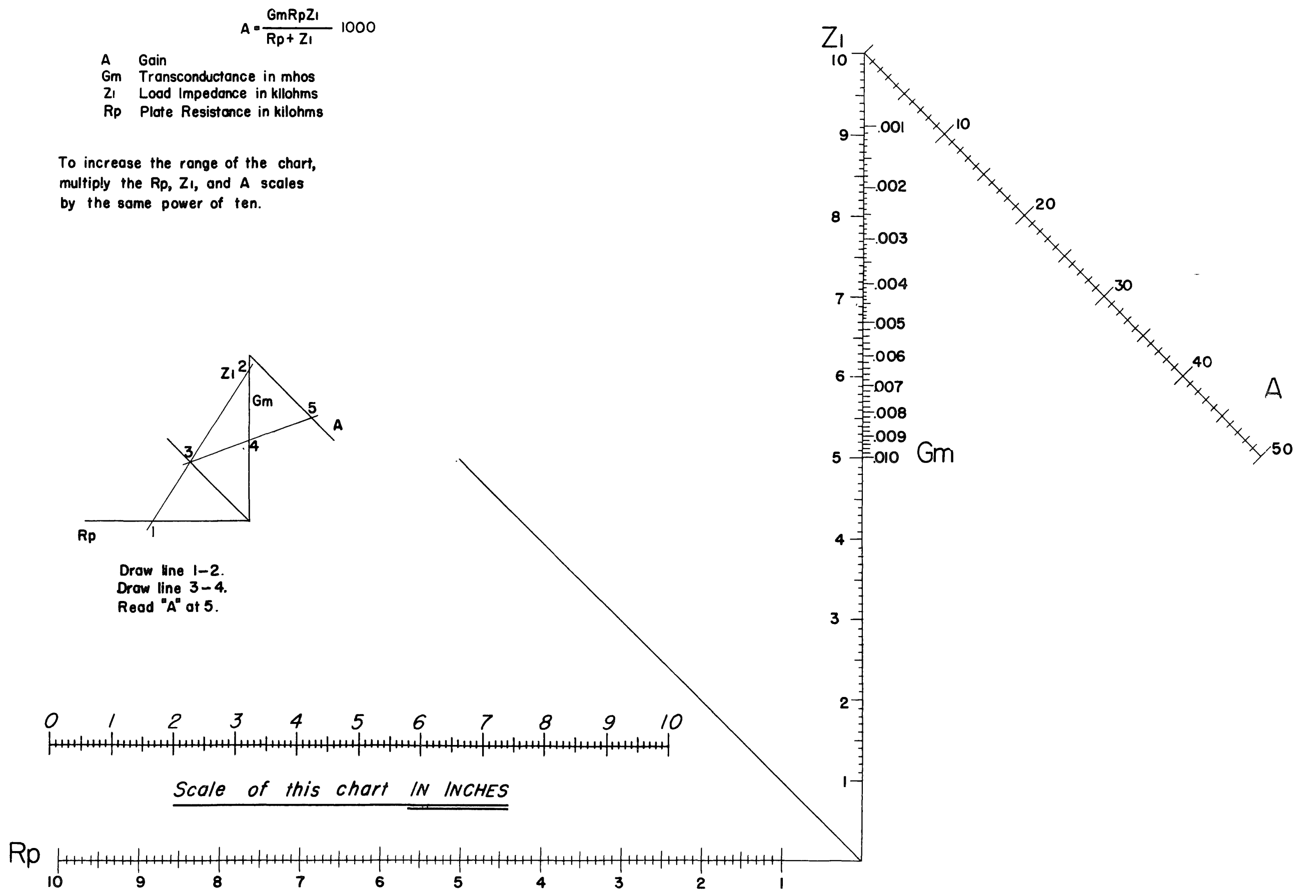


Figure 4–24.

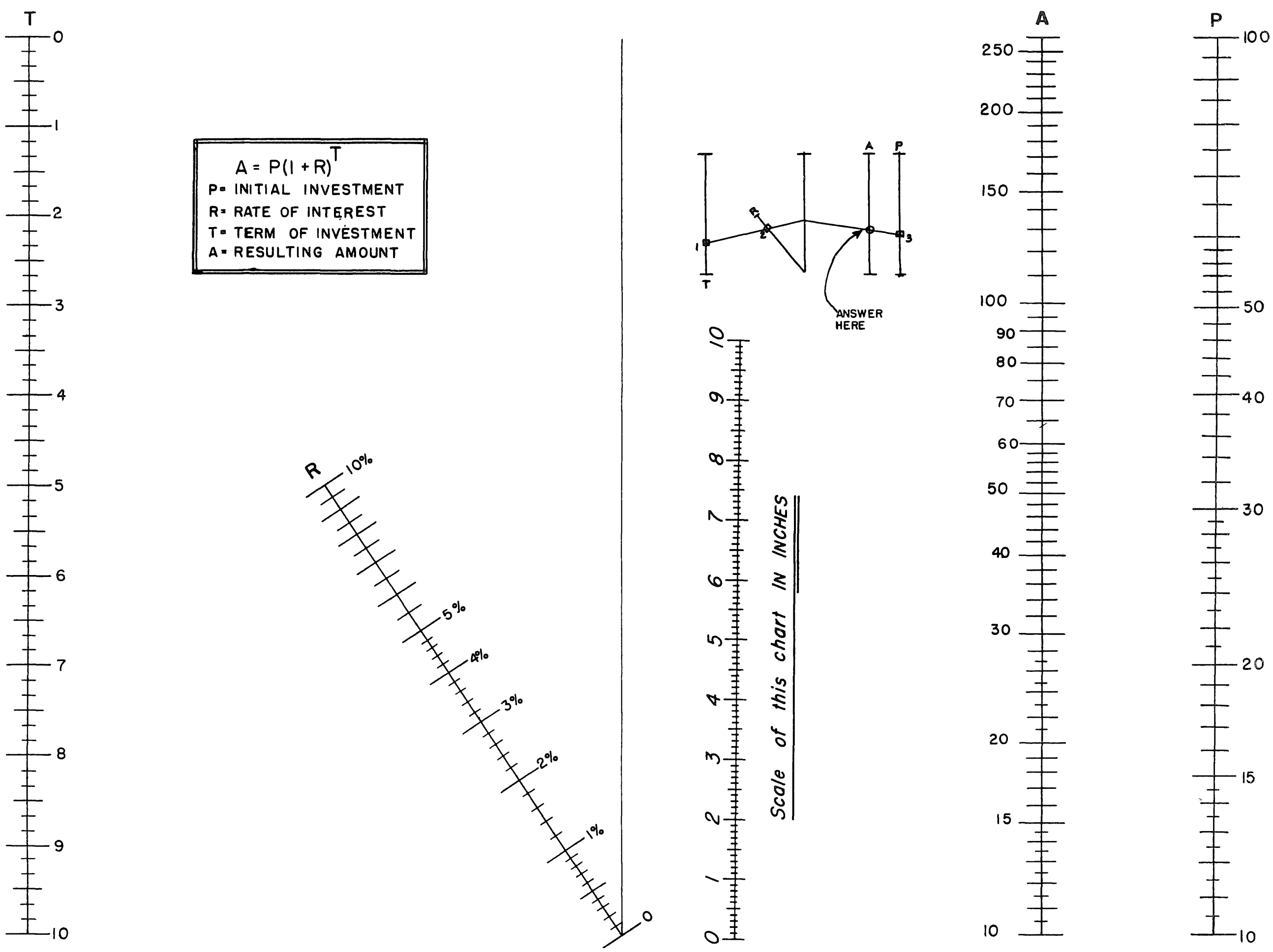

T
A
P
R
$A = P(1 + R)^T$
P= INITIAL INVESTMENT
R= RATE OF INTEREST
T= TERM OF INVESTMENT
A= RESULTING AMOUNT
ANSWER HERE
Scale of this chart IN INCHES
10%
5%
4%
3%
2%
1%
0

Figure 4–25.

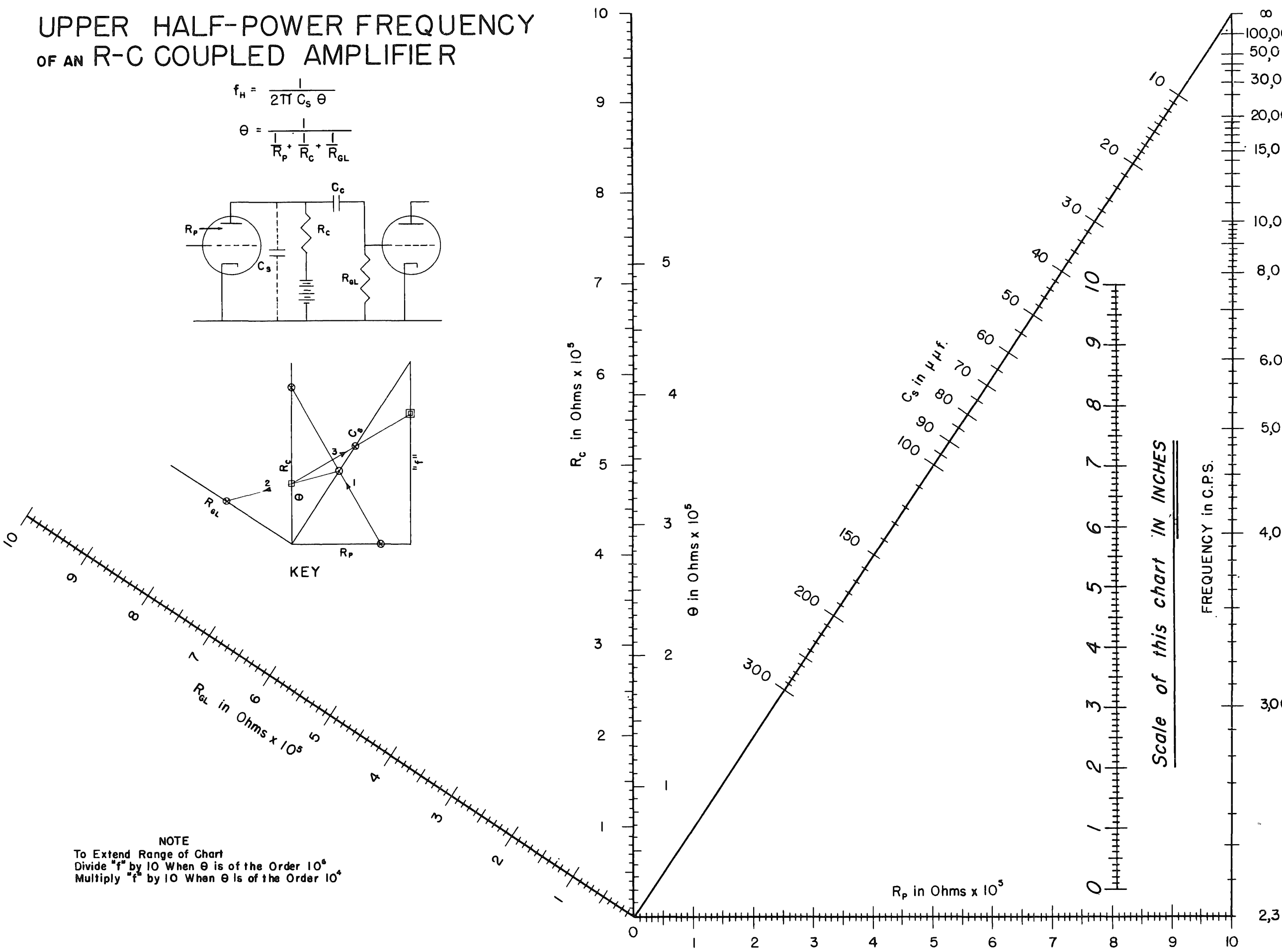

UPPER HALF-POWER FREQUENCY
OF AN R-C COUPLED AMPLIFIER
$f_H = \frac{1}{2\pi C_s \theta}$
$\theta = \frac{1}{\frac{1}{R_P} + \frac{1}{R_C} + \frac{1}{R_{GL}}}$
KEY
R_C in Ohms x 10^5
θ in Ohms x 10^5
C_s in $\mu\mu f$.
R_{GL} in Ohms x 10^5
R_P in Ohms x 10^3
FREQUENCY in C.P.S.
Scale of this chart IN INCHES
NOTE
To Extend Range of Chart
Divide "f" by 10 When θ is of the Order 10^6
Multiply "f" by 10 When θ is of the Order 10^4

Figure 4–27.

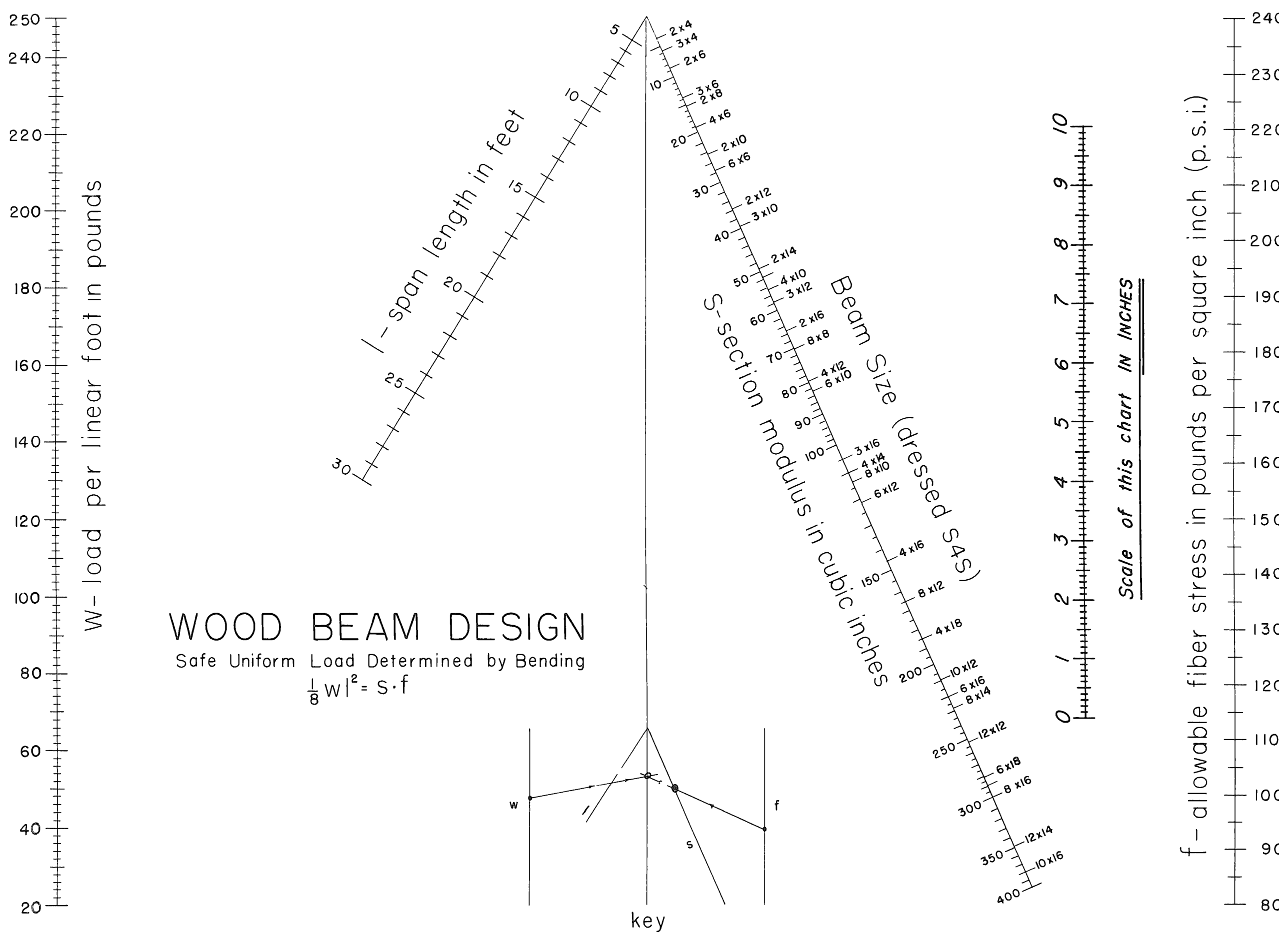

WOOD BEAM DESIGN
Safe Uniform Load Determined by Bending
$\frac{1}{8} w l^2 = S \cdot f$
W- load per linear foot in pounds
l - span length in feet
S-section modulus in cubic inches
Beam Size (dressed S4S)
Scale of this chart IN INCHES
f - allowable fiber stress in pounds per square inch (p. s. i.)
key
w
l
s
f
250
240
220
200
180
160
140
120
100
80
60
40
20
5
10
15
20
25
30
2x4
3x4
2x6
3x6
2x8
4x6
2x10
6x6
2x12
3x10
2x14
4x10
3x12
2x16
8x8
4x12
6x10
3x16
4x14
8x10
6x12
4x16
8x12
4x18
10x12
6x16
8x14
12x12
6x18
8x16
12x14
10x16
150
200
250
300
350
400
0
1
2
3
4
5
6
7
8
9
10
2400
2300
2200
2100
2000
1900
1800
1700
1600
1500
1400
1300
1200
1100
1000
900
800

Figure 4–26.

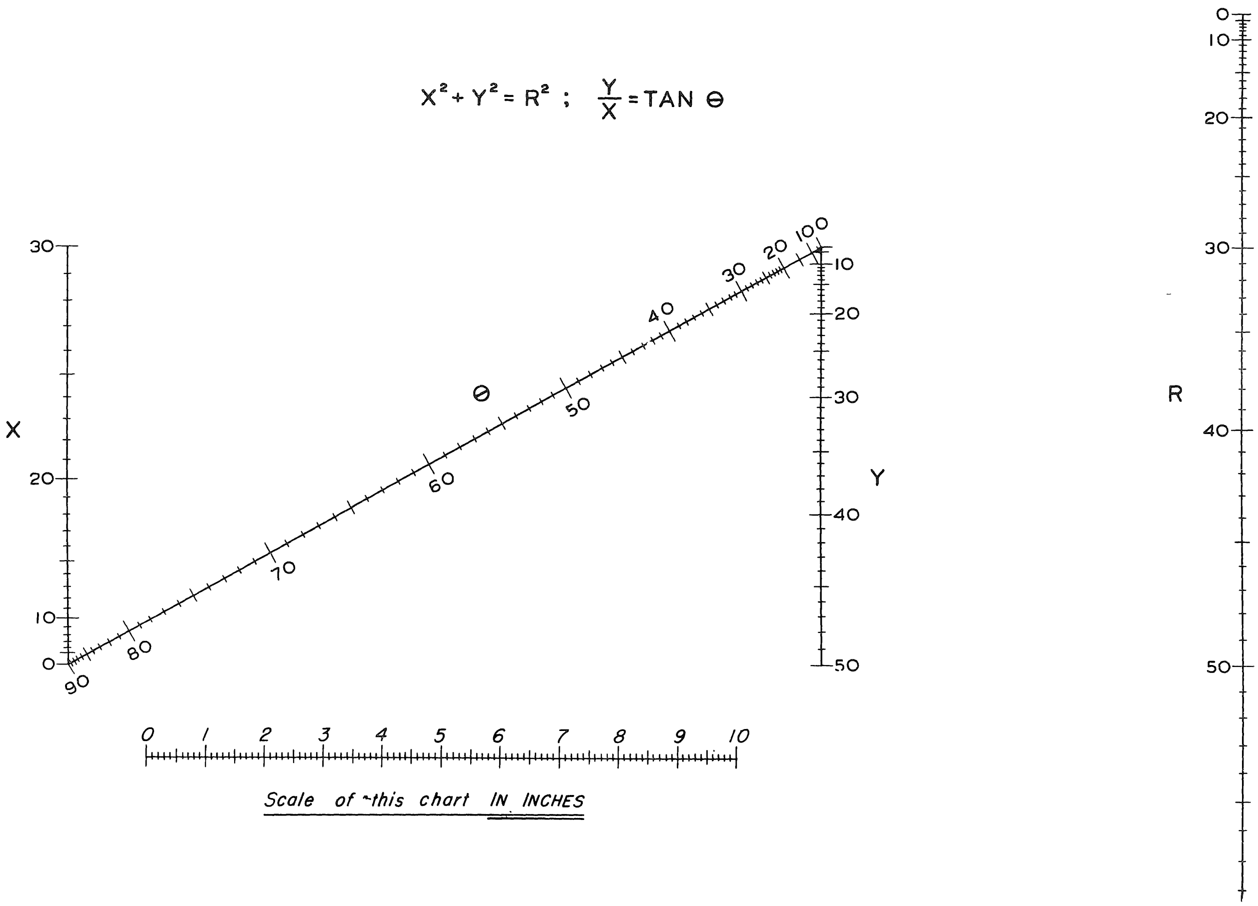
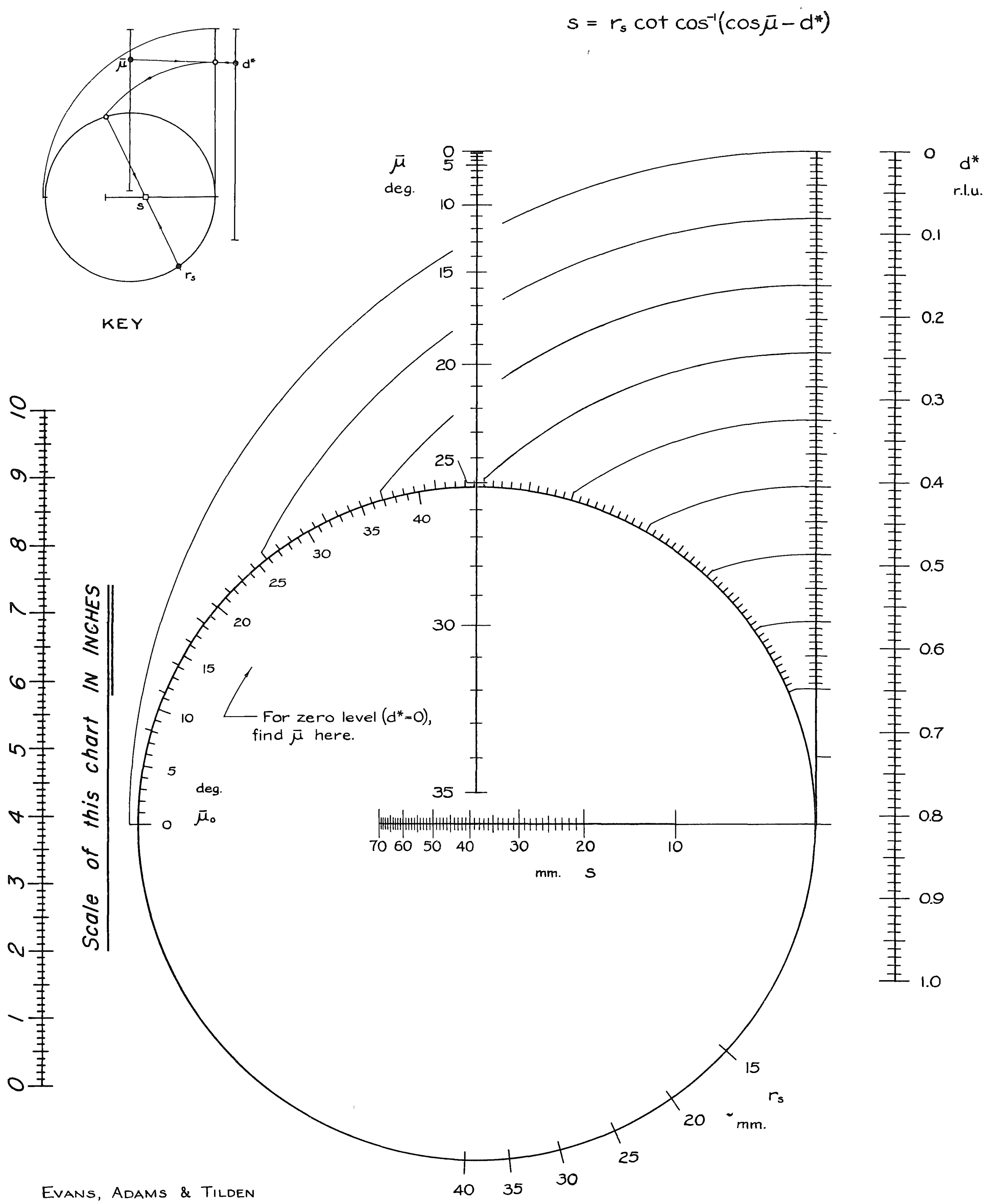

$X^2 + Y^2 = R^2$; $\frac{Y}{X}$ = TAN Θ
X
30
20
10
0
Θ
90
80
70
60
50
40
30
20
100
10
Y
R
Scale of this chart IN INCHES
0 1 2 3 4 5 6 7 8 9 10

Figure 4–28.

Nomogram for Setting the Buerger Precession Camera



Evans, Adams & Tilden
Section of Graphics
Massachusetts Institute of Technology

Figure 4–29.

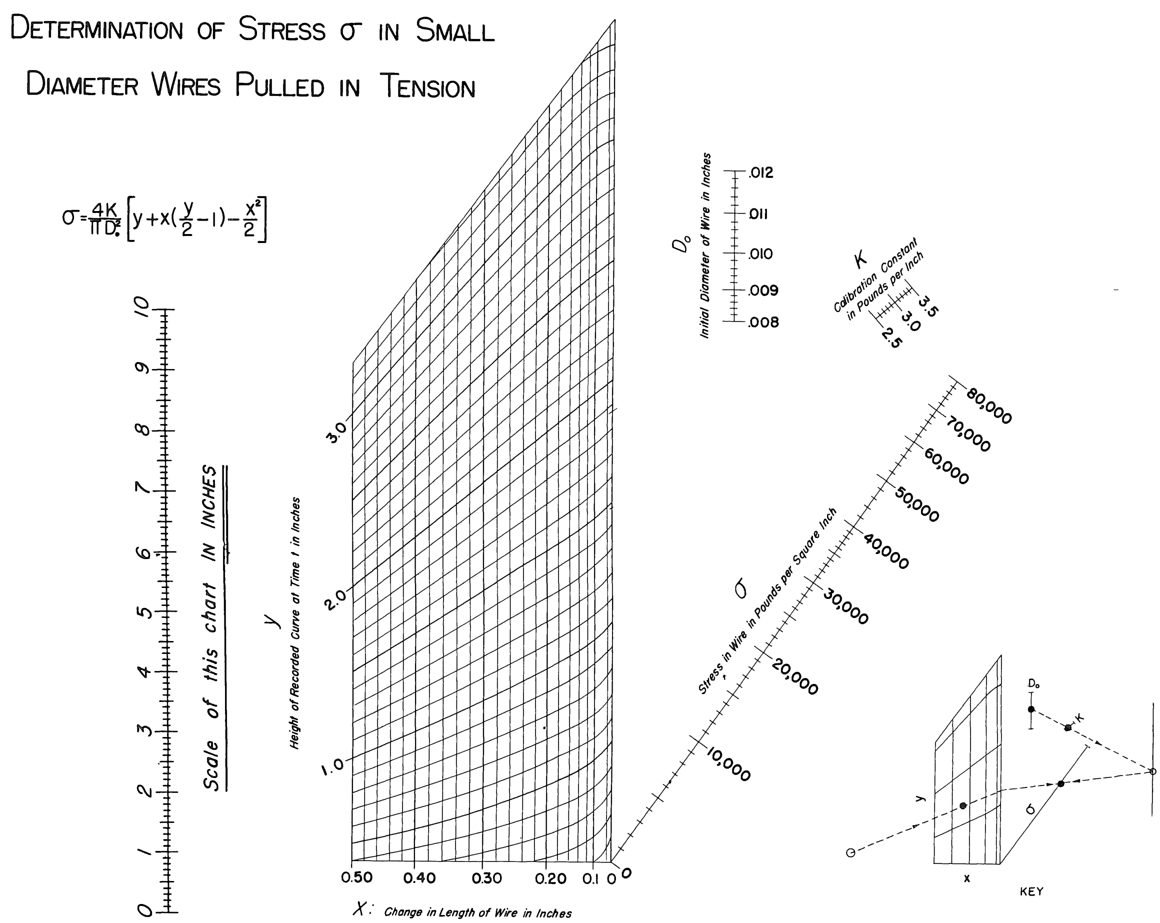
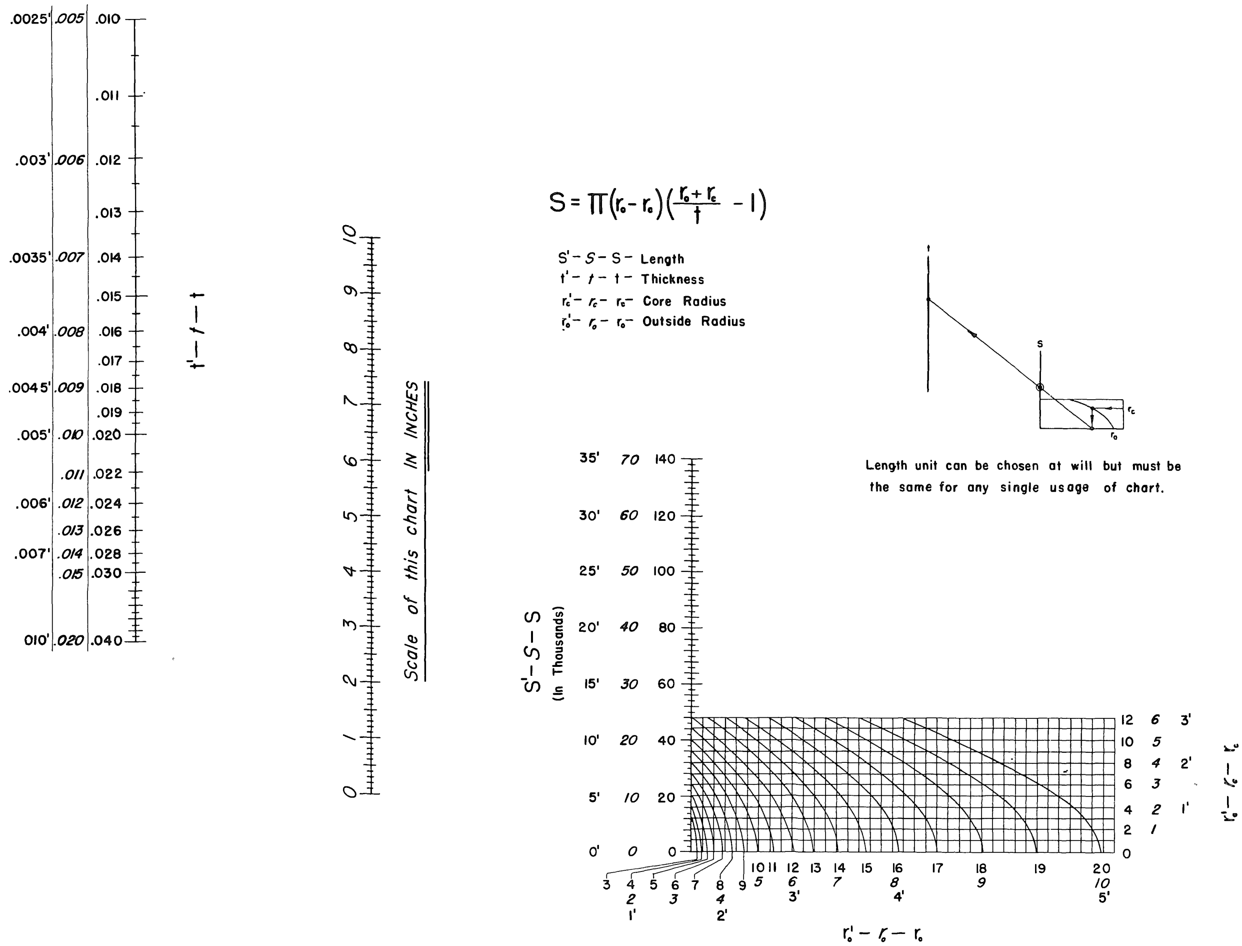


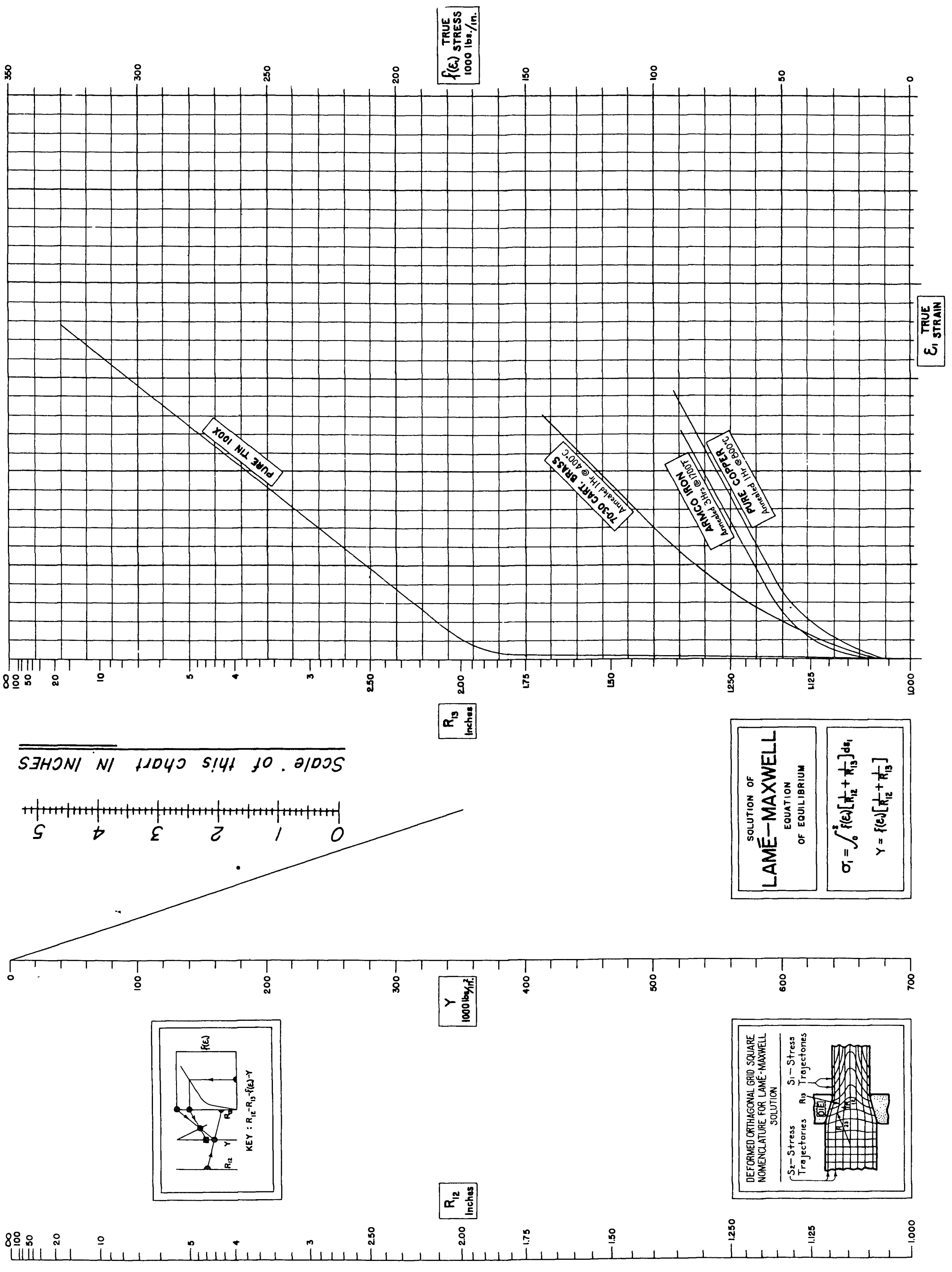
Figure 4–30.

Page 74, Figure 4-30. Insert a vertical straight line along the righthand side of the page, 20 scaled inches from the key circle in the lower lefthand corner. The use of this line is indicated in the key and is fundamental in the use of the chart.



S = π(r₀ - r꜀)((r₀ + r꜀)/t - 1)
S' - S - S - Length
t' - t - t - Thickness
r꜀' - r꜀ - r꜀ - Core Radius
r₀' - r₀ - r₀ - Outside Radius
Length unit can be chosen at will but must be the same for any single usage of chart.
Scale of this chart IN INCHES
t' — t — t
S' - S - S
(In Thousands)
r₀' — r₀ — r₀
r꜀' — r꜀ — r꜀

Figure 4–31.



f(ε) TRUE STRESS 1000 lbs./in.
ε TRUE STRAIN
PURE TIN 100X
70-30 CART. BRASS
PURE COPPER
ARMCO IRON
R13 Inches
Scale of this chart IN INCHES
SOLUTION OF LAMÉ-MAXWELL EQUATION OF EQUILIBRIUM
Y 1000 lbs/in²
R12 Inches
DEFORMED ORTHOGONAL GRID SQUARE NOMENCLATURE FOR LAMÉ-MAXWELL SOLUTION

Figure 4–32.

CHAPTER 5

THREE-DIMENSIONAL NOMOGRAMS*

5–1. *Three-Dimensional Nomogram Theory.* The method used to construct two-dimensional "alignment" charts, or nomograms, can be extended for three-dimensional "coplanar" diagrams. The theories for the two types of charts are parallel and are shown side by side below:

Three points are in alignment when:

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Assume that a function

$$F(U, V, W) = 0$$

can be put in the form

$$\begin{vmatrix} U_1 & U_2 & 1 \\ V_1 & V_2 & 1 \\ W_1 & W_2 & 1 \end{vmatrix} = 0$$

where U_1, U_2 are functions of U only, and similarly for V and W. Interpret U_1, U_2 as the X and Y coordinates of a plane curve. Then the curve can be plotted on X, Y axes and calibrated in U. Similarly for V and W. Values of U, V and W which satisfy the original equation also satisfy the determinant and hence, are "aligned."

Four points are coplanar when:

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0 \qquad (5\text{–}1)$$

Assume that a function

$$F(U, V, W, T) = 0$$

can be put in the form

$$\begin{vmatrix} U_1 & U_2 & U_3 & 1 \\ V_1 & V_2 & V_3 & 1 \\ W_1 & W_2 & W_3 & 1 \\ T_1 & T_2 & T_3 & 1 \end{vmatrix} = 0 \qquad (5\text{–}2)$$

where U_1, U_2, U_3 are functions of U only, and similarly for V, W, and T. Interpret U_1, U_2, U_3 as the X, Y, and Z coordinates of a space curve. Then the curve can be plotted on X, Y, Z axes and calibrated in U. Similarly for V, W and T. Values of U, V, W and T which satisfy the original equation satisfy the determinant and hence, are "coplanar."

If the equation for which the chart is made is F(U, V, W, T) = 0, values of any three variables (such as U, V, and W) will determine a plane and the value of the fourth variable, which in this instance is T, is found at the point where this plane intersects the T-scale.

Example 5–1. The equation

$$P = \frac{bh}{H - h} \qquad (5\text{–}3)$$

can be placed in the form ((B))

$$\begin{vmatrix} 0 & 1 & 0 & P \\ -1 & 1 & 0 & -b \\ 0 & H & 1 & 0 \\ h & 0 & 1 & 0 \end{vmatrix} = 0 \qquad (5\text{–}4)$$

*Based on an article in *Product Engineering,* August, 1955.

and then in the canonical form ((C))

$$\begin{vmatrix} 0 & 1 & P & 1 \\ -1 & 1 & -b & 1 \\ 0 & \dfrac{H}{1+H} & 0 & 1 \\ h & 0 & 0 & 1 \end{vmatrix} = 0 \qquad (5\text{–}5)$$

Figure 5–1 shows a pictorial sketch for this canonical form complete with graduated scales. For P = 3/4, b = 1/2, h = 1, (5–3) gives H = 5/3, which checks with the Figure 5–1.

In Figure 5–1, joining P, b and h values has pictured the plane of the solution. To find where this plane cuts the H scale, a point X_3 where line Pb cut the XY plane was marked, joined to h, and the line observed to cut the H scale at the answer. This shows how a three-dimensional nomogram can sometimes break down into a compound, two-dimensional one, for these two collineations are a direct steal from the treatment given this equation in Problems 3–1 to 3–5. From (3–36),

$$\begin{vmatrix} 0 & P & 1 \\ H & -b & 1 \\ h & 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & P & 1 \\ -1 & -b & 1 \\ -X_3 & 0 & 1 \end{vmatrix} = 0 \qquad (5\text{–}6)$$

$$X_3 = \frac{h}{H} \qquad (5\text{–}7)$$

Equations (5–6) and (5–7) are the bases of compound two-dimensional diagrams in their respective planes—similar to the treatment given Problems 3–3 and 3–4. It is interesting to note that Figure 5–1, a pictorial representation of a three-dimensional nomogram together with the method of solution, can be used on the page *as it stands* as a compound, two-dimensional diagram. The significance of this will be discussed later.

Space diagrams have been suggested in the past and drawn pictorially, but they do not seem to have been used for harder problems because of the difficulty of handling space figures on the two-dimensional page. This can be overcome with the aid of such descriptive geometry techniques as now follow.

5–2. *The Method of Auxiliary Line and Parallel Join.* Here is a technique that can be used effectively when the four variables have straight, vertical scales as shown in Figure 5–2. Although the equation for this illustration is simple, U + V + W = T, more difficult problems can also be solved by the same method, as shown in Section 5–5.

Considering first an equation that can be solved with a two-dimensional alignment diagram, it has been shown that U + V = W can be put in the determinant form (2–22)

$$\begin{vmatrix} 0 & uU & 1 \\ G & vV & 1 \\ \dfrac{G}{u+v} & \dfrac{uvW}{u+v} & 1 \end{vmatrix} = 0 \qquad (5\text{–}8)$$

Coordinate Interpretation:

X Y

where u, v, and G are constants.

In similar manner, the equation of Figure 5–2, U + V + W = T, can be put in determinant form and given a coordinate interpretation.

$$\begin{vmatrix} 0 & 0 & uU & 1 \\ G & 0 & vV & 1 \\ 0 & K & wW & 1 \\ \dfrac{uwG}{uv+vw+wu} & \dfrac{uvK}{uv+vw+wu} & \dfrac{uvwT}{uv+vw+wu} & 1 \end{vmatrix} = 0 \qquad (5\text{–}9)$$

Coordinate Interpretation:

X Y Z

where u, v, w, G and K are constants with roles like those in (2–22).

A space nomogram for this determinant is pictured in Figure 5–2 for u = 2, v = 1, w = 3, G = 5, and K = 5. Hence the T-scale is located at X = 2.731, Y = 0.9; and the scale factor of the T-scale is 0.546. In Figure 5–2(b), a solution plane is shown for U = 3, V = 2, and W = 1. Pictorially, it appears that this plane cuts the T-scale at T = 6. Figure 5–2(c) shows pictorially a descriptive geometry "cutting plane" device for finding precisely the intersection of the T-scale with such a plane. The cutting plane α has been passed through the T-scale parallel to the plane of the V and W scales, cutting the UW plane (Y, Z plane) in the vertical line T′. The orthographic projection of the entire space figure onto the U, W, (Y, Z) plane is now used. Here the line T″V″p is parallel to the known line WVp, also Vp = V and the solution takes the form shown in Figure 5–2(d). A conventional double-alignment diagram (Section 4–6) could also solve this particular equation, but the method shown here will be useful later.

5–3. *The Method of Numbered Line Pairs.* This is a second general method that can be used effectively. Figure 5–3(a) shows pictorially a three-dimensional nomogram for a function of U, V, W and T where the U, V and W scales are vertical, straight lines at three corners of a rectangle and the T-scale is a helix with a horizontal axis.

Projections of this curve on the two coordinate planes are shown. The projection on the YZ plane is defined by the combined behavior of the Y and Z functions in the determinant form of the equation. Since Y = T + 2, T = Y − 2, $(Y - 2)^2 + (Z - 2)^2 = (1.5)^2$ and the YZ projection is a circle, center at (2, 2), radius 1.5. In the X, Y plane, Y = T + 2, T = Y − 2, Y − 2 = 1.5 sin (X − 0.75) and the X, Y projection is a sine curve of period two units, amplitude 1.5 units, zero point (0.75, 2). Graduations of T will appear on this curve according to the equation T = Y − 2.

Values of U, V and W could be assigned such that the plane determined by them would not cut the T-helix at all, indicating there is no solution in T for these values of U, V, W. Or, a plane cutting the helix and lying parallel to its axis could determine an infinite number of solutions. In Figure 5–3(a), the values of U = 7.20, V = 11.20, W = 5.15 are shown determining a solution plane. This plane cuts the helix cylinder in an ellipse, helix and ellipse meeting at points where the plane cuts the helix—at values of T which are solutions. The boldline portions of the helix lie above the plane and end at the ellipse—at the solution T-values. (Calibrations have been shown only for these bold portions of the T-curve.) There are actually some fourteen intersections, seven of which are shown. They certainly could not be found readily by trial and error.

With descriptive geometry, it is possible to represent this space diagram and find its solution by means of the projections of these curves as in Figure 5–3(b). This shows the YZ and XY projections of the T-curve previously discussed and also twelve numbered horizontal section planes on edge in the circular view. Their parallel lines of intersection with the cylinder are shown in the lower view, numbered correspondingly. A horizontal plane through U = 7.20 will cut a horizontal masterline from the U, V, W solution plane and is shown by its two projections in Figure 5–3(b) (section line). Each of the twelve horizontal section planes cuts lines parallel to this from the U, V, W plane which can be drawn in the lower view and numbered as soon as the masterline has been established. Then in the lower view, a line from the cylinder and a line from the plane with the same number are coplanar and determine a point common to both the U, V, W plane and the cylinder—namely a point on the desired ellipse. The projected ellipse will cut the projected T-curve at values of T as shown, which are solutions of the given equation.

Example 5–2. Find and check a solution to the equation for which the chart of Figure 5–3(b) was made. (In Figure 5–3(a), one of these solutions appears pictorially to lie in line with W = 2.) Referring to this one in Figure 5–2(b), its value is read as T = −0.55. When this is put into the determinant of Figure 5–3(a), the following values result which check quite well.

$$\begin{vmatrix} 5 & 0 & 3.60 & 1 \\ 0 & 5 & 1.05 & 1 \\ 0 & 0 & 5.15 & 1 \\ 1.88 & 1.45 & 3.40 & 1 \end{vmatrix} = 0$$

5–4. *Method of Doubly-Indexed Scales.* The third descriptive geometry technique involves calibrating each nomographic scale in a given orthographic view, both in the value of its variable and also (along the other side of the scale stem) in its space coordinate perpendicular to the plane of that view. Let an equation be given by

$$\begin{vmatrix} U & 5 + \frac{1}{5}(U - 5)^2 & U & 1 \\ V + 1 & V & 2V & 1 \\ W - 1 & 10 - \frac{1}{10}(W - 10)^2 & W & 1 \\ 10 - T & T & T & 1 \end{vmatrix} = 0 \quad (5\text{–}10)$$

Coordinate Interpretation:

X Y Z

In Figure 5–4 two views are given of U, V, W and T curves for the above equations. V and T have straight scales, the other two are curves. In the upper (front, YZ) view, a vertical calibration is the value of the variable at that point, the inclined calibration is the X-coordinate of the curve at that point. Now the X, Y view can be discarded.

Example 5–3. If U = 2, V = 3, W = 6, what value does T have? The values of U, V, and W have been circled and joined to indicate a solution plane. Several methods are available for finding that value of T where the T-scale is cut by the plane. For example, an "auxiliary view" which shows the plane on edge will show it cutting the T-scale at this answer value. Such a view could be found in the direction of "constant X-value" of the solution plane because this direction is parallel to the Y, Z plane. In the present case, for the plane V = 3, X = 4; U = 2, X = 2; W = 6, X = 5, one finds by proportional division the point on line U = 2, W = 6, such that X = 4. Joining this point with V = 3, X = 4, gives the *direction* of the strategic edge view. In the auxiliary view taken in this direction (the double arrow) the "levels" represent values of X and can be chosen to any convenient scale. In this view, the U, V, W plane is seen on edge cutting the T curve at the answer-value of T, point "Q", where one reads T = 5.75. If this value is placed along with the others in (5–10), one has

$$\begin{vmatrix} 2 & 5 - \frac{9}{5} & 2 & 1 \\ 4 & 3 & 6 & 1 \\ 5 & 10 - \frac{16}{10} & 6 & 1 \\ 4.25 & 5.75 & 5.75 & 1 \end{vmatrix} = 0$$

which checks out very closely.

5–5. *Applications.* To illustrate the application of the first technique, Section 5–2, consider the following fifth degree equation

$$x^5 + A''x^4 + A'x^3 + B'x^2 + C'x + D' = 0. \quad (5\text{–}11)$$

This can have the fourth power of x removed by reducing the roots by a real value, when it becomes

$$x^5 + Ax^3 + Bx^2 + Cx + D = 0. \quad (5\text{–}12)$$

This can be put in the canonical nomographic form

$$\begin{vmatrix} 0 & 0 & -A & 1 \\ 15 & 0 & -B & 1 \\ 0 & 15 & -C & 1 \\ \frac{15x^2}{x^3 + x^2 + x} & \frac{15x}{x^3 + x^2 + x} & \frac{(D + x^5)}{x^3 + x^2 + x} & 1 \end{vmatrix} = 0 \quad (5\text{–}13)$$

Coordinate Interpretation:

X Y Z

where A, B, C range from −10 to 10 and the chart is 15 units wide and 20 units high. The reader should check the determinant by expansion. The A, B, C scales are uniform and lie upon straight, vertical lines. For constant x, D enters only into Z, so a constant x-value is represented by a vertical line with known X, Y coordinates graduated uniformly in D. Taken together, these x-lines form a vertical, cylindrical surface whose X, Y projection has the equation

$$X = \frac{15x^2}{x^3 + x^2 + x}; \qquad Y = \frac{15x}{x^3 + x^2 + x} \quad (5\text{–}14)$$

This is a smooth, convex curve in the X, Y plane. Curves of constant D wind over the vertical cylindrical surface rising from it. When values of A, B, C, D are specified and x sought, a plane in space is determined by the first three which cuts the cylindrical surface in a smooth curve. Each intersection of the letter with the specified D-curve on that surface lies on a vertical x answer line. Figure 5–5 shows the two-dimensional representation and use of this nomogram. It is based on Figure 5–2(d) which was: (1) a projection of the space scales onto the UW plane, plus (2) a T′ line derived from the T scale-stem by a section plane α parallel to the VW plane. Correspondingly in the present case: (1) the entire space figure is projected onto the AC (Y, Z) plane, and (2) each x-line is carried into an x′-line by a section plane passing through it and parallel to the plane of scales B and C. Each pair of x (solid) and x′ (dotted) lines is correlated by being given the same serial numbers (vertical and inclined respectively), and is then used exactly as the pair of T and T′ lines were in Figure 5–2(c), (d). Variable A of Figure 5–5 corresponds to U, B to V, C to W, and X to T. Each point on an x-line obtained this way gives one point on a curve, namely on the curve of intersection of the x, D cylinder and the A, B, C plane as projected on the AC plane (YX-plane). Wherever this projected curve cuts the projected D curve, there lies a vertical solution-line in x. In the working nomogram, this projected curve of the intersection always lies within the sector ACB, passing from C to A and arching toward B.

Practical use of the chart is quite rapid because the only portion of the projected intersection curve which needs to be drawn is that which stands some chance of cutting the D-curve. This portion can be identified by inspection. The chart is made for positive roots of the quintic but yields negative roots on reversal of the signs of B and D. Multiple roots are indicated by tangency of the intersection curve with the D-line. Horner's or Newton's methods are natural complements of the chart getting their basic root values from it before refining them.

The chart is applicable to fourth degree equations for if a fourth degree equation has the third power missing, it can be multiplied by $x = 0$ to become a reduced fifth degree equation for which this chart is made in which the fourth power is missing and $D = 0$.

Example 5–4. Given equation, Figure 5–5, to find the roots of

$$x^5 - 3x^3 + 4x^2 + 8x - 1 = 0 \qquad (5\text{–}15)$$

On establishing lines AC and BC, it is clear from the path of $D = -1$ that there will be only one real, positive root. Two or three points establish as much of the curve as is necessary and the root $x = 0.13$ (refined value $x = 0.1294$).

Reversing signs of B and D, the AC line remains the same and curve $D = +1$ cuts through a limited portion of the new sector. A few points show that the projection of the intersection curve will not cut $D = +1$. Hence there are no negative roots.

PROBLEMS

PROBLEM 5–1. Place the equation $P = \dfrac{bh}{H - h}$ in a four-rowed, determinant canonical form ((C)) different from (5–5). Make a pictorial sketch to show that the diagram works. Derive any two-dimensional compound diagrams to which it may be equivalent.

PROBLEM 5–2. Derive the equations for a three-dimensional diagram for the complete quartic. Sketch it and show that it works.

PROBLEM 5–3. Verify the real roots of the quintic by using Figure 5–6.

$$x^5 - 9x^3 + 10x^2 - 3x + 0.2 = 0$$

Improve them by Horner's or Newton's method. Check by showing that the sum of the roots is zero and the product is -0.200.

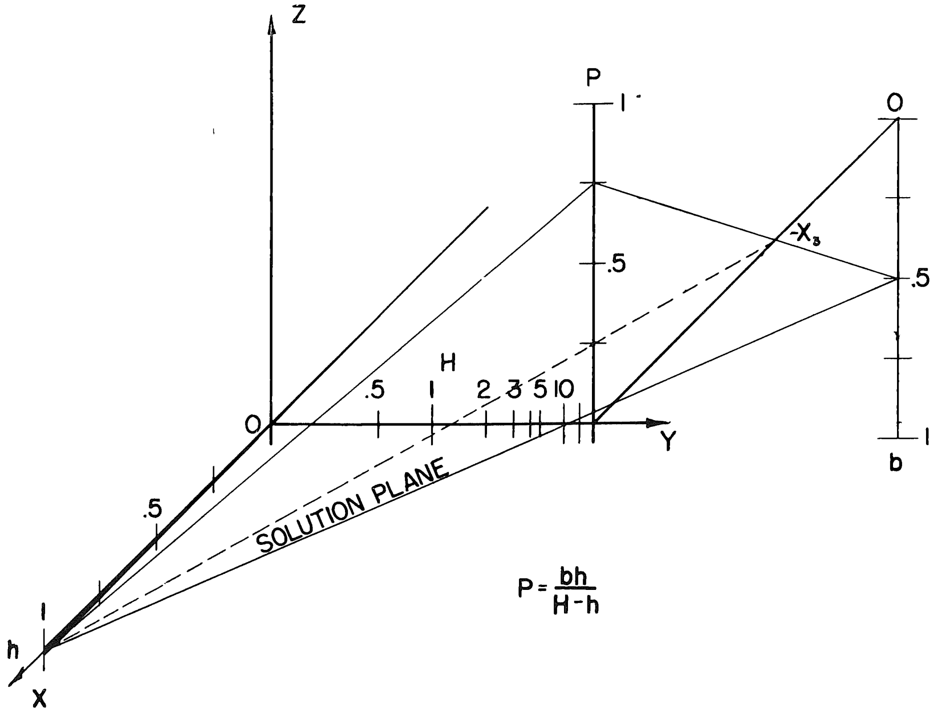


Figure 5–1.

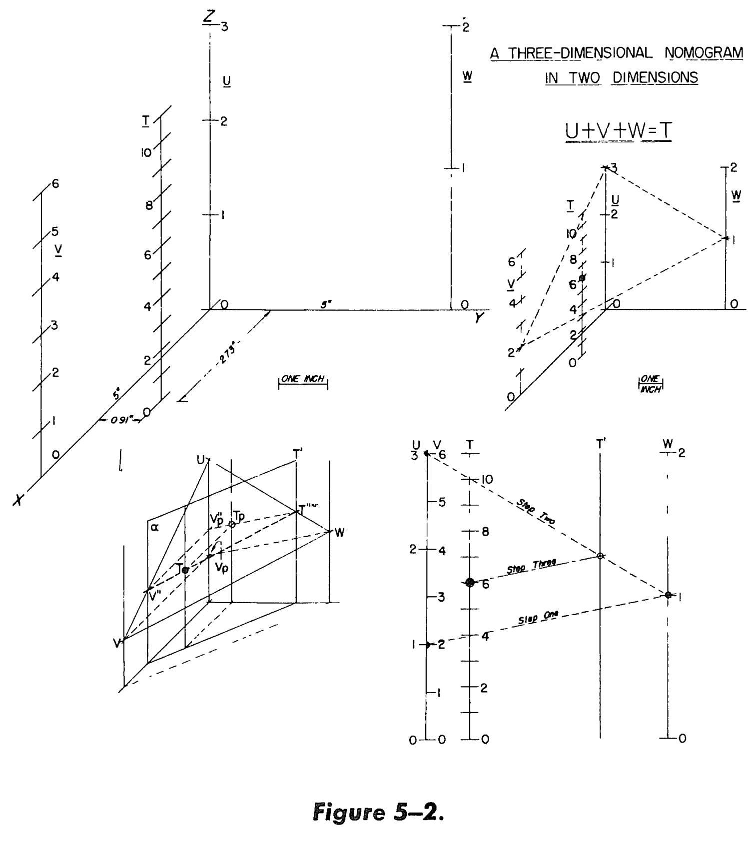
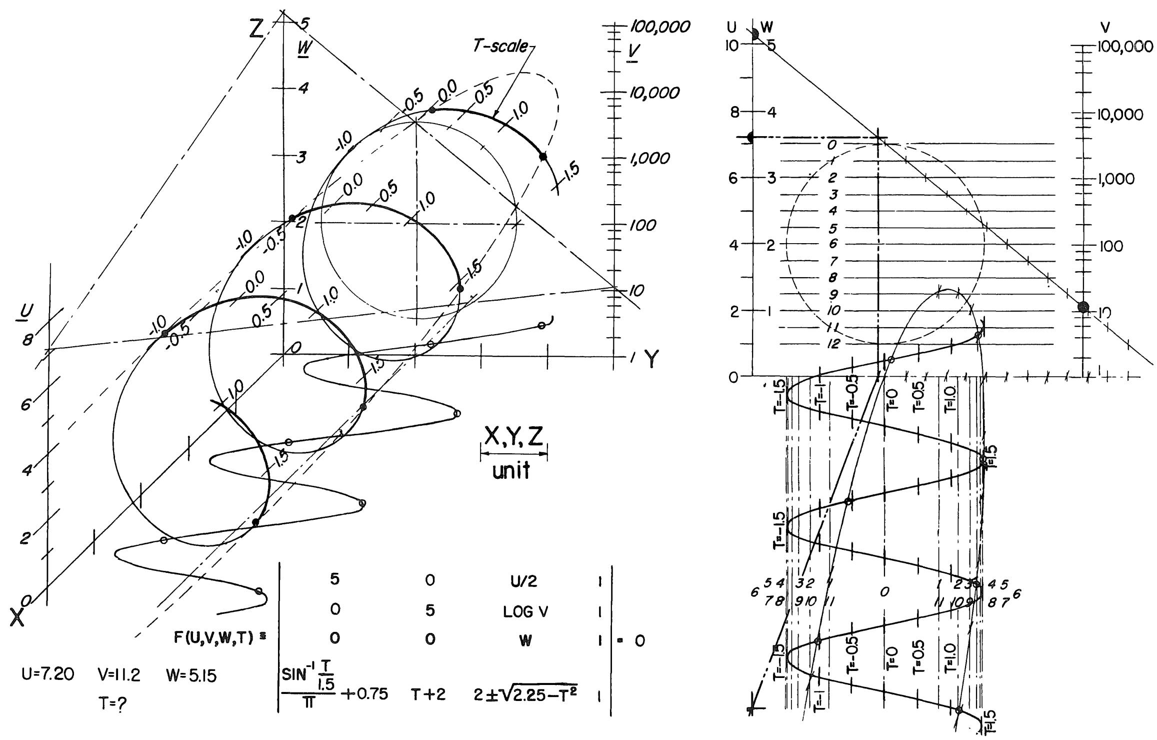
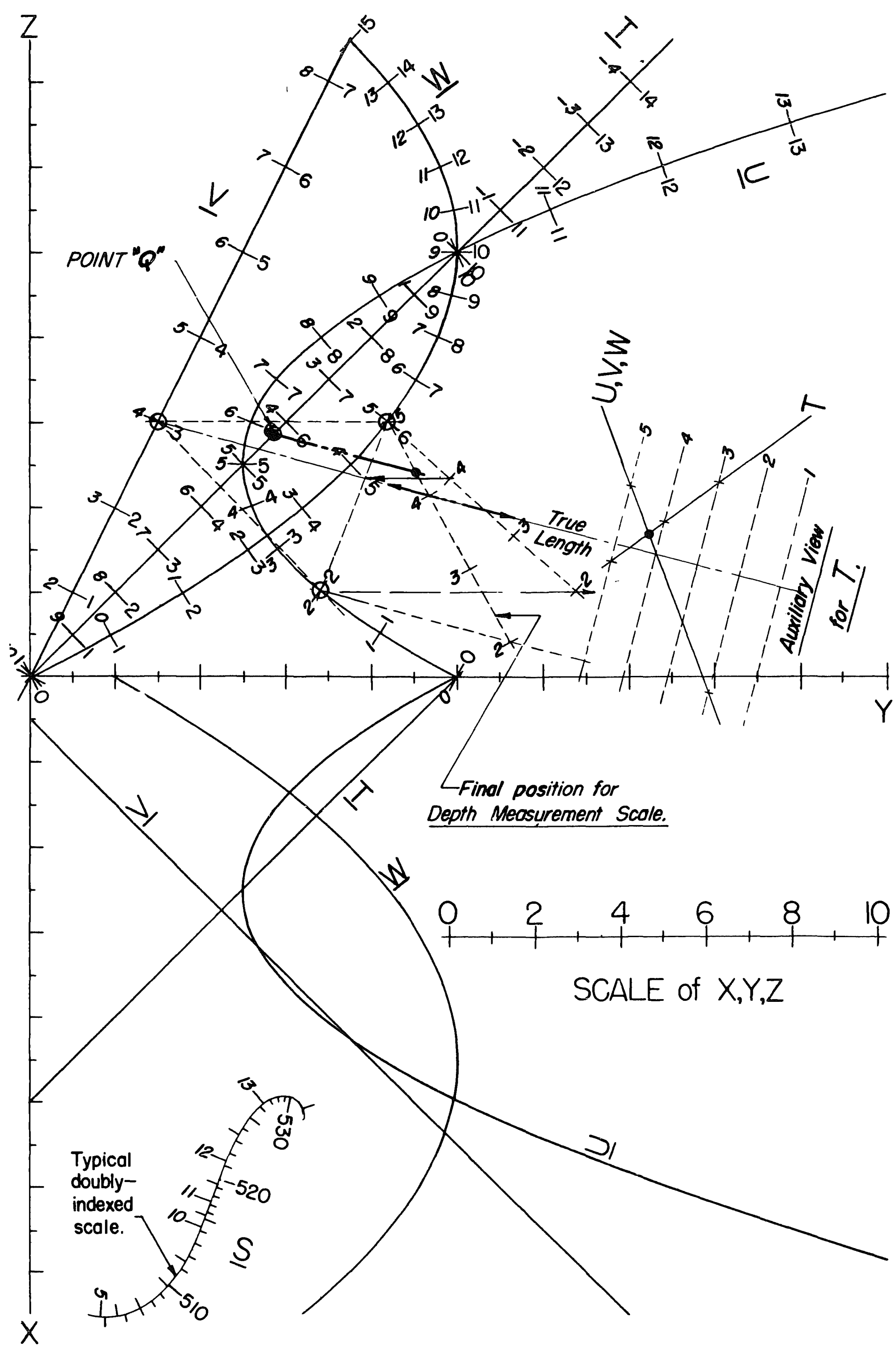


Figure 5–2.



Z
X
Y
W
U
V
T-scale
X,Y,Z unit
100,000
10,000
1,000
100
10
F(U,V,W,T) ≡
5 0 U/2 1
0 5 LOG V 1
0 0 W 1
SIN⁻¹ T/1.5 / π +0.75 T+2 2±√2.25−T² 1
= 0
U=7.20 V=11.2 W=5.15
T=?
T=-1.5
T=-1
T=-0.5
T=0
T=0.5
T=1.0
T=1.5

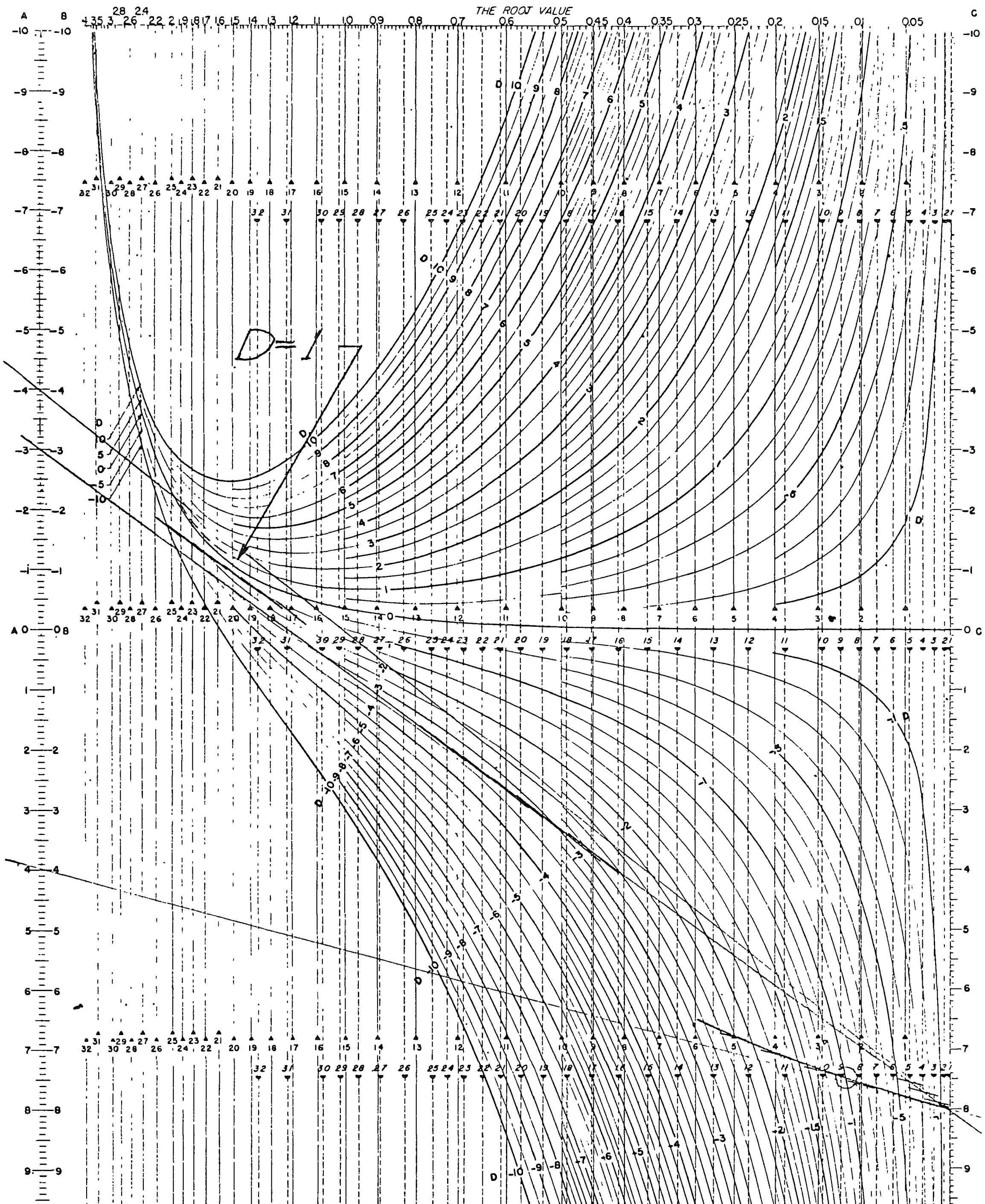
Figure 5–3.



POINT "Q"
True Length
Auxiliary View for T.
Final position for Depth Measurement Scale.
SCALE of X,Y,Z
Typical doubly-indexed scale.

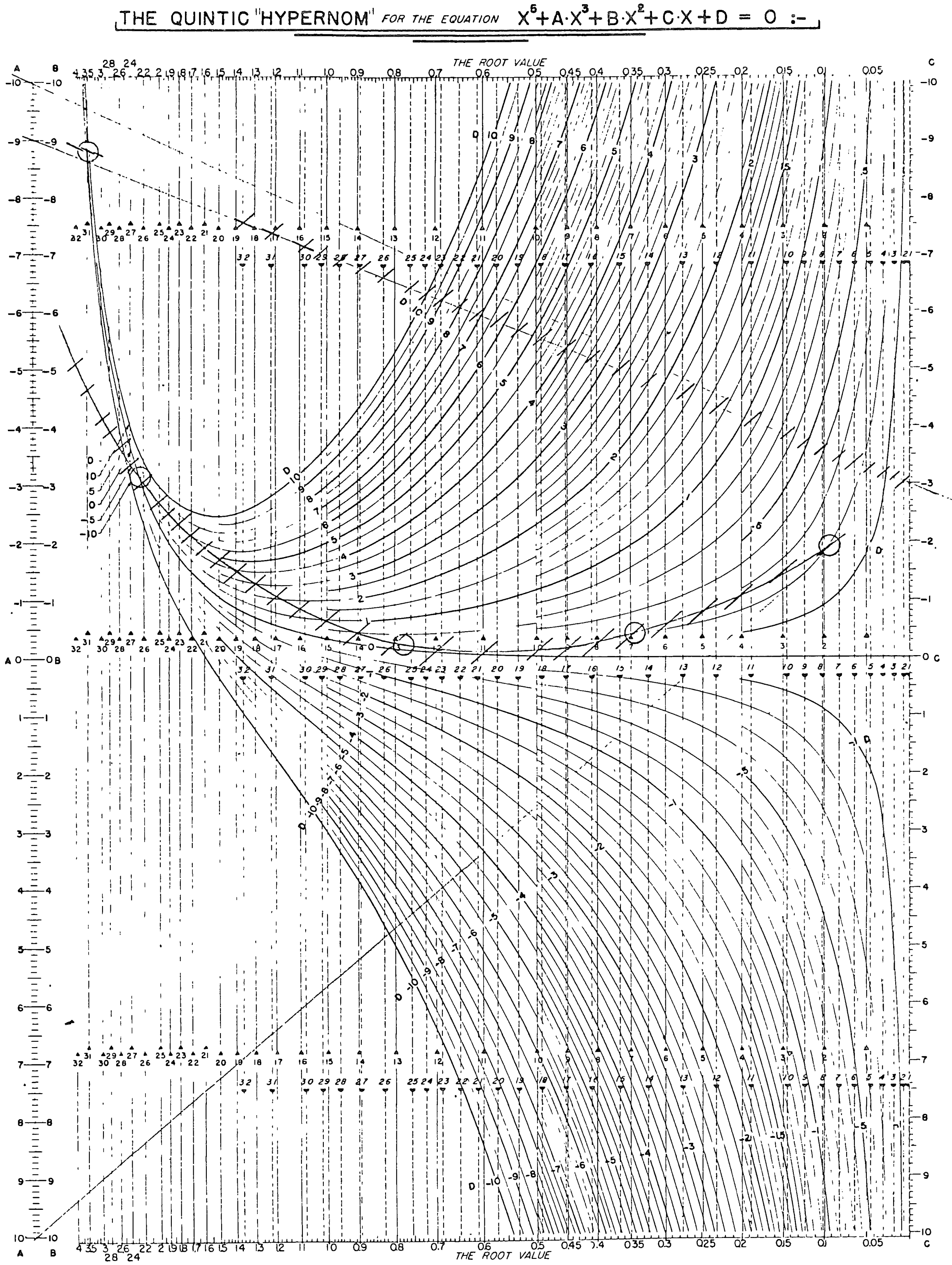
Figure 5–4.

THE QUINTIC "HYPERNOM" FOR THE EQUATION $X^5 + A \cdot X^3 + B \cdot X^2 + C \cdot X + D = 0$:-



$$X^5 - 3X^3 + 4X^2 + 8X - 1 = 0$$

Figure 5–5.



$$X^5-9X^3+10X^2-3X+0.2=0$$

Figure 5–6.

CHAPTER 6

CENTRAL PROJECTION IN NOMOGRAPHY

6–1. *A Central Projection of One Plane onto Another Plane Preserves Collineation.* (Figure 6–1)

By a central projection of a plane α onto a plane β, one means that planes α and β, fixed somewhere in space, are intersected by rays through some fixed point 0 of space, and hence that to every point A of α, there corresponds uniquely some point A′ of β, where line AA′ passes through 0. If A, B, C are three points of α collinear in line L, line L and point 0 determine a plane whose intersection with β is a line L′. Hence A′, B′, C′ are collinear and points aligned in α are aligned in β. The central projection onto plane β of any diagram in plane α is the diagram consisting of the projection of its points onto plane β. This projected diagram may have a very different aspect from the original diagram in α. An *alignment diagram* in α will, however, be projected into an *alignment diagram* in β since every collineation in α is preserved under the projection into β. The projected alignment diagram in β, it is worth repeating, may bear small physical resemblance to the alignment diagram of α. The ability to preserve the alignment property but vary the physical aspect of the diagram is useful. An acquaintance with the elements of central projection will obviously be necessary. Otherwise, as noted in the foreword, many seemingly complicated relations will appear chaotic rather than as related applications of a small number of far-reaching principles.

6–2. *One-to-One Correspondence.* Two sets of elements can be said to be in one-to-one correspondence if there exists some law, procedure or arrangement whereby when an element of one set is named, one and only one element of the other is identified with it, and *conversely*. Two such sets of elements might be all the lines of the plane and all the points of the plane, etc.

Consider all the points on the non-negative X-axis as split into two groups:

$$0 \leqq x_1 < 1 \text{ and } 1 < x_2 \qquad (6\text{–}1)$$

These can be put into one-to-one correspondence, for instance, by the equation

$$x_1 = \frac{1}{x_2}\,;\ \text{or } x_2 = \frac{1}{x_1} \qquad (6\text{–}2)$$

One regards the point $x_1 = x_2 = 1$ as lying in both sets and being self-corresponding. It is assumed that there is one and only *one point* at ∞ on the line, *this being in correspondence with* $x_1 = 0$.

One can establish this correspondence geometrically by using the curve $y = 1/x$, Figure 6–2.

6–3. *Infinity in the Line, Plane and Space.* The notion that a line is a continuous set of finite points with a single point at ∞ turns out to be practical. The infinite point lies at *either* end of the line. This way of thinking can be extended so that, in the plane, a family of parallel lines has a single point in common—the point at infinity in their common direction. All of such infinite points of the plane make up the line at infinity of the plane. The statement is sometimes made that this "line" at infinity should be regarded at best as a "circle" at infinity because it "surrounds" the observer. This should be countered in the following way. If one is at point P of the plane, Figure 6–3, then, no matter what slope a line through P may have, it will be seen to meet any given line, L, of the plane at some point Q_1, Q_2, . . . The fact that the line at ∞ also "has a point in every direction from P" should, then, no longer bother the student.

Extending these motions to three-dimensional space, a family of parallel lines has a single point in common at infinity in their direction. Every plane has its own infinite line shared by every plane parallel to it. All of such lines together make up *the plane at infinity.* Thus a one-dimensional set of points, a line, has a point at infinity; a two-dimensional set of points, a plane, has a line at infinity; a three-dimensional set of points, space, has a plane at infinity. These notions can be extended to hyperspace. They can be dealt with algebraically by using homogeneous coordinates. (See Appendix.) They are a convenient way of handling problems of central projection.

6–4. *Central Projection of the Points of a Line into the Points of a Line.* In Figure 6–4, line α, line β and point 0 all lie in a plane. There is a one-to-one correspondence between the points of line α and those of line β, as A to A′, B to B′, established by requiring that lines connecting corresponding points, like lines AA′ and BB′, are always drawn through 0. This is the central projection of line α on line β or line β on line α.

Special attention should be given certain correspondences, namely (1) where the two given lines intersect at C and C′, these two points being identical; (2) where a "ray of correspondence" lies parallel to line α cutting it at ∞_α, cutting line β in point G and causing points ∞_α and G to correspond, (3) where a "ray of correspondence" lies parallel to line β, cutting it at ∞_β, cutting line α in point F and causing points ∞_β and F to correspond. This arrangement assumes that there is a single point at ∞ on each line as described in Section 6–3.

6–5. *Central Projection of a Plane into a Plane. Infinite Lines.* An extension of central projection from the line to the plane is natural. Figure 6–1 has already presented the general idea and Figure 6–5 shows further that there is a line of points common to both planes, which are in self-correspondence. Rays of correspondence or projection through point 0 parallel to plane α, cutting it in ∞, put the points of the infinite line in correspondence with points of G. Hence the statement: the infinite line of plane α is projected into line G of plane β. Correspondingly, the infinite line of plane is projected into line F of plane α.

6–6. *Central Projection Pictorially. Cognate Types of Alignment Diagram.* We can make pictorial drawings of many central projections which will be as accurate as graphical procedure permits. Although such a projection provides an accurate way, within graphical limits, of changing the shape of a chart, it is more useful in *visualized* form to hint at what this outcome could be and to prompt definite algebraic steps to bring this about.

Figure 6–6(a) shows that a pictorial can suggest Figure 4–16(a) from Figure 4–16(d) and supply the new advanced relations for the more general diagram of Figure 4–16(a). Figure 6–6(b) shows the same sort of thing for Figure 4–18(a). Figure 6–6(c) shows projective relations between Figures 4–17(a) and (b). Figure 6–6(d) shows projective relations between Figures 4–19(a) and (b). Figure 6–6(c) relates projectively Figures 4–8 and 4–9. Thus the pictorial projection can probe for cognate forms.

When nomograms are related to one another through some central projection, they are said to be cognate types of nomogram. Three straight lines are either (a) concurrent or (b) non-concurrent, both configurations being preserved by any central projection. If the three lines are thought of as scale stems for alignment diagrams, then (a) gives rise to the familiar diagram for (1) three parallel or (2) three concurrent lines, (1) and (2) being related by *projecting the point of concurrency* of (2) to infinity in (1). Figure 6–6(c) shows that the various N-diagrams are projectively related through a basic diagram consisting of three non-concurrent lines, it being usually more convenient to project the intersection of two of them to infinity and to make them parallel. The circular, elliptical, parabolic and hyperbolic diagrams for multiplication are a group cognate to each other but to none of the above. One can think of the vertex of a cone as a center of projection with these various conic sections being scale stems for nomograms projected from the scale stem of the circular nomogram onto planes cutting the cone. Using later material, a picture or sketch of such projections can suggest the kind of changes necessary in one canonical form to bring about a new canonical form for a better diagram.

6–7. *Central Projection by Engineering Drawing.* A central projection can also be represented and carried through with graphical accuracy by means of conventional engineering drawing. Figure 6–7 shows the same central projection pictured in figure 6–6(d).

6–8. *Central Projection by Analytic Geometry.* The analytic equations for a central projection can be found for certain useful cases as follows:

In Figure 6–8(a), a general straight line is shown with P_0 a running point upon it.

$$\frac{x_2 - x_o}{x_1 - x_o} = \frac{y_2 - y_o}{y_1 - y_o} = \frac{z_2 - z_o}{z_1 - z_o}$$

$$\frac{x_2 - x_o}{x_1 - x_o} = \frac{z_2 - z_o}{z_1 - z_o}$$

$$\frac{y_2 - y_o}{y_1 - y_o} = \frac{z_2 - z_o}{z_1 - z_o} \qquad (6\text{–}3)$$

In Figure 6–8(b), planes I and II are the zy and xy planes respectively, P_0 the center of projection and P_1 and P_2 intersection points with planes I and II. Then

$$x_0 = k,\ y_0 = l,\ z_0 = m$$

$$x_1 = 0,\ y_1 = Y_1,\ z_1 = X_1$$

$$x_2 = X_2,\ y_2 = Y_2,\ z_2 = 0 \qquad (6\text{–}4)$$

This results in

$$\frac{X_2 - k}{0 - k} = \frac{0 - m}{X_1 - m}$$

$$\frac{Y_2 - l}{Y_1 - l} = \frac{0 - m}{X_1 - m}$$

$$X_2 = \frac{kX_1}{X_1 - m}$$

$$Y_2 = \frac{lX_1 - mY_1}{X_1 - m} \tag{6–5}$$

$$X_1 = \frac{mX_2}{X_2 - k}$$

$$Y_1 = \frac{lX_2 - kY_2}{X_2 - k} \tag{6–6}$$

This central projection turns out to be very useful and more general than might at first be suspected, because it will be found that the angle between planes I and II, here 90°, can be varied without affecting the form of the equations if "affine" coordinates of P_0 are used.

6–9. *Parallel and Orthographic Projection. Rotation and Translation of the Plane.* Let the center of projection move out to infinity in space. Then a *parallel projection* results, that is, a projection using rays having a common direction from that infinite point where the center now lies. When this common direction is at right angles to one of the two given planes, an orthographic projection upon it results. Thus parallel and orthographic projections are special cases of central projection.

Now imagine two superposed planes with duplicate Cartesian coordinates. If the upper one is rotated and translated, a point in one is related to the point it touches in the other one through the conventional equations of rotation and translation of points in the plane. (6–8). Now let the upper plane be moved away from the first plane while remaining parallel to it. Then parallel rays in the direction of this motion create the same correspondence between the points of the two planes as before their separation, namely (6–8). Figure 6–9.

New and old coordinates under a pure rotation are given by the familiar formulas, with inverses, Figure 6–10,

$$x = x' \cos\theta - y' \sin\theta;\ x' = x\cos\theta + y \sin\theta$$
$$y = x' \sin\theta + y' \cos\theta;\ y' = -x \sin\theta + y\cos\theta \tag{6–7}$$

The inverse is derived from the original most easily by replacing θ by $(-\theta)$. If a translation is added to the rotation, one obtains,

$$x = x' \cos\theta - y \sin\theta + x_o;$$
$$x' = x\cos\theta + y\sin\theta + x'_o$$
$$y = x'\sin\theta + y'\cos\theta + y_o;$$
$$y' = -x\sin\theta + y\cos\theta + y'_o \tag{6–8}$$

The added constant of translation here, it is convenient to remember, is the coordinate of the *old* origin in the *new system.* Likewise, the angle θ is reckoned positively from the new to the old prime direction.

6–10. *Central Projection Between Inclined Planes with "Natural" Coordinate Systems.* One wishes now to project centrally between *inclined* planes. By "natural" coordinates is meant that the Y-axes of both the systems of coordinates *coincide,* value for for value, in the line of intersection of the two planes. In Figure 6–11, P_1 in plane I is projected into P_2 of plane II through a center of projection 0. The angle between the planes is ϕ. The Cartesian coordinates of the two planes have also been set up so that a plane through 0 perpendicular to their line of intersection cuts the latter at the common origin of the two natural systems.

An orthographic view taken in the direction y_1, y_2 (Figure 6–12(a)) showing this line of intersection on end, gives

$$\frac{r_2/\sin\phi}{x_1} = \frac{x_2 - r_1/\sin\phi}{x_2}, \text{ or}$$

$$x_2 = \frac{x_1 r_1}{x_1 \sin\phi - r_2} \tag{6–9}$$

An orthographic view taken in the direction of x_1, (right side view above), showing this axis on end, gives

$$\frac{-y_1}{r_1} = \frac{y_2}{x_2\sin\phi - r_1} \tag{6–10}$$

$$y_2 = \frac{y_1}{r_1}(r_1 - x_2 \sin\phi) \tag{6–11}$$

$$y_2 = \frac{y_1}{r_1}\left[r_1 - \frac{x_1 r_1 \sin\phi}{x_1 \sin\phi - r_2}\right] \tag{6–12}$$

$$y_2 = -r_2 y_1/(x_1 \sin\phi - r_2) \tag{6–13}$$

6–11. *Central Projection Independent of Planes' Angle. Affine Coordinates.*

Equations (6–9) and (6–13) can be written

$$x_2 = \frac{x_1 \dfrac{r_1}{\sin\phi}}{x_1 - \dfrac{r_2}{\sin\phi}}\ ;\ y_2 = \frac{-\,y_1 \dfrac{r_2}{\sin\phi}}{x_1 - \dfrac{r_2}{\sin\phi}} \qquad (6\text{–}14)$$

Make the substitution

$$R_1 = \frac{r_1}{\sin\phi}$$

$$R_2 = \frac{r_2}{\sin\phi} \qquad (6\text{–}15)$$

Then

$$x_2 = \frac{x_1 R_1}{x_1 - R_2}\ ;\ y_2 = \frac{-y_1\ R_2}{x_1 - R_2} \qquad (6\text{–}16)$$

The quantities R_1 and R_2 are shown in Figure 6–12(a) and are slanting or "affine" coordinates, They "follow" the *angle ϕ between the planes,* so that equations (6–16) are true for *all* angles ϕ. Comparing Figures 6–8(b) and 6–12,

$$k = R_1,\ l = 0,\ m = R_2$$

and equations (6–5), and (6–16) are identical, the former being a special case of the latter. These relations are verified graphically in Figure 6–13. Here plane I (seen on edge) is shown in fixed position while plane II (also seen on edge) occupies two different positions corresponding to two values of angle ϕ. Let 0_1 be a center of projection for the first position of plane II, projecting line ∞ II into line F of plane I and line ∞ I into line G of plane II, these four lines appearing as points. Corresponding points in the two positions of plane II are connected with equal angle arcs of rotation. Then if a projection preserves the same correspondence between the points of I and II, that is the same transformation (6–16), *for both positions of plane II,* it would have to be centered at points 0_1 and 0_2 respectively. Affine coordinates for this point R_1 and R_2 have been labelled. The figure *verifies graphically* that center 0_2 *does* project A_o, B_o, and C_o into the same points A_2, B_2, C_2 of plane II as does center 0_1. (See Appendix for further treatment of this central projection.)

Hence one can picture plane I and plane II as lying at right angles without loss of generality. This is an example of the remark at the end of Section 6–1. A second example now follows.

6–12. *Inversion of the Plane.* Pictorial visualizing shows the following property of *every* central projection, Figure 6–14. Let L_1 and L_2 be corresponding lines under projection through 0 parallel to the intersection, Y, of planes I and II. Then plane I is divided into four strips, two A strips being inside of the L_1Y area, two B strips outside the L_1Y area.

Plane II is similarly divided but now the A strips are outside the L_2Y area and the B strips are inside. Hence it is possible to shift material by a central projection from outer to inner positions and conversely. It is possible, for instance, to shift a vertical scale K lying in B_1'' to a vertical position in B_2''. *This would reverse the direction of such a scale.* The position in B_2'' would be preferable if scale K carried the dependent variable, the position in B_1'' if line L carried the dependent variable. To provide the change in determinant form, achieving this same result, one would attempt to reverse the sign of this scale in the form ((C)) by starting with determinant changes in form ((A)). See Sections 4–6, 4–7 and many others for examples. These remarks bring together earlier material dealing with dependent and independent variables, central projection, determinant changes, etc., and later material on the projective transformation and projective operators. The approach here has been largely descriptive. In the next chapter it becomes quantitative.

PROBLEMS

PROBLEM 6–1. In Figure 6–15, alignment diagram scales for the equation $U \cdot V = W$ appear in the y, z plane, plane I. A center for projection is given at 0. Duplicate this layout, then check the tri angular diagram for the scales projected into the x, y plane, plane II. Mark all zero and 00 points, ten typical graduations, and indicate all positive and negative ranges of U, V and W that are not already shown. Check that this triangular diagram *"works."*

PROBLEM 6–2. In orthographic views lay out the same diagram for the above equation including ten values on each of the scales. Carry through the projection by engineering drawing as in Section 6–7. Show three alignments on the original and projected diagrams.

PROBLEM 6–3. In Figure 6–16, alignment diagram scale stems for the equation $U \cdot V = W$ appear in the y, z plane, plane I. A center for projection is given at 0. Sketch the triangular diagram of the scale stems projected into the x, y plane, plane II. Show all zero values, infinite values, and plus and minus ranges for the variables. Establish an alignment of values in the plane diagram I, project them into the diagram of plane II.

PROBLEM 6–4. Using the pictorial figure just derived establish ten values on each scale of plane I and project them into the scales of plane II. *Now choose a new center of projection 0′ which will take the intersection of the W and V* scales off to infinity when projected back into plane I. Carry back the ten points of each scale and show that the V-scale now turns out to be uniform—which is natural since the roles of U and V are symmetric in $U \cdot V = W$ and are now interchanged compared to the original diagram of plane I.

PROBLEM 6–5. Show pictorially how a quadrilateral of any shape and size in plane I can be taken into a rectangle of specified dimensions (but not location) in plane II by a suitable projection. Let the planes be arranged as in Figure 6–16. Use a straight edge on all lines. Derive an exact center for the projection. See Figure 13–13 for solution.

PROBLEM 6–6. Make a sketch along the lines of Figure 6–8(b) and derive the equations (6–5) by plane geometry.

PROBLEM 6–7. Make a sketch illustrating the central projection of the following points:

(0,0)	to	(0,0)
(0,20)	to	(0,20)
(15,0)	to	(15,0)
(15,20)	to	(15,30)

Show by plane geometry that the center of projection required to do this will have coordinates $k = -30$; $l = 0$; $m = 45$. See Section 7–4.

PROBLEM 6–8. Check by engineering drawing and prove by plane geometry that changing the size of angle ϕ in the space figure shown on edge in Figure 6–13 does *NOT* cause any shift of projected points A_1, B_1, C_1 etc., *parallel* to the dihedral edge E as they move to positions A_2, B_2, C_2. Their positions in plane II_2 remain identical to those in plane II_1.

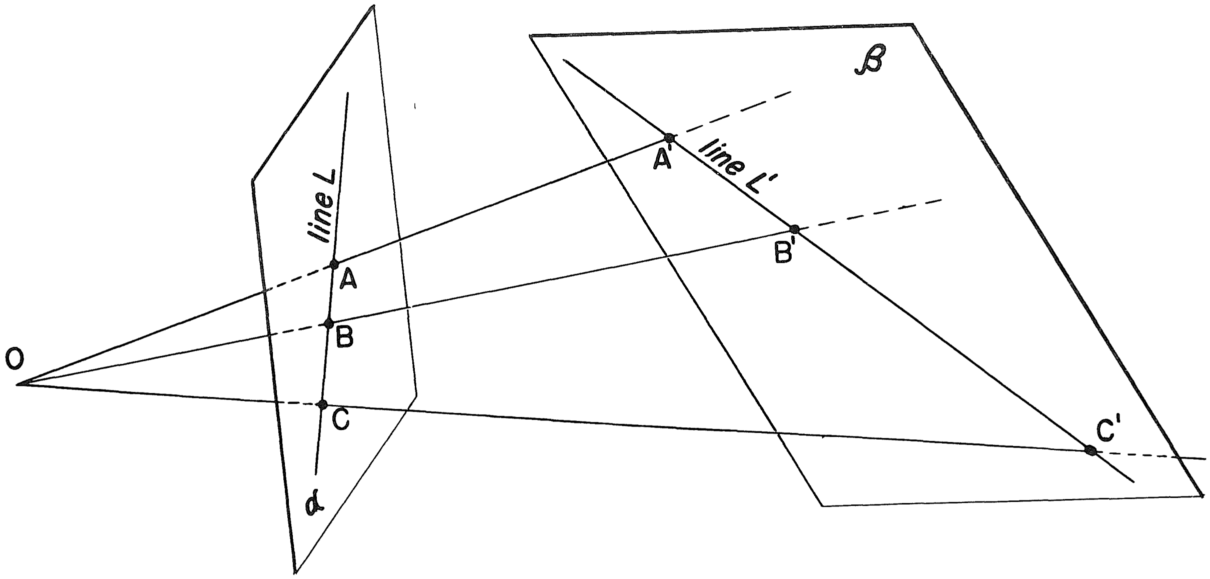


Figure 6–1.

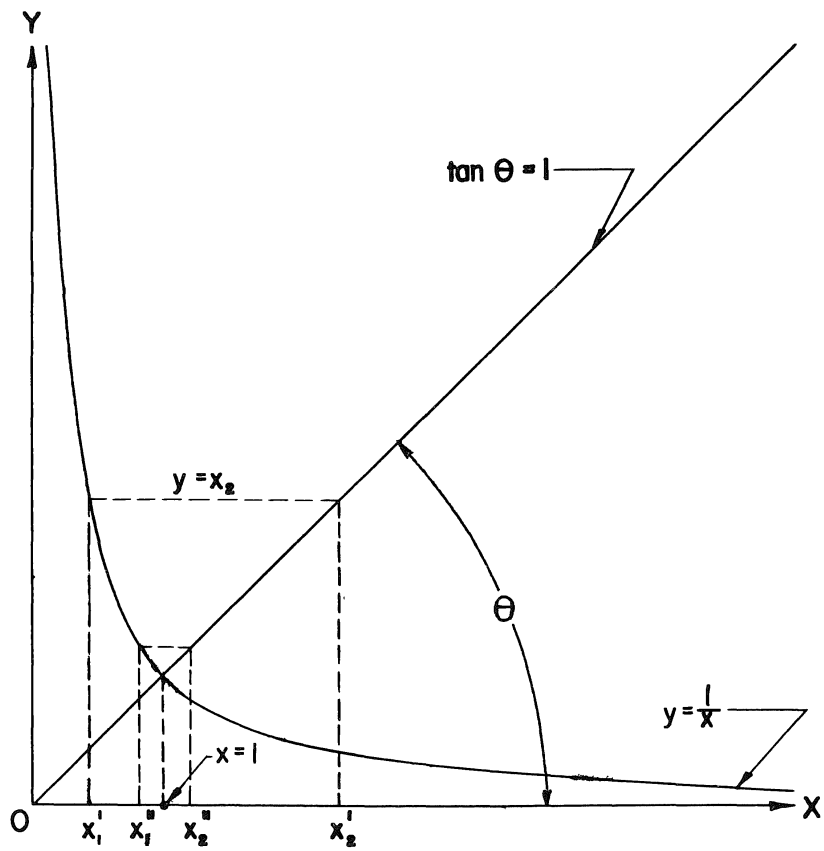


Figure 6–2.

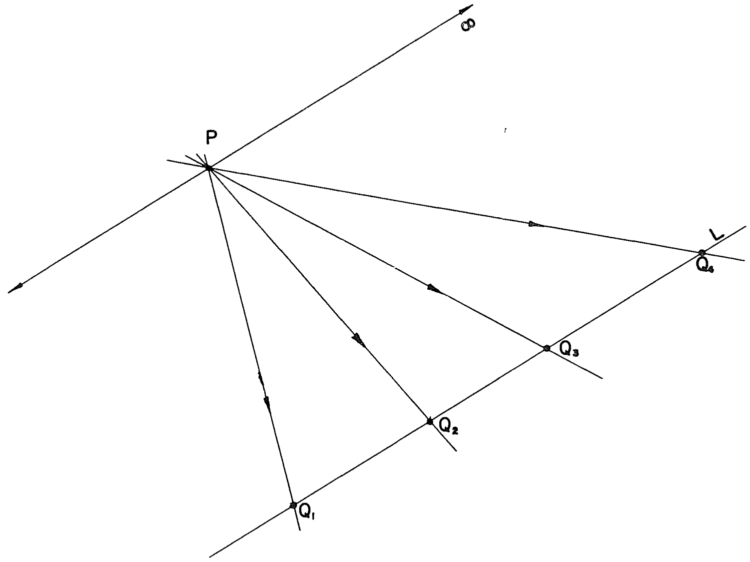


Figure 6–3.

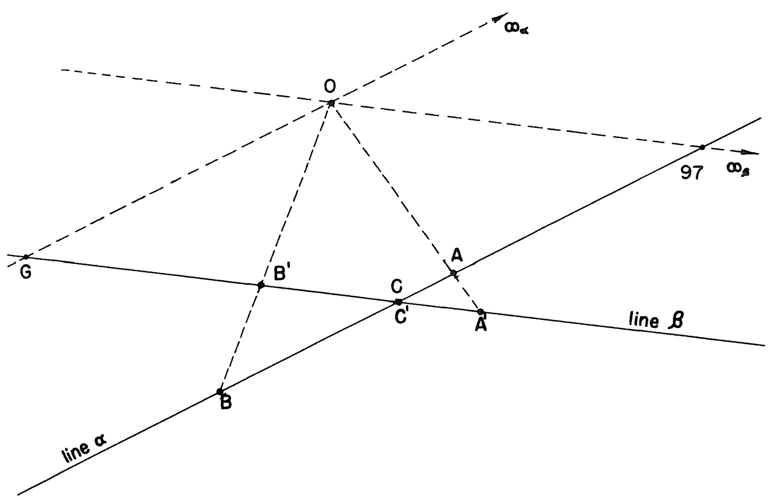


Figure 6–4.

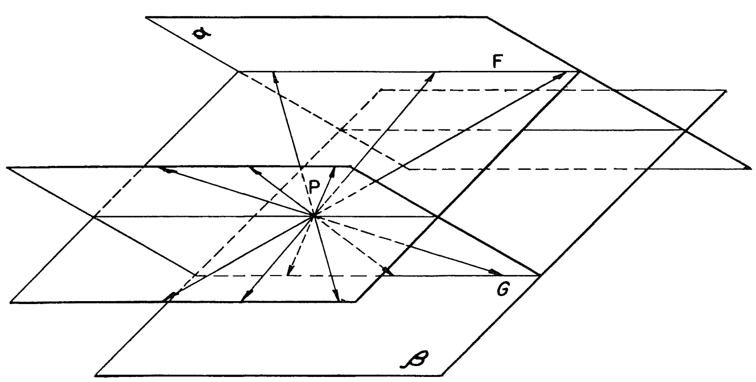


Figure 6–5.

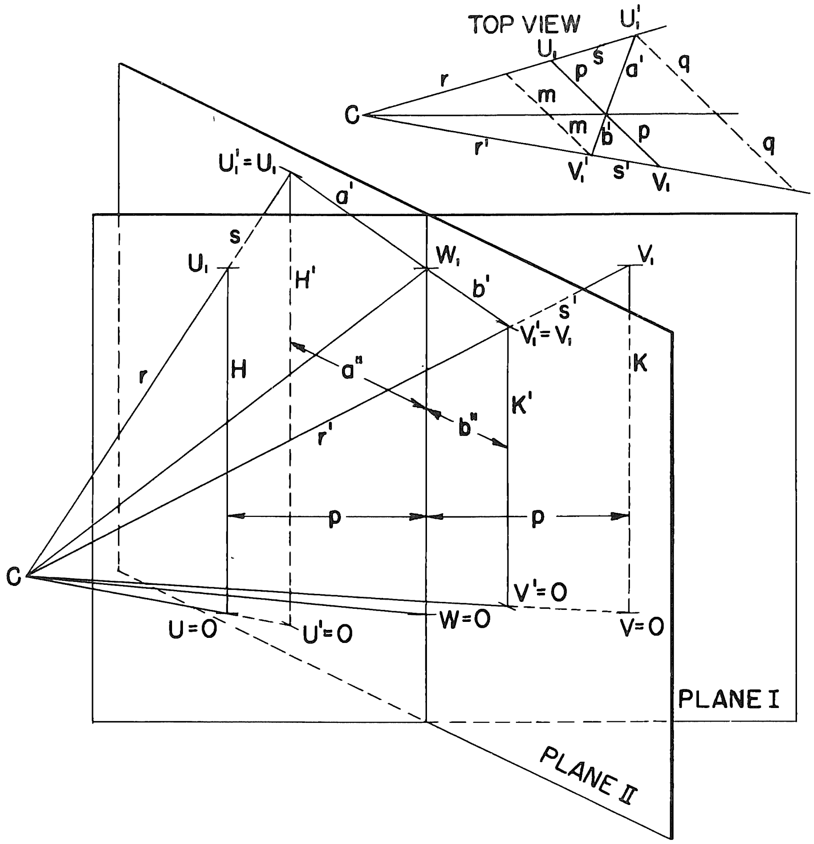


Figure 6–6a.

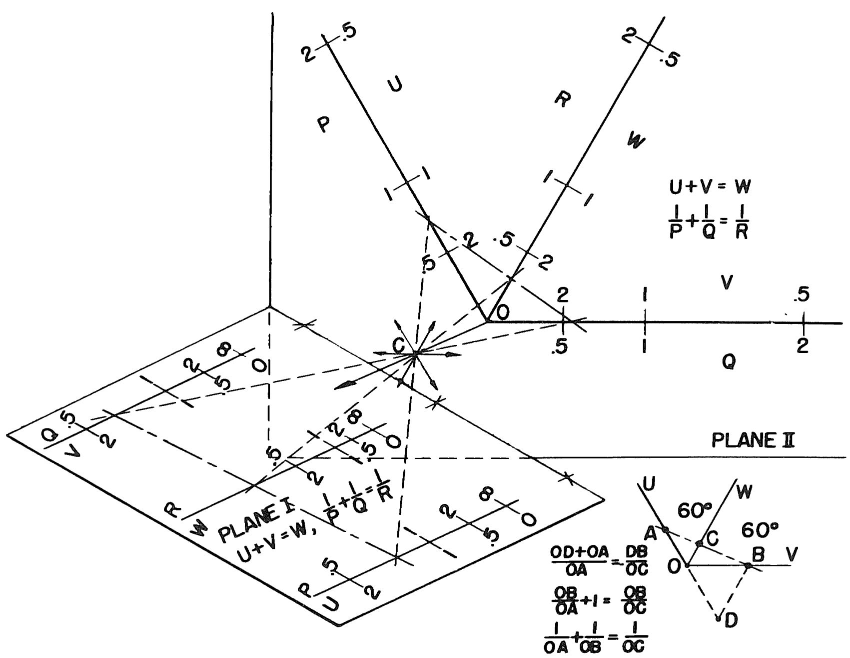


Figure 6–6b.

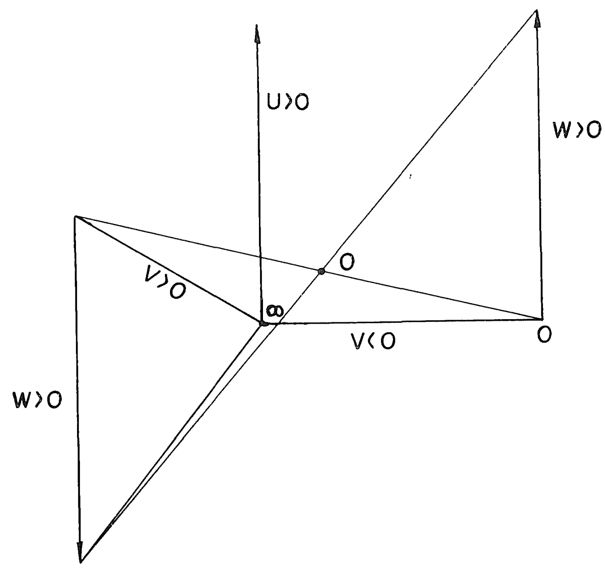


Figure 6–6c.

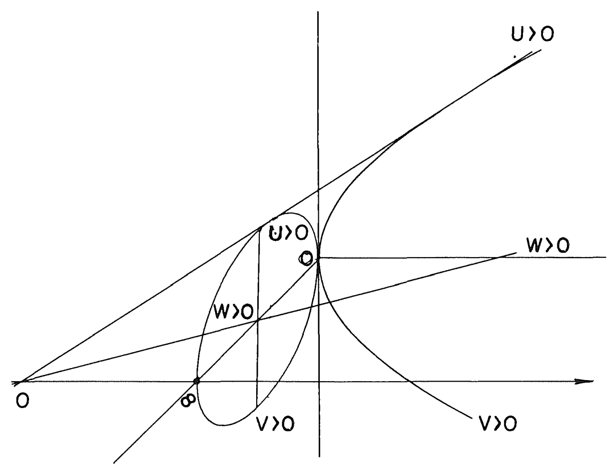


Figure 6–6d.

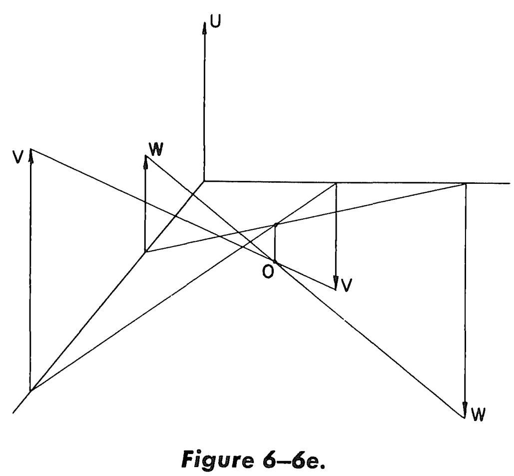


Figure 6–6e.

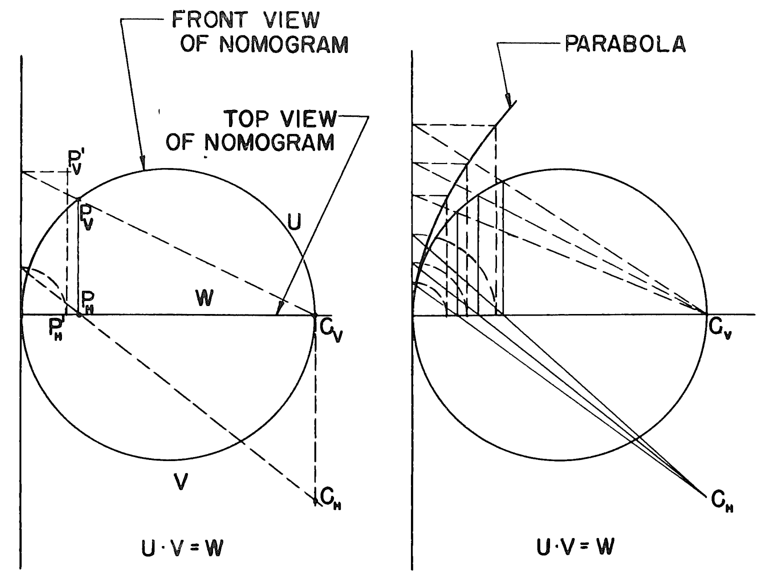


Figure 6–7.

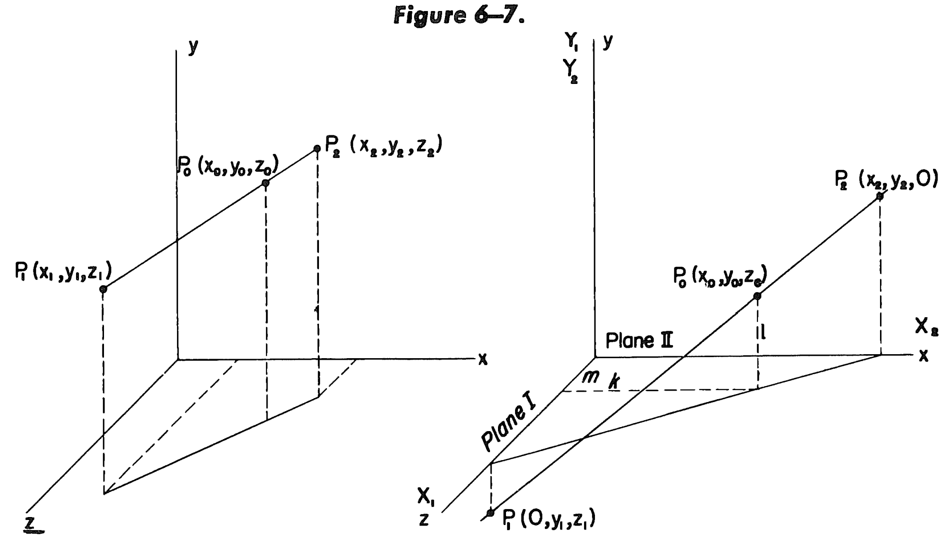


Figure 6–8.

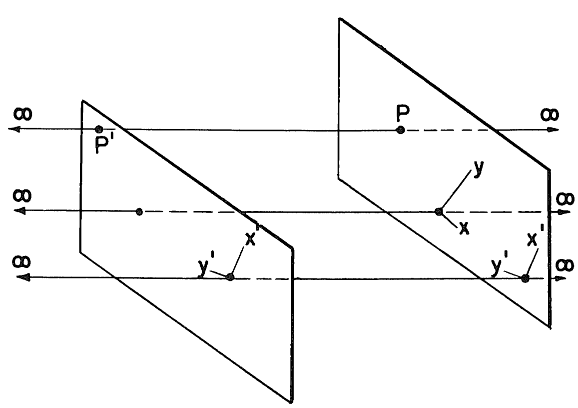


Figure 6–9.

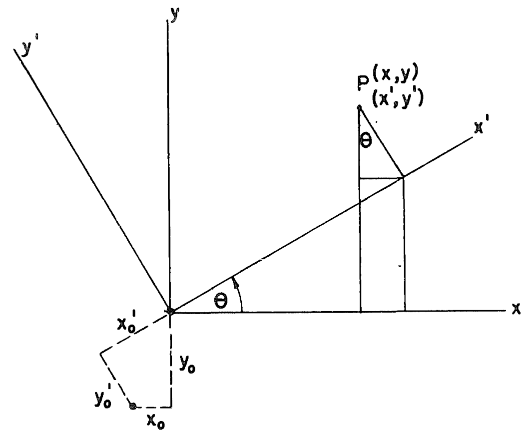


Figure 6—10.

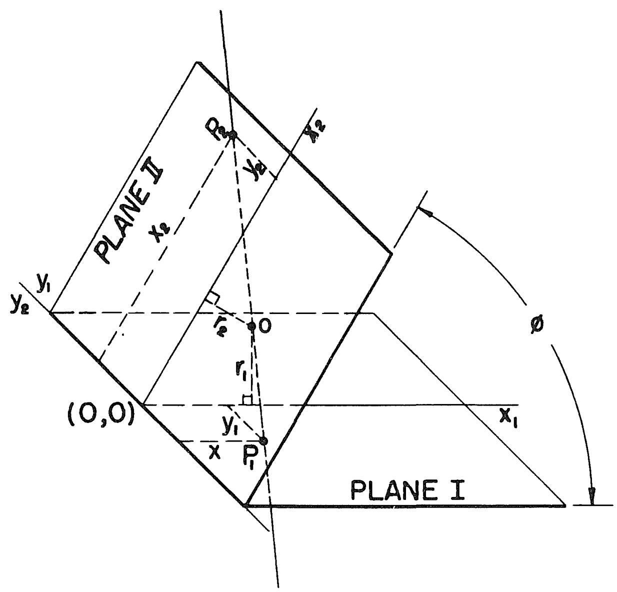


Figure 6—11.

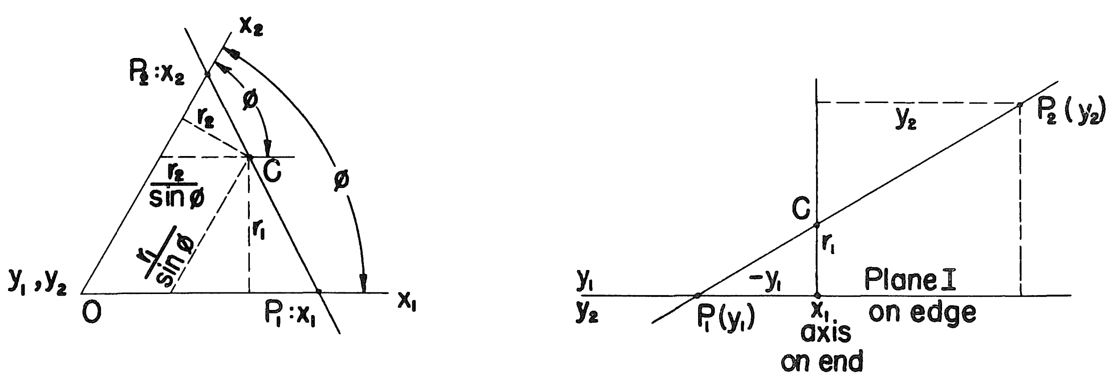


Figure 6—12.

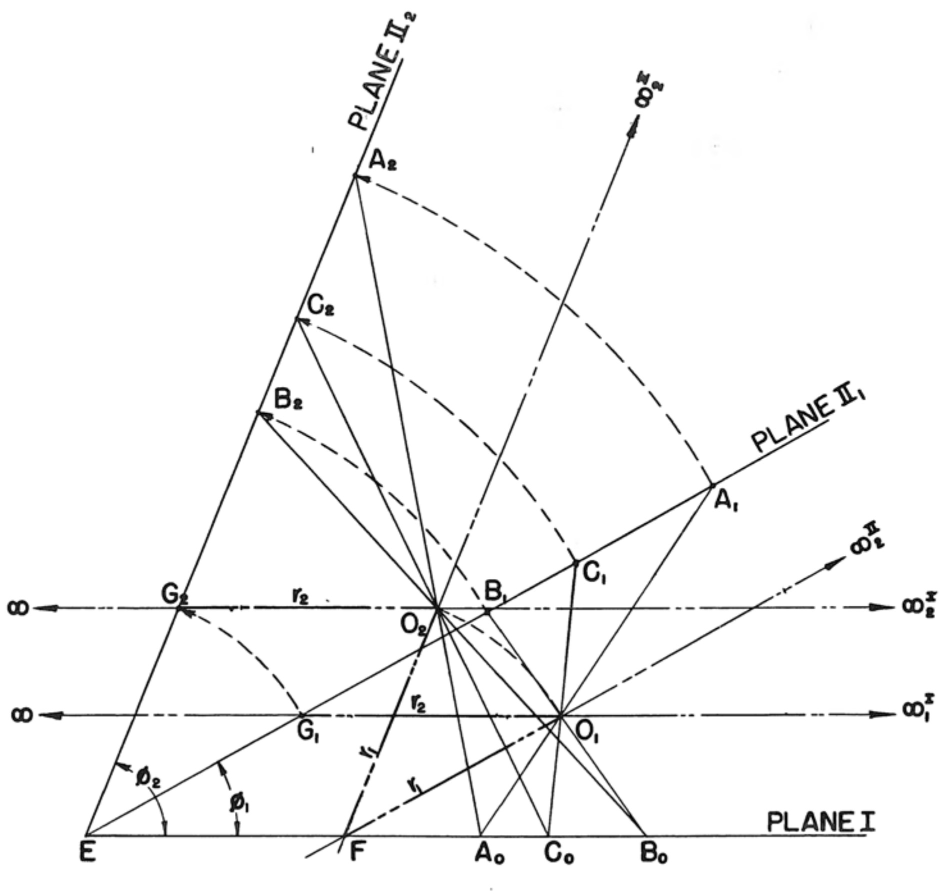


Figure 6–13.

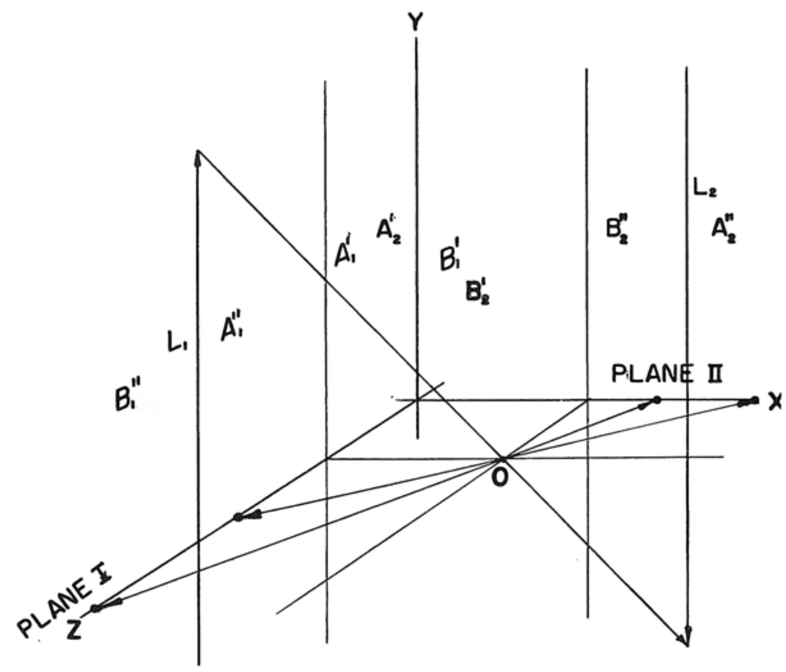


Figure 6–14.

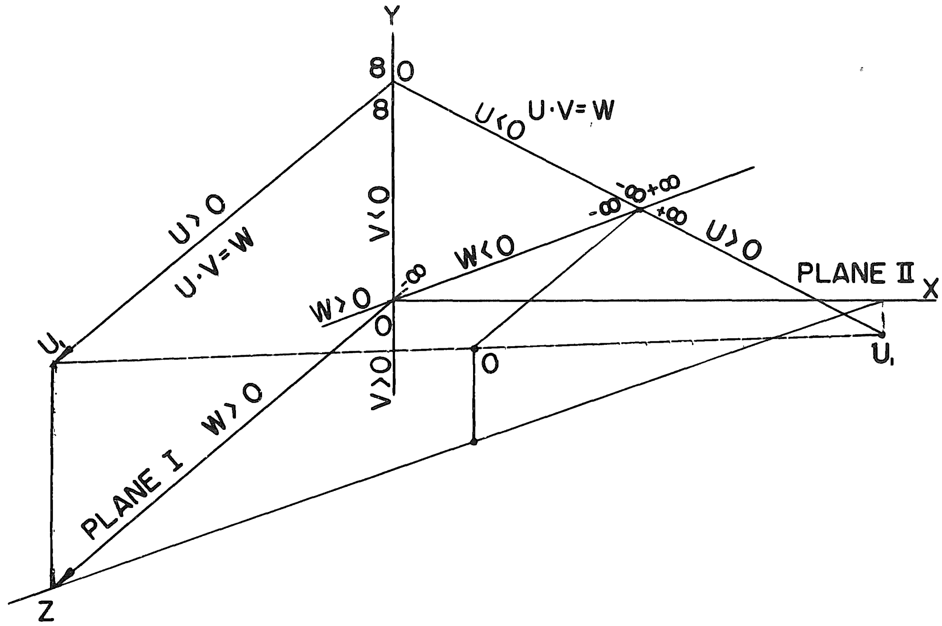


Figure 6–15.

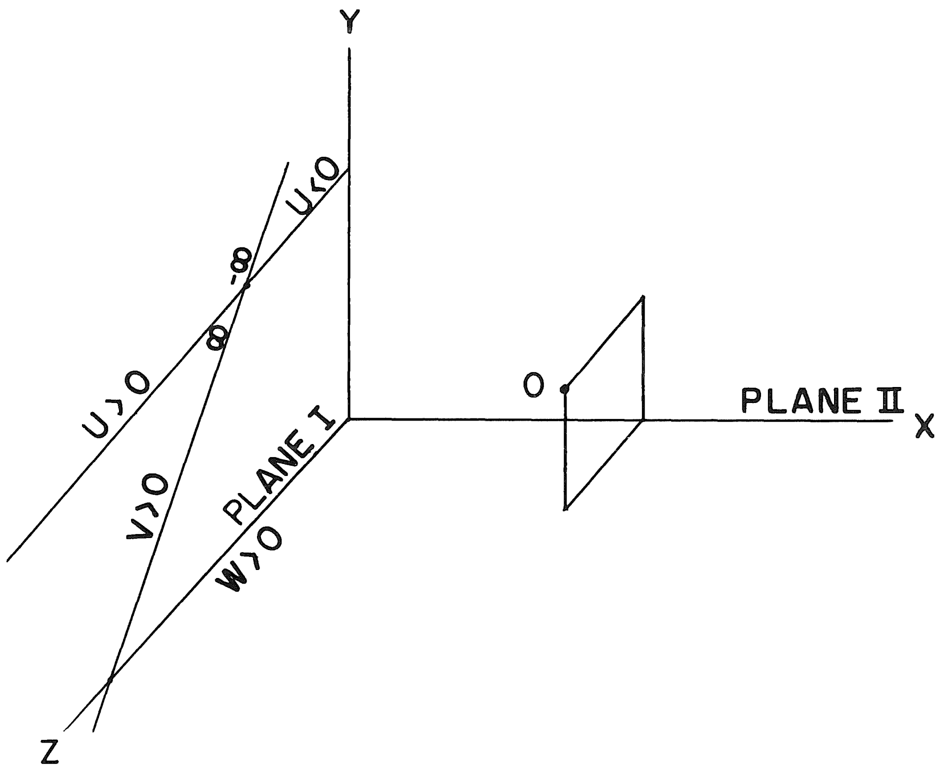


Figure 6–16.

CHAPTER 7

THE GENERAL PROJECTIVE TRANSFORMATION

7–1. *Form of the Projective Transformation.* Let the points of plane II be related to those of plane I by the equations

$$x'' = \frac{a_1x' + b_1y' + c_1}{a_3x' + b_3y' + c_3} \qquad y'' = \frac{a_2x' + b_2y' + c_2}{a_3x' + b_3y' + c_3} \tag{7–1}$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \equiv |\,abc\,| \neq 0 \tag{7–2}$$

This relation is called the general projective transformation because it will always be found to give rise to the same transformation or correspondence of points in plane I to those in plane II, as a central projection. Conversely every central projection will be found to give rise to a correspondence or relation like (7–1). Equations (6–5), (6–6), (6–7), (6–8), (6–9), (6–13), (6–16) are examples. The restriction $|\,abc\,| \neq 0$ is vital, for it can be shown that it assures that the center of projection lies outside of either plane.

The line $a\,x' + b\,y' + c = 0$ in plane I causes (7–1) to blow up and hence corresponds to the ∞-line of plane II. If the *inverse* transformation were easily found, the line of plane II corresponding to the ∞-line of plane I could be read off. For purposes like this, it is helpful to study the transformation (7–1).

7–2. *The Projective Transformation as an Operator.* Figure 6–1 showed that a central projection of an alignment diagram of plane I onto plane II was again an alignment diagram. Consider an equation in canonical determinant form (but with rows and columns interchanged from their conventional positions):

$$F(U, V, W) \equiv ((C')) \equiv \begin{vmatrix} U_1' & U_2' & 1 \\ V_1' & V_2' & 1 \\ W_1' & W_2' & 1 \end{vmatrix} \equiv \begin{vmatrix} U_1' & V_1' & W_1' \\ U_2' & V_2' & W_2' \\ 1 & 1 & 1 \end{vmatrix} = 0 \tag{7–3}$$

The interpretation being employed now is,

$$((C')) \equiv \begin{vmatrix} X_1' & X_2' & X_3' \\ Y_1' & Y_2' & Y_3' \\ 1 & 1 & 1 \end{vmatrix} \equiv 0 \qquad \begin{matrix} X_1' \equiv U_1' \\ Y_1' \equiv U_2' \text{ etc} \end{matrix}$$

U_1' is a function of U only, etc. (7–4)

Correspondingly the transformed alignment diagram in plane II is assumed to have the canonical form

$$F(U, V, W) \equiv ((C'')) \equiv \begin{vmatrix} X_1'' & X_2'' & X_3'' \\ Y_1'' & Y_2'' & Y_3'' \\ 1 & 1 & 1 \end{vmatrix} \equiv \begin{vmatrix} U_1'' & V_1'' & W_1'' \\ U_2'' & V_2'' & W_2'' \\ 1 & 1 & 1 \end{vmatrix} = 0 \tag{7–5}$$

Applying (7–1) to (7–3)

$$F(U,V,W) \equiv ((C'')) \equiv \begin{vmatrix} \dfrac{a_1U_1' + b_1U_2' + c_1}{a_3U_1' + b_3U_2' + c_3} & \dfrac{a_1V_1' + b_1V_2' + c_1}{a_3V_1' + b_3V_2' + c_3} & \dfrac{a_1W_1' + b_1W_2' + c_1}{a_3W_1' + b_3W_2' + c_3} \\ \dfrac{a_2U_1' + b_2U_2' + c_2}{a_3U_1' + b_3U_2' + c_3} & \dfrac{a_2V_1' + b_2V_2' + c_2}{a_3V_1' + b_3V_2' + c_3} & \dfrac{a_2W_1' + b_2W_2' + c_2}{a_3W_1' + b_3W_2' + c_3} \\ 1 & 1 & 1 \end{vmatrix} = 0 \qquad (7\text{-}6)$$

Under the interpretation (7–5), (7–6) yields parametric curves in U, V, W which make up the new, projected alignment diagram for $F(U, V, W) = 0$. A more compact way of making the transformation lies in handling only the constants of it. These are written in a form, $\triangle$, already observed as a condition (7–2), namely

$$\triangle \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$$

The action of the transformation upon a canonical form ((C′)) is expressed by writing down that form preceded by the $\triangle$ of the transformation.

$$\triangle \cdot ((C')) \equiv ((C'')) \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ & \text{row} & \end{vmatrix} \cdot \begin{vmatrix} U_1' & V_1' & W_1' \\ U_2' & V_2' & W_2' \\ 1 & 1 & 1 \\ & \text{column} & \end{vmatrix} = 0 \qquad (7\text{-}7)$$

The result of (7–7) is (7–6), the three-rowed determinant ((C″)). This is a product determinant formed by conventional determinant multiplication of $\triangle$ and ((C′)); that is, each element of ((C″)) is an "inner product" of a *row* of $\triangle$ and a *column* of ((C′)). The *row number* of the element being formed in the product is the *number of the row* being used from $\triangle$ to form the element. The *column number* of the element being formed in the product is the *number of the column* being used from ((C′)) to form the element. Hence

$$\triangle\,((C')) \equiv \begin{vmatrix} a_1U_1' + b_1U_1' + c_1 & a_1V_1' + b_1V_1' + c_1 & a_1W_1' + b_1W_1' + c_1 \\ a_2U_2' + b_2U_2' + c_2 & a_2V_2' + b_2V_2' + c_2 & a_2W_2' + b_2W_2' + c_2 \\ a_3U_3' + b_3U_3' + c_3 & a_3V_3' + b_3V_3' + c_3 & a_3W_3' + b_3W_3' + c_3 \end{vmatrix} = 0 \qquad (7\text{-}8)$$

Each column of (7–8) can be divided by the element of its bottom row, which gives, by (7–6)

$$\triangle \cdot ((C')) = ((C'')) \qquad (7\text{-}9)$$

See Examples (7–2), (7–3), (7–4), (7–5).

Summary: (1) a projective transformation is an operation, (7–1); (2) the operation can be symbolized by an operator (7–2), $\triangle \equiv |\,abc\,| \neq 0$; (3) the operator $\triangle$ multiplies a canonical form ((C′)) for a diagram in plane I to yield the canonical form ((C″)) for the projected diagram in plane II by the familiar rules of determinant multiplication; (4) $\triangle$ is never zero, ((C′)) always is zero, ((C″)) is always zero.

It is possible to make several central projections in succession each represented by the $\triangle$ of its projective transformation. The *order* in which these central projections are carried through is important. Let us assume this order to be $\triangle_1, \triangle_2, \triangle_3 \ldots \triangle_j$. This order is preserved leftwards of the canonical form ((C′)) being projected

$$\triangle_j \cdots \triangle_3 \cdot \triangle_2 \cdot \triangle_1 \cdot ((C')) = ((C'')) \qquad (7\text{-}10)$$

A pair of adjacent $\triangle$'s can operate on each other under the same rules used for $\triangle$ and ((C′)), the product $\triangle$ remaining in their place. Thus, the $\triangle$'s are "associative but not commutative."

7–3. *Some General Rules of Nomography Operators. Identity and Inverse Operators.* The right hand sides of both equations (7–1) can be divided top and bottom by the same quantity without altering the transformation. Any one of the nine constants can thus be reduced to unity. Thus, the general projective transformation has nine homogeneous, or proportional, parameters. Assume one of the constants has been made unity. The transformation now has eight independent parameters. Altering one alters the transformation and the correspondence between points in the planes arising from the implied central projection. Since the *nine* elements of the transformation are proportional, the elements of $\triangle$ taken from this transformation can be altered by multiplication of them all (or division) by a constant. *No other determinant change of $\triangle$ is permissible. Evaluation of an operator as though it were an ordinary determinant is meaningless.* A product canonical form such as (7–8) can have each column divided by the lowest element but any other determinant change will alter the true significance of the product. The determinant operator I

$$I \equiv \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \qquad (7\text{–}11)$$

when applied to a canonical form ((C′)) leaves it unaltered. In ordinary multiplication this would be like multiplying a quantity by unity. I is accordingly called the *identity* operator. It can be shown that I is unique in this property. It is also clear

$$I = \begin{vmatrix} K & 0 & 0 \\ 0 & K & 0 \\ 0 & 0 & K \end{vmatrix} \qquad K \neq 0 \qquad (7\text{–}12)$$

If a given transformation of coordinates from plane I to plane II is represented by $\triangle$, then the inverse transformation from plane II to plane I is the one which "undoes" the work of the first. It is written symbolically $\triangle^{-1}$ to convey the impression that $\triangle^{-1} \cdot \triangle$ (that is, $\triangle$ followed by its inverse, or undoing) is an operator of zeroth order *or* one with null effect. It has just been observed, however, that this is the identity operator, so it is sensible to write for any

$$\triangle^{-1} \cdot \triangle = \triangle \cdot \triangle^{-1} \equiv I \equiv \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \qquad (7\text{–}13)$$

(Care should be taken not to confuse $\triangle^{-1}$ with $\frac{1}{\triangle}$)

$$\triangle^{-1} \neq \frac{1}{\triangle}$$

Given $\triangle$, there will be times when it will be useful to know $\triangle^{-1}$ explicitly. *For our purposes,* the inverse determinant is, by definition,

$$\triangle^{-1} \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^{-1} \equiv \begin{vmatrix} \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} & -\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} & \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \\ -\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} & \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} & -\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \\ \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} & -\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} & \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \end{vmatrix}$$

The body diagonal of $\triangle$ is the set of elements a_1, b_2, c_3. The elements a_2 and b_1, a_3 and c_1, b_3 and c_2 can be called minor images or "reflections" of one another in the body diagonal. Each member of the body diagonal is its own reflection. The inverse of $\triangle$ can be described as a three-rowed determinant consisting of the signed minors of the mirror images or reflections of the corresponding elements of $\triangle$. Thus the element of $\triangle^{-1}$ corresponding to b_1 is the signed minor of a_3, etc. The fundamental property of $\triangle^{-1}$, (7–13), can now quickly be checked.

$$\triangle \cdot \triangle^{-1} \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \cdot \begin{vmatrix} \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} & -\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} & \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \\ -\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} & \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} & -\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \\ \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} & -\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} & \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \end{vmatrix}$$

$$\equiv \begin{vmatrix} a_1\begin{vmatrix} b_2c_2 \\ b_3c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2c_2 \\ a_3c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2b_2 \\ a_3b_3 \end{vmatrix}; & -a_1\begin{vmatrix} b_1c_1 \\ b_3c_3 \end{vmatrix} + b_1\begin{vmatrix} a_1c_1 \\ a_3c_3 \end{vmatrix} - c_1\begin{vmatrix} a_1b_1 \\ a_3b_3 \end{vmatrix}; & a_1\begin{vmatrix} b_1c_1 \\ b_2c_2 \end{vmatrix} - b_1\begin{vmatrix} a_1c_1 \\ a_2c_2 \end{vmatrix} + c_1\begin{vmatrix} a_1b_1 \\ a_2b_2 \end{vmatrix} \\ a_2\begin{vmatrix} b_2c_2 \\ b_3c_3 \end{vmatrix} - b_2\begin{vmatrix} a_2c_2 \\ a_3c_3 \end{vmatrix} + c_2\begin{vmatrix} a_2b_2 \\ a_3b_3 \end{vmatrix}; & -a_2\begin{vmatrix} b_1c_1 \\ b_3c_3 \end{vmatrix} + b_2\begin{vmatrix} a_1c_1 \\ a_3c_3 \end{vmatrix} - c_2\begin{vmatrix} a_1b_1 \\ a_3b_3 \end{vmatrix}; & a_2\begin{vmatrix} b_1c_1 \\ b_2c_2 \end{vmatrix} - b_2\begin{vmatrix} a_1c_1 \\ a_2c_2 \end{vmatrix} + c_2\begin{vmatrix} a_1b_1 \\ a_2b_2 \end{vmatrix} \\ a_3\begin{vmatrix} b_2c_2 \\ b_3c_3 \end{vmatrix} - b_3\begin{vmatrix} a_2c_2 \\ a_3c_3 \end{vmatrix} + c_3\begin{vmatrix} a_2b_2 \\ a_3b_3 \end{vmatrix}; & -a_3\begin{vmatrix} b_1c_1 \\ b_3c_3 \end{vmatrix} + b_3\begin{vmatrix} a_1c_1 \\ a_3c_3 \end{vmatrix} - c_3\begin{vmatrix} a_1b_1 \\ a_3b_3 \end{vmatrix}; & a_3\begin{vmatrix} b_1c_1 \\ b_2c_2 \end{vmatrix} - b_3\begin{vmatrix} a_1c_1 \\ a_2c_2 \end{vmatrix} + c_3\begin{vmatrix} a_1b_1 \\ a_2b_2 \end{vmatrix} \end{vmatrix}$$

$$\equiv \begin{vmatrix} \begin{vmatrix} a_1b_1c_1 \\ a_2b_2c_2 \\ a_3b_3c_3 \end{vmatrix} & \begin{vmatrix} a_1b_1c_1 \\ a_1b_1c_1 \\ a_3b_3c_3 \end{vmatrix} & \begin{vmatrix} a_1b_1c_1 \\ a_2b_2c_2 \\ a_1b_1c_1 \end{vmatrix} \\ \begin{vmatrix} a_2b_2c_2 \\ a_2b_2c_2 \\ a_3b_3c_3 \end{vmatrix} & \begin{vmatrix} a_1b_1c_1 \\ a_2b_2c_2 \\ a_3b_3c_3 \end{vmatrix} & \begin{vmatrix} a_1b_1c_1 \\ a_2b_2c_2 \\ a_2b_2c_2 \end{vmatrix} \\ \begin{vmatrix} a_3b_3c_3 \\ a_2b_2c_2 \\ a_3b_3c_3 \end{vmatrix} & \begin{vmatrix} a_1b_1c_1 \\ a_3b_3c_3 \\ a_3b_3c_3 \end{vmatrix} & \begin{vmatrix} a_1b_1c_1 \\ a_2b_2c_2 \\ a_3b_3c_3 \end{vmatrix} \end{vmatrix} \equiv \begin{vmatrix} |abc| & 0 & 0 \\ 0 & |abc| & 0 \\ 0 & 0 & |abc| \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \equiv I \qquad (7\text{–}14)$$

Example 7–1. The transformation (6–5)

$$X_2 = \frac{KX_1}{X_1 - m}; \qquad Y_2 = \frac{lX_1 - mY_1}{X_1 - m}$$

has an inverse (6–6) which was derived there by direct solution for X_1, Y_1. Check this result using determinant operator methods:

$$\triangle = \begin{vmatrix} K & 0 & 0 \\ l & -m & 0 \\ 1 & 0 & -m \end{vmatrix}; \triangle^{-1} = \begin{vmatrix} m^2 & 0 & 0 \\ lm & -Km & 0 \\ m & 0 & -Km \end{vmatrix} = \begin{vmatrix} m & 0 & 0 \\ l & -K & 0 \\ 1 & 0 & -K \end{vmatrix} \qquad (7\text{–}15)$$

so

$$X_1 = \frac{mX_2}{X_2 - K}; \qquad Y_1 = \frac{lX_2 - KY_2}{X_2 - K}$$

which agrees with (6–6).

7–4. *Four Points Carried into Four Points by the Projective Transformation.* The transformation (7–1) sets up a correspondence between the points of plane I and plane II and hence between their coordinates. If the coordinates of a point in plane I are inserted into the right hand side of (7–1), those of the corresponding point in plane II will result on the left. One can specify by this equation that a point P_1 with certain coordinates in plane I shall be carried into a point P_2 with certain coordinates in plane II by putting the coordinates of those points into their places in (7–1). This creates a relationship between the eight independent constants of the equation consisting of two linear equations. If we repeat the process four times we get eight simultaneous, linear equations which will permit us to solve for the eight constants. It is often possible to arrange that enough of the coordinates in plane I and plane II shall be zero so that it is not hard to solve this rather large number of equations for the eight constants. Let us assume, for instance, that the origin of coordinates in plane I is carried by the transformation (7–1) into the origin of coordinates in plane II. When these coordinates are placed in their proper positions in (7–1), one has the pair of relations

$$0 = \frac{a_1 \cdot 0 + b_1 \cdot 0 + c_1}{a_3 \cdot 0 + b_3 \cdot 0 + c_3} \qquad 0 = \frac{a_2 \cdot 0 + b_2 \cdot 0 + c_2}{a_3 \cdot 0 + b_3 \cdot 0 + c_3} \qquad (7\text{–}16)$$

or $\quad 0 = c_1 \qquad 0 = c_2$

Make a second assumption that the point (0, 20) of plane I is carried into the point (0, 20) of plane II. Then (7–1) becomes, including results from (7–16),

$$0 = \frac{a_1 \cdot 0 + b_1 \cdot 20 + 0}{a_3 \cdot 0 + b_3 \cdot 20 + c_3} \qquad 20 = \frac{a_2 \cdot 0 + b_2 \cdot 20 + 0}{a_3 \cdot 0 + b_3 \cdot 20 + c_3} \qquad (7\text{–}17)$$

or $\quad b_1 = 0 \qquad b_3 \cdot 20 + c_3 = b_2$

Let a third assumption be that point (15, 0) of plane I is carried into point (15, 0) of plane II. Then

$$15 = \frac{a_1 \cdot 15 + 0 + 0}{a_3 \cdot 15 + b_3 \cdot 0 + c_3} \qquad 0 = \frac{a_2 \cdot 15 + b_2 \cdot 0 + 0}{a_3 \cdot 15 + b_3 \cdot 0 + c_3} \qquad (7\text{–}18)$$

or $\quad a_3 \cdot 15 + c_3 = a_1 \qquad a_2 = 0$

Let the fourth and last assumption be that point (15, 20) of plane I is carried into (15, 30) of plane II, then

$$15 = \frac{a_1 \cdot 15 + 0 \cdot 20 + 0}{a_3 \cdot 15 + b_3 \cdot 20 + c_3} \qquad 30 = \frac{0 \cdot 15 + b_2 \cdot 20 + 0}{a_3 \cdot 15 + b_3 \cdot 20 + c_3} \tag{7-19}$$

or

$$a_3 \cdot 15 + b_3 \cdot 20 + c_3 = a_1 \qquad 30(a_3 \cdot 15 + c_3) + 30 \cdot 20\, b_3 = b_2 \cdot 20$$

Using above results these become

$$a_1 + b_3 \cdot 20 = a_1 \qquad 30\, a_1 + 0 = b_2 \cdot 20$$

$$b_3 = 0 \qquad a_1 = \frac{2c_3}{3}$$

$$b_2 = c_3 \qquad 15 \cdot a_3 = \left(\frac{20c_3}{30} - c_3\right)$$

$$a_3 = -\,c_3/(3 \cdot 15) \tag{7-20}$$

By insisting that the projective transformation (7–1) should carry four specific points of plane I into four specific points of plane II, it has thus been possible to find the values of eight of the nine constants required to do this in terms of the ninth—c_3. These results are:

$$\begin{array}{lll} a_1 = 2c_3/3 & b_1 = 0 & c_1 = 0 \\ a_2 = 0 & b_2 = c_3 & c_2 = 0 \\ a_3 = -\,c_3/45 & b_3 = 0 & c_3 = c_3 \end{array} \tag{7-21}$$

If the value of c_3 is taken arbitrarily as 1, then all nine have definite values. The transformation then becomes

$$X'' = \frac{\frac{2}{3} X' + 0 \cdot Y' + 0}{\frac{-X'}{45} + 0 \cdot Y' + 1}$$

$$X'' = \frac{30X'}{-X' + 45}$$

$$Y'' = \frac{0 \cdot X' + 1 \cdot Y' + 0}{\frac{-X'}{45} + 0 \cdot Y' + 1}$$

$$Y'' = \frac{45Y'}{-X' + 45} \tag{7-22}$$

Expressed in operator form, this central projection can be written in either of the forms

$$\triangle \equiv \begin{vmatrix} 2/3 & 0 & 0 \\ 0 & 1 & 0 \\ -1/45 & 0 & 1 \end{vmatrix}$$

or

$$\triangle \equiv \begin{vmatrix} -30 & 0 & 0 \\ 0 & -45 & 0 \\ 1 & 0 & -45 \end{vmatrix} \tag{7-23}$$

with inverse

$$\triangle^{-1} \equiv \begin{vmatrix} 45 & 0 & 0 \\ 0 & 30 & 0 \\ 1 & 0 & 30 \end{vmatrix} \tag{7-24}$$

Both $\triangle$ and $\triangle^{-1}$ are of the form of the operator shown in equation (7–15)

where for $\triangle$	and for $\triangle^{-1}$
$k = -\,30$	$k = 45$
$l = 0$	$l = 0$
$m = +\,45$	$m = -\,30$

The central projection has been sketched in Problem 6–7, where it was solved by similar triangles and equations (7–22) derived.

The carrying of four points of plane I into four points of plane II can be very useful. Among other things, it enables a desirable *non-rectangular portion* of a nomogram in plane I to be carried into a rectangular diagram of specified proportions and position in plane II. The operator representing the central projection, when applied to the canonical form ((C′)) defining nomogram I in plane I, yields the canonical form ((C″)) defining nomogram II in plane II. The desirable nomogram in plane II can be chosen to have two sides on the axes of coordinates in that plane, while the non-rectangular, undesirable quadrilateral in plane I can also have one vertex at the origin and another on an axis of coordinates without loss of generality. In such ways, the maximum number of zeros in eight simultaneous equations are used and the solution made practical. A translation of the plane is useful in operator form. It can be found from its equation or by carrying four points into four translated points.

Example 7–2. A translation in the x direction by the amount *a*, would have the equations

$$x_2 = x_1 + a$$
$$y_2 = y_1 \tag{7–25}$$

and hence the operator

$$\triangle \equiv \begin{vmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \tag{7–26}$$

Applied to (1–4), this operator works as expected

$$\begin{vmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} X_1 & X_2 & X_3 \\ Y_1 & Y_2 & Y_3 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} X_1 + a & X_2 + a & X_3 + a \\ Y_1 & Y_2 & Y_3 \\ 1 & 1 & 1 \end{vmatrix} = 0 \tag{7–27}$$

The circular nomogram for multiplication appears in (1–17) with infinite points at the origin and zero points to the right of the origin. Let the x-coordinates be reversed in sign, placing these to the left of the origin and let the diameter of the circle be "a". Now translate the plane by the amount a, using (7–26) in order to find the canonical form of the circular nomogram when the zero points are at the origin and the infinite points to the right.

$$\begin{vmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} \frac{-a}{1+U^2} & \frac{-a}{1+V^2} & \frac{-a}{1+W} \\ \frac{aU}{1+U^2} & \frac{-aV}{1+V^2} & 0 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} \frac{-a}{1+U^2} + a & \frac{-a}{1+V^2} + a & \frac{-a}{1+W} + a \\ \frac{aU}{1+U^2} & \frac{-aV}{1+V^2} & 0 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \frac{aU^2}{1+U^2} & \frac{aV^2}{1+V^2} & \frac{aW}{1+W} \\ \frac{aU}{1+U^2} & \frac{-aV}{1+V^2} & 0 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$$= \begin{vmatrix} \frac{aU^2}{1+U^2} & \frac{+aU}{1+U^2} & 1 \\ \frac{aV^2}{1+V^2} & \frac{-aV}{1+V^2} & 1 \\ \frac{aW}{1+W} & 0 & 1 \end{vmatrix} = 0 \tag{7–28}$$

The same result could have been obtained by multiplying the row of 1's by *a* and adding to the top row of X values in the original canonical form (7–27).

The operator

$$\begin{vmatrix} k & 0 & 0 \\ 0 & -m & 0 \\ 1 & 0 & -m \end{vmatrix} \text{ is a special case of (6–5) } \begin{vmatrix} k & 0 & 0 \\ l & -m & 0 \\ 1 & 0 & -m \end{vmatrix}$$

and able to relate many of the cognate forms of each type of diagram we have considered.

Example 7–3. If the central projection comes from the point $k = B/2$, $l = 0$, $m = B/2$ then two related (cognate) forms arise, one given, one derived, for the three-parallel-line diagram for addition which are well known to us. Figures 6–6(e) and 4–16. We apply the operator above described to the known canonical form to obtain the new canonical form, that is, by (7–9).

$$\begin{vmatrix} B/2 & 0 & 0 \\ 0 & -B/2 & 0 \\ 1 & 0 & -B/2 \end{vmatrix} \times \begin{vmatrix} 0 & B & \dfrac{u}{u+v}B \\ uU & vV & \dfrac{uv\,W}{u+v} \\ 1 & 1 & 1 \end{vmatrix} =$$

$$\begin{vmatrix} 0 & \dfrac{B^2}{2} & \dfrac{B^2}{2}\cdot\left\{\dfrac{u}{u+v}\right\} \\ \dfrac{-B}{2}uU & \dfrac{-B}{2}vV & \dfrac{-B}{2}\cdot\left\{\dfrac{uv\,W}{u+v}\right\} \\ \dfrac{-B}{2} & B\left(\dfrac{-B}{2}\right) & B\left\{\left(\dfrac{u}{u+v}\right)-\dfrac{1}{2}\right\} \end{vmatrix} \equiv$$

$$\begin{vmatrix} 0 & B & \dfrac{Bu}{u-v} \\ uU & -vV & \dfrac{-uv}{u-v}W \\ 1 & 1 & 1 \end{vmatrix} = 0$$

Example 7–4. A canonical form for the circular nomogram has been derived in (7–28). The zero point for all three scales here is at origin, and the three infinite points are at (a, 0). It is interesting to project this familiar nomogram now from a point $k = -a$, $l = 0$, $m = +a$. Figures 7–1 and 1–10.

$$\begin{vmatrix} -a & 0 & 0 \\ 0 & -a & 0 \\ 1 & 0 & -a \end{vmatrix} \cdot \begin{vmatrix} \dfrac{aU^2}{1+U^2} & \dfrac{aV^2}{1+V^2} & \dfrac{aW}{1+W} \\ \dfrac{aU}{1+U^2} & \dfrac{-aV}{1+V^2} & 0 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} \dfrac{-a^2U^2}{1+U^2} & \dfrac{-a^2V^2}{1+V^2} & \dfrac{-a^2W}{1+W} \\ \dfrac{-a^2U}{1+U^2} & \dfrac{a^2V}{1+V^2} & 0 \\ \dfrac{aU^2}{1+U^2}-a, & \dfrac{aV^2}{1+V^2}-a, & \dfrac{aW}{1+W}-a \end{vmatrix}$$

$$\equiv \begin{vmatrix} aU^2 & aV^2 & aW \\ aU & -aV & 0 \\ 1 & 1 & 1 \end{vmatrix} = 0 \qquad (7\text{–}29)$$

Example 7–5. The familiar three-parallel-line alignment diagram for addition and the equally familiar three-concurrent-line diagram for the sum of reciprocals are cognate types. In the three-parallel-line diagram, the point of concurrency is at infinity. An operator which projects the point of concurrency from a local position to an infinite position, or conversely, will carry one form ((C)) into the other. (Figure 7–2) This was shown pictorially once before in Figure 6–6(b). Using equations (6–5), (1–16), with k = a, l = 0, m = c,

$$\begin{vmatrix} a & 0 & 0 \\ 0 & -c & 0 \\ 1 & 0 & -c \end{vmatrix} \cdot \begin{vmatrix} -\frac{U}{2}+c & V+c & \frac{W}{2}+c \\ \frac{\sqrt{3}}{2}U & 0 & \frac{\sqrt{3}}{2}W \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} -a\frac{U}{2}+ac & aV+ac & a\frac{W}{2}+ac \\ -c\frac{\sqrt{3}}{2}U & 0 & -c\frac{\sqrt{3}}{2}W \\ -\frac{U}{2}+c-c & V+c-c & \frac{W}{2}+c-c \end{vmatrix} \tag{7–30}$$

$$= \begin{vmatrix} a-\frac{ac\cdot 2}{U} & a+\frac{ac}{V} & a+\frac{2ac}{W} \\ c\sqrt{3} & 0 & -c\sqrt{3} \\ 1 & 1 & 1 \end{vmatrix} = 0 \tag{7–31}$$

This is the familiar three-parallel-line diagram for a sum of reciprocals with the inner scale equidistant from the outer scales and using a scale factor half as great as theirs. U, V and W are infinite along an ordinate *a* units from the Y-axis. The inner, dependent variable is V, so that U has a direction opposite to V and W, Figure 4–16(c).

7–6. *Interpreting the Constants of the Operator.* The projective operator from (6–5)

$$\begin{vmatrix} k & 0 & 0 \\ l & -m & 0 \\ 1 & 0 & -m \end{vmatrix}$$

was special. k, *l* and m were Cartesian coördinates of the center of projection and the coordinate systems of plane I and plane II fitted into and were a part of this same Cartesian system, Figure 6–8. Equation (7–1) also related the coordinates of a point in plane I with those of a point in plane II, but here the coordinate systems of plane I and plane II were not as simply related. Their origins were distinct, their Y-axes also, and each system was rotated and translated to a general position. The front, top and auxiliary views of such a pair of coordinate systems in planes I and II are shown in Figure 7–3, together with a center of projection 0 and a pair of convenient, intermediate coordinate systems like those used in Figure 6–11. The origins of these two intermediate systems coincide at the foot of a perpendicular from the center of projection 0 onto the line L common to planes I and II. Their Y-axes also coincide in line L. Let the intermediate systems be designated by X_1, Y_1 and X_2, Y_2. A translation and rotation will carry the original system x′, y′ of plane I into the intermediate system X_1, Y_1. Let this be written

$$X_1 = x' \cdot \cos\phi - y' \cdot \sin\phi + X_1^{\circ}$$
$$Y_1 = x' \cdot \sin\phi + y' \cdot \cos\phi + Y_1^{\circ} \tag{7–32}$$

or, in operator form

$$\begin{vmatrix} \cos\phi & -\sin\phi & X_1^{\circ} \\ \sin\phi & \cos\phi & Y_1^{\circ} \\ 0 & 0 & 1 \end{vmatrix} \tag{7–33}$$

A translation and rotation will carry the intermediate system X_2, Y_2 into the original system of plane II, x'', y''. Let this be written

$$x'' = X_2 \cdot \cos\theta + Y_2 \cdot \sin\theta + x''_o$$
$$y'' = -X_2 \cdot \sin\theta + Y_2 \cdot \cos\theta + y''_o \qquad (7\text{–}34)$$

or, in operator form

$$\begin{vmatrix} \cos\theta & \sin\theta & x''_o \\ -\sin\theta & \cos\theta & y''_o \\ 0 & 0 & 1 \end{vmatrix} \qquad (7\text{–}35)$$

The passage from one intermediate system X_1, Y_1 to another X_2, Y_2, in rectangular or affine coordinates (6–5), (6–16), has already been shown to be in operator form

$$\begin{vmatrix} R_1 & 0 & 0 \\ 0 & -R_2 & 0 \\ 1 & 0 & -R_2 \end{vmatrix} \qquad (7\text{–}36)$$

The projection of a point in plane I (x', y') into a point in plane II (x'', y'') can be done by first finding the intermediate coordinates X_1, Y_1 in plane I by (7–32), (7–33), then projecting these into the intermediate points X_2, Y_2 on plane II, by (7–36) and then finding from these the final coordinates x'', y'' (7–34), (7–35). Using operators applied in order from right to left, these projections can be expressed, Section 7–2, in the product form.

$$\triangle = \begin{vmatrix} \cos\theta & \sin\theta & x''_o \\ -\sin\theta & \cos\theta & y''_o \\ 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} R_1 & 0 & 0 \\ 0 & -R_2 & 0 \\ 1 & 0 & -R_2 \end{vmatrix} \cdot \begin{vmatrix} \cos\phi & -\sin\phi & X_1{}^\circ \\ \sin\phi & \cos\phi & Y_1{}^\circ \\ 0 & 0 & 1 \end{vmatrix} \qquad (7\text{–}37)$$

The determinant resulting from this multiplication is the one that does the entire job of projecting x', y' into x'', y'', that is, it is the operator that performs the transformation (7–1), (7–2)

$$\triangle \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad (7\text{–}38)$$

On carrying out this determinant multiplication (7–37), the quantities a_1 to c_3 will be found to have the values given in Figure 7–3. Fortunately, many practical problems can be set up in the more convenient coordinates X_1, Y_1; X_2, Y_2, related by (7–36). Figure 7–3 illustrates and corroborates much of this material and shows true size orthographic projections of planes I and II. Eight independent parameters have had their values recorded there, namely

$$x''_o = -4.15 \qquad Y_1{}^\circ = 0.20$$

$$X_1{}^\circ = 4.34 \qquad R_2 = \frac{4.79}{\sin 54^\circ 20'}$$

$$R_1 = \frac{1.74}{\sin 54^\circ 20'} \qquad \theta = 20^\circ 40'$$

$$y''_o = 2.83 \qquad \phi\,[=\gamma] = 29^\circ 40'$$

The angle between the planes is shown but is not one of the parameters because of the use of affine coordinates R_1 and R_2 (Section 6–11, Figure 6–13). The eight degrees of freedom in the specification of the central projection equal in number the independent parameters of the projective transformation (7–1) that expresses this transformation analytically (Section 7–3). The values of nine *homogeneous* constants of the projection shown there turn out to be respectively,

$$\begin{vmatrix} 0.63 & -1.07 & 11.9 \\ -0.27 & -1.72 & -2.58 \\ 0.71 & -0.40 & 1.78 \end{vmatrix}$$

A quadrilateral in plane I is shown projected into a rectangle of plane II. If these nine constants are used in (7–1), it will be found to carry the quadrilateral into the rectangle.

These details are given to show how real a case can be. The chief need is for general understanding and appreciation rather than frequent derivation and uses, though these are sometimes very helpful.

PROBLEMS

PROBLEM 7–1. Derive the inverse, D^{-1}, of the operator

$$D = \begin{vmatrix} k & 0 & 0 \\ l & -m & 0 \\ 1 & 0 & -m \end{vmatrix}$$

Show that

$$D \cdot D^{-1} = I$$

$$D^{-1} \cdot D = I$$

PROBLEM 7–2. Derive the operator D representing the transformation carrying points from plane I to plane II as indicated

Plane I		Plane II
(0,0)	to	(0,0)
(0,20)	to	(0,20)
(15,0)	to	(15,20)
(15,−20)	to	(15,0)

1. Write down and solve the eight simultaneous equations of form (7–1) with the above values substituted into them.
2. Express the central projection in equation form like (7–22).
3. Express the projection in operator form and and derive its inverse. Find the coordinates of the center of projection. See (7–15). Compare with Problem 6–9.

PROBLEM 7–3. Derive the operator D representing the transformation carrying points from plane I to plane II as indicated

Plane I		Plane II
(0,0)	to	(0,0)
(0,20)	to	(0,20)
(15,0)	to	(15,20)
(15,20)	to	(15,0)

1. Write down and solve the eight simultaneous equations of form (7–1) with the above values substituted into them.
2. Express the central projection in equation form like (7–22).
3. Express the projection in operator form and derive its inverse. Find the coordinates of the center of projection. See (7–15). Compare with Problem 7–2.

PROBLEM 7–4.

1. Write down the operator D_1 that reverses the sign of x throughout the plane.
2. Write down the operator D_2 that increases each X-value by *a*.
3. Form the resulting operator $D_2 \cdot D_1 = D_3$.
4. Derive a canonical form for the circular nomogram by applying D_3 to (7–28).
5. Show why this form is that of (1–17).

PROBLEM 7–5. Derive the operator representing the transformation carrying points from plane I to plane II as indicated.

Plane I		Plane II
(0,0)	to	(a,0)
(a,0)	to	(0,0)
(a/2,a/2)	to	(a/2,a/2)
(a/2,−a/2)	to	(a/2,−a/2)

Show why this is the same operator as was obtained in Problem 7–4.

PROBLEM 7–6. The operator D_3 in Problem 7–4 carried (7–28) back to the form of (1–17). The definition of D_3 shows that it follows the steps of Example 7–2, which is the reverse of this, so D_3 must be its own inverse. Check this fact by: 1) deriving the inverse of D_3, namely D_3^{-1}, and 2) deriving the product $D_3^{-1} \cdot D3$.

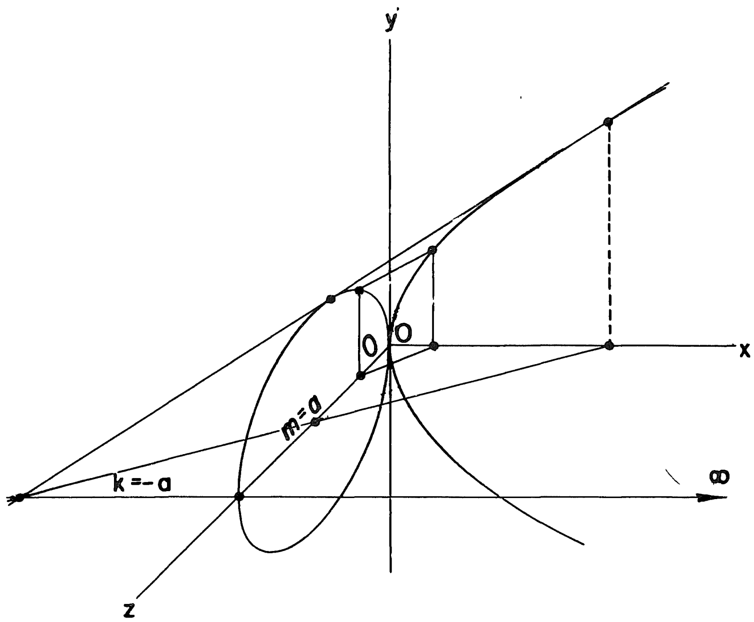


Figure 7–1.

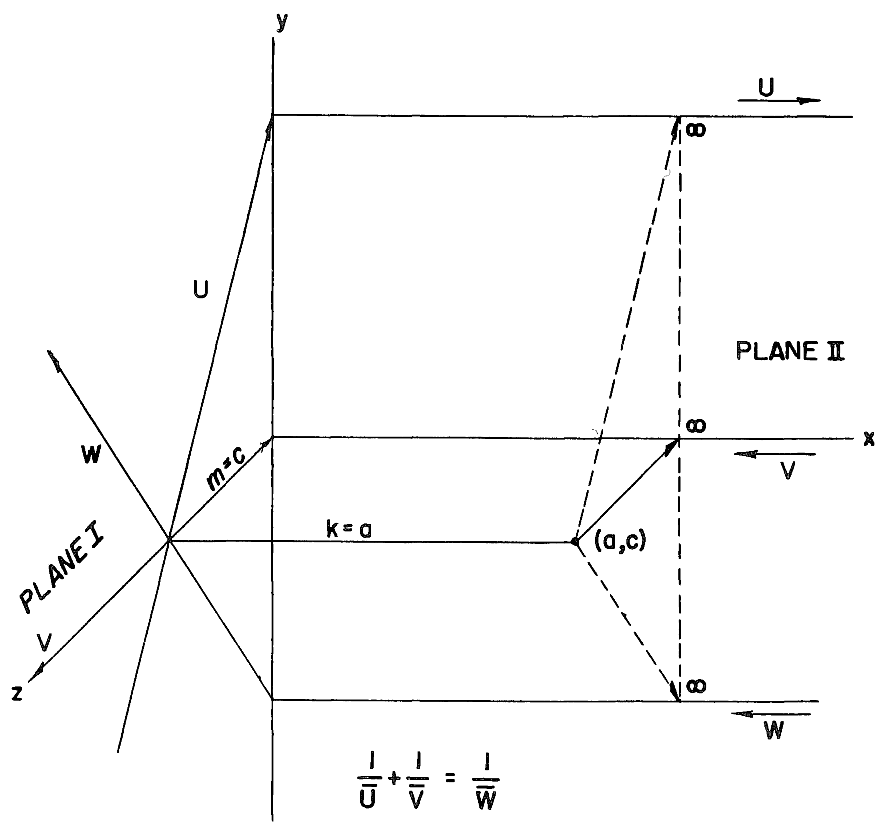


Figure 7–2.

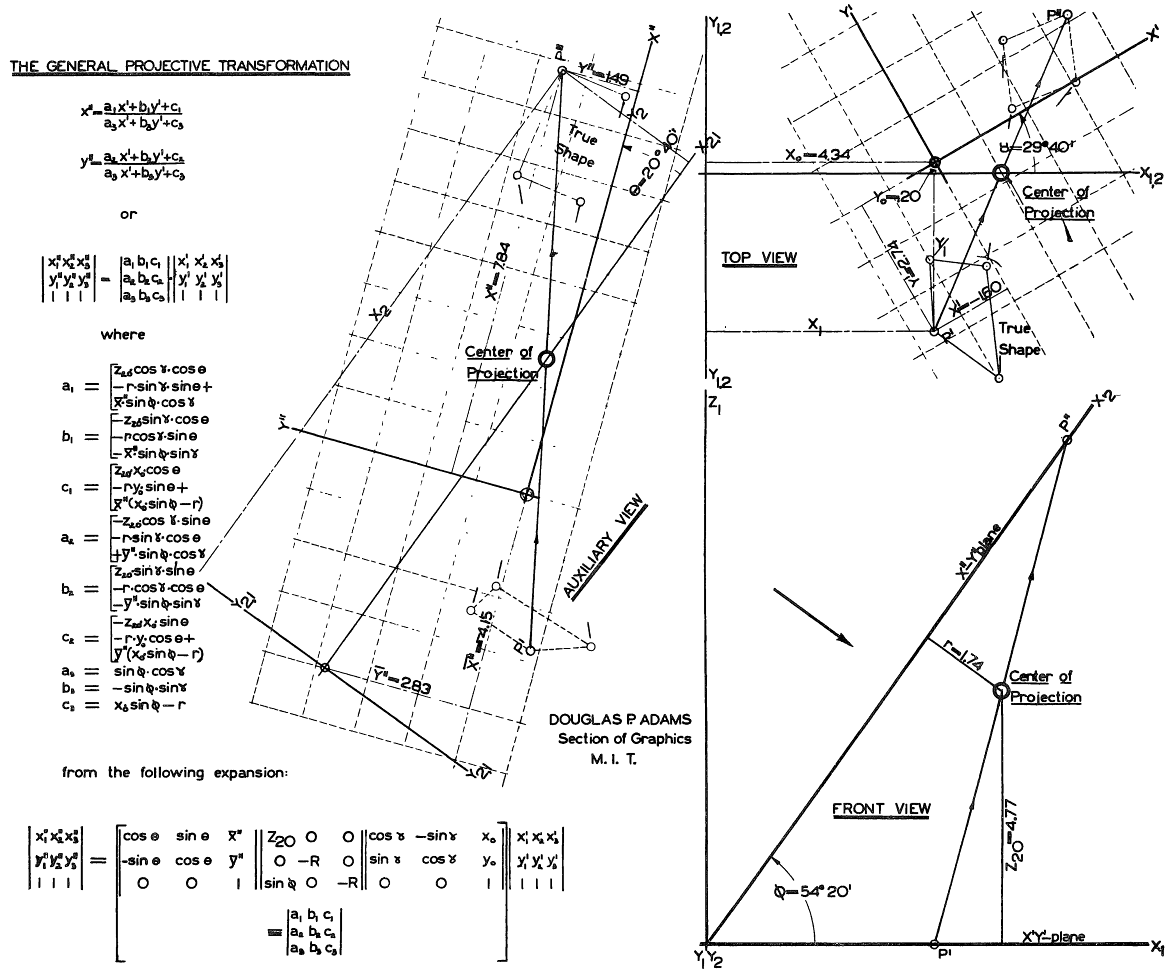

THE GENERAL PROJECTIVE TRANSFORMATION
x'' = (a1x'+b1y'+c1)/(a3x'+b3y'+c3)
y'' = (a2x'+b2y'+c2)/(a3x'+b3y'+c3)
or
where
a1 = z2,0 cos γ·cos θ − r·sin γ·sin θ + x̄''sin φ·cos γ
b1 = −z2,0 sin γ·cos θ − r cos γ·sin θ − x̄''sin φ·sin γ
c1 = z2,0 x0 cos θ − r y0 sin θ + x̄''(x0 sin φ − r)
a2 = −z2,0 cos γ·sin θ − r sin γ·cos θ + ȳ''·sin φ·cos γ
b2 = z2,0 sin γ·sin θ − r·cos γ·cos θ − ȳ''·sin φ·sin γ
c2 = −z2,0 x0 sin θ − r·y0 cos θ + ȳ''(x0 sin φ − r)
a3 = sin φ·cos γ
b3 = −sin φ·sin γ
c3 = x0 sin φ − r
from the following expansion:
DOUGLAS P. ADAMS
Section of Graphics
M. I. T.
AUXILIARY VIEW
TOP VIEW
FRONT VIEW
Center of Projection
True Shape
X0=4.34
Y0=1.20
θ=29°40'
φ=54° 20'
r=1.74
Z20=4.77
Y''=1.49
X''=7.84
X̄''=4.15
Ȳ''=2.83
θ=20°
X''Y''-plane
X'Y'-plane

Figure 7–3.

CHAPTER 8

DUALITY OF POINT AND LINE

POINT COORDINATES AND LINE COORDINATES

8–1. *Corresponding Properties of Points and Lines.* There is an intimate relationship between the ordinary, much-used network chart and the alignment diagram. An interesting study of points and lines reveals the basis of this relationship and helps the network chart contribute simply and directly to the alignment diagram.

In the plane, the point seems like a simpler element than the line. We are used to thinking of the line as "consisting of its points," so it seems more massive than any one of the points within it. A different attitude toward the line, one which treats it as an equal or-partner with the point, can be useful. The equation of a line in point coordinates can be written

$$Ax + By = 1 \qquad (8\text{–}1)$$

This equality states that the equation of the line is satisfied by the coordinates x, y of each and every point lying on it.

If x, y are allowed to be constants in (8–1) and A and B are permitted to vary, (8–1) then defines a point x, y. It says that the equation (8–1) of the point x, y is satisfied by the coordinates A, B of each and every line through the point. A, B will seem still more natural in the role of coordinates of a line when it is observed that they are the reciprocals of the intercepts of the line.

Our former notion of the point as being *simpler* than the line was unfounded; in fact, the two are strictly comparable things in two dimensions. The line is defined by all the points on it; the point is defined by all the lines through it. Further statements for point and line are as follows:

Three points lie on the same line if:

$$Ax_1 + By_1 = 1$$
$$Ax_2 + By_2 = 1$$
$$Ax_3 + By_3 = 1$$

Three lines pass through the same point if:

$$A_1x + B_1y = 1$$
$$A_2x + B_2y = 1$$
$$A_3x + B_3y = 1 \qquad (8\text{–}2)$$

The condition of collineation of three points accordingly is:

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

The condition of concurrency of three lines accordingly is:

$$\begin{vmatrix} A_1 & B_1 & 1 \\ A_2 & B_2 & 1 \\ A_3 & B_3 & 1 \end{vmatrix} = 0 \qquad (8\text{–}3)$$

The equation of a line determined by two points can be written, in point coordinates:

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

The equation of a point determined by two lines can be written, in line coordinates:

$$\begin{vmatrix} A & B & 1 \\ A_1 & B_1 & 1 \\ A_2 & B_2 & 1 \end{vmatrix} = 0 \qquad (8\text{–}4)$$

8–2. *Dual Figures in Point and Line.* The statements opposite one another above pass into one another if the words *point* and *line, collineation* and *concurrency,* are interchanged. Such statements can be said to be dual in point and line. Dual figures (Figure 8–1) are possible, also, such as:

(a) Two *points* and the *line* joining them.	(b) Two *lines* and *point* common to them.
(c) *Point* B on *line* m, *line* L determined by *points* A and B.	(d) *Line* b through *point* M, *point* L determined by *lines* a and b.
(e) Three *points* and the three *lines* determined by them in pairs.	(f) Three *lines* and the three *points* determined by them in pairs.

Since each of the latter figures is a triangle, this figure is one dual to itself, or self-dual.

8–3. *Network Chart and Alignment Diagram are Dual Figures.* (Figure 8–2.) An alignment diagram and a network chart that has only straight lines on it, if both are for the same equation, are dual figures. The reasoning runs as follows: In the network chart for the equation $F(U, V, W) = 0$, consisting only of straight lines, if there is a point through which a *line* U_1, a *line* V_1 and a *line* W_1 go, then those three values satisfy the equation, that is, $F(U_1, V_1, W_1) = 0$; on the alignment diagram, if there is a *line* on which a *point* U_1, a *point* V_1, and a *point* W_1, lie, then those three values satisfy the equation, that is, $F(U_1, V_1, W_1) = 0$. These are dual statements, since interchanging *line* and *point, point* and *line* carries either one into the other.

If one had a network chart representing an equation or empirical data, without any knowledge of nomography an alignment diagram could be made for that equation or empirical data if a method were known for creating a diagram dual to the network chart, for this dual diagram would be the desired alignment diagram. Such methods are given later.

8–4. *Duality of Point and Line by Equality of Coordinates.* The points and lines of Figure 8–1 were few in number. One naturally wonders if all the lines in the plane and all the points in the plane can possibly be related by a duality. If the relation is to be useful, it would have to occur in such a way that each time three lines were concurrent the three points related to them would be collinear and vice versa. At least one way to bring this dual relation about is immediately apparent from (8–3). To every point with coordinates x, y, let there correspond a line with coordinates A, B according to the law that $x = B$, $y = A$. The dual correspondence between point and line is then unique and one-to-one. If three points are collinear and the three lines corresponding to them are concurrent,

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 \quad \text{then by (8–3)} \quad \begin{vmatrix} A_1 & B_1 & 1 \\ A_2 & B_2 & 1 \\ A_3 & B_3 & 1 \end{vmatrix} = 0 \qquad (8–5)$$

The converse is true. This handy duality can be called *Coordinate Duality.*

The line coordinates A, B used here came from the convenient form (8–1) of the equation of the line

$$Ax + By = 1$$

where A and B are the reciprocals of the intercepts of the line on the axes. This permits plotting the dual line from the point or the dual point from the line with relative ease and provides quick testing of the above properties. In Figure 8–3(a), (b), (c), the rectangular hyperbola $y = 1/x$ has been carefully plotted. If the ordinate of a point M ($x = B$, $y = A$) (Figure 8–3(a)) is carried to this curve, the abscissa at the curve will be its reciprocal, and hence the x-intercept of the dual line. Let the abscissa value of a point M be carried to the curve to furnish an ordinate value, the y-intercept of the dual line. In Figure 8–3(b) any three points P_1, P_2, P_3 which are collinear will be found to give rise to three lines L_1, L_2, L_3 which are concurrent. (We note in passing that in this particular case the point Q_4 of their concurrency will correspond by the same rules to that line m_4 of collineation.)

In near-coordinate duality can the point dual to a line ever lie upon the line itself? Can the line dual to a point ever pass through the point itself? The dual to point x_1, y_1, line $A_1x + B_1y = 1$, becomes

$$y_1x + x_1y = 1 \qquad (8–6)$$

If the point lies upon it, its coordinates satisfy its equation

$$y_1x_1 + x_1y_1 = 1$$

$$x_1y_1 = 1/2 \qquad (8\text{–}7)$$

Hence in near-coordinate duality, the locus of points such that they lie upon their dual lines, and conversely, *is another rectangular hyperbola halfway to the one originally plotted.* The locus is defined by these points, yet also by their dual lines for these are its tangents and the curve is their envelope. Dual point and line here are tangent point and tangent line and this curve $xy = 1/2$ is a self-dual figure under this duality. Figure 8–3(c) Two such tangent points and lines are shown there. Note also that the chord thru the tangent points is dual to the point determined by the tangent lines.

The points at infinity and the line at infinity are included in the pattern of coordinate duality (Section 6–3).

A dual point or line that seems hard to identify can sometimes be found as the limit of a succession of readily identifiable dual points or lines. No dual point or line can long resist this approach.

Example 8–1. Find the dual point Q of the line L: $Ax + By = 0$. Figure 8–4. The line can be regarded as the limit, as $k \to 0$, of the lines $Ax + By = K$, all of which have the slope $\lambda = -A/B$, or as the limit of the lines $A/K + B/K = 1$. The dual points of such lines have coordinates $(B/K, A/K)$ and they all lie in a line through the origin with slope A/B. Point Q, as the limit of these points, as $K \to 0$, is the point at infinity in the direction A/B. Figure 8–4 shows the graphical derivation of the same result.

Example 8–2. Another way to use coordinate duality but not have to draw or use the curve $y = 1/x$ is to graduate the X- and Y-axes uniformly in A and B and reciprocally (on the other side) in B and A. Coordinates of points can be read on the reciprocal scale and intercepts of lines on the uniform scale (or conversely), each pair A, B then giving rise to dual elements. In Figure 8–5 the two sets of lines have been separated for clarity. Example 8–1 shows that in dualizing a network chart its lines and points should avoid the origin so that their duals will not be too far out in the alignment diagram.

8–5. *Dual Point and Line for a Conic. Pole and Polar by Graphical Methods.* A second way of setting up a useful dual correspondence between the points and the lines of the plane comes from the notion of pole and polar to a conic. (Figure 8–6.) Under this notion, tangents to a conic are drawn from a point P called the pole. If points Q and R are the points of tangency, then line QR is said to be the polar line to pole P. It appears graphically that if three poles P_1, P_2, P_3 are collinear in line j their three polar lines L_1, L_2, L_3 will be concurrent in point K. Point K is the pole for polar line j. Thus a point "interior" to the conic has a polar line (which will be "exterior") tho tangents cannot be drawn to the conic from this point. The pole and polar scheme furnishes a unique, one-to-one correspondence between every point and every line of the plane. The points at infinity and the line at infinity are dual respectively to lines through the center and the center itself. (Figure 8–4.) It is equally clear that the conic is a self-dual figure under this scheme, tangent point and tangent line being dual elements. (Either the tangent points or the tangent lines define the conic.) This perfect duality of point and line of the plane works this way only from the conic curve.

8–6. *Pole and Polar Algebraically.* The algebraic forms for pole and polar take simple form. For the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

pole and polar turn out to have the form

Pole: Point P, $(x_1\ y_1)$

$$\text{Polar: Line L, } \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1. \qquad (8\text{–}8)$$

Tangent point and tangent line have this form, being familiar examples of pole and polar. There are correspondingly simple equations for all the other conics. For instance, for the rectangular hyperbola

$$xy = 1/2 \qquad (8\text{–}9)$$

Tangent point and tangent line and hence pole and polar take the form

Pole: Point P, $(x_1\ y_1)$

Polar: Line L, $x_1y + y_1x = 1$ (8–10)

(See Problem 8–2.) The coordinate duality of Section 8–4 now appears as a conic duality using for conic the rectangular hyperbola $y = 1/2x$, since this is the locus of self-dual points under that duality. (8–6) (8–7). Figure 8–3(c).

8–7. *Practical Duality Methods. The Scanner.* Deriving an alignment diagram as the dual to a network chart must be a practical method to be useful. In Section 8–5 it was necessary to draw a conic and then construct tangents to it. The algebraic form of Section 8–6 did not lend itself well to graphical methods. The coordinate duality of Section 8–4 has several advantages but none of these methods have as many advantages as the duality that now follows.

If the conic used is the *parabola,* the dual process becomes simpler. Figure 8–7(a): let m be a line through a pole P parallel to the axis and G, K and Q the points where line m cuts the parabola, directrix and polar line L respectively. Then PG = GQ and line KF is perpendicular to polar line L. See also Appendix, Sections 13–6, 13–7. Hence L can be drawn if point P is given, or conversely. This method requires drawing the parabola and then using it. The algebraic relation for both tangent point and tangent line and pole and polar states that for the parabola $y^2 = 2mx$, we have

$$\text{Pole: Point P, } (x_1, y_1)$$
$$\text{Polar: Line L, } yy_1 = (mx + x_1) \qquad (8\text{–}11)$$

This permits an alternative procedure in which the *direction* of L is found once more as perpendicular to the direction KF but the *placement* of L comes from the fact that its x-intercept in (8–11) has the negative value of the x-coordinate of Point P. Thus the drawing and use of the parabola itself is avoided—a point and a line (the focus and directrix of the parabola) are all that is necessary for the dualizing to begin. Figure 8–7(b).

Simplification can be carried further and mechanized. Figure 8–9. Since the x-coordinate of the pole equals the x-intercept of the polar line with sign reversed, one can omit reversal of this sign and thereby interchange left for right in the dual alignment diagram. The y-coordinate of the pole is the height of line m, or, by Figure 8–7(a), $y_1 = m \cdot \tan\phi$ and $y_1 = m/\tan\theta$. Let S be a transparent plastic sheet with lower and right edges square, Figure 8–9. Line 0X is parallel to the lower edge and parallel to and distant m from congruent, uniform symmetrically placed y-scales. A third congruent y-scale lies along the right edge. A line N of the network chart seen passing through 0 cuts a horizontal y-scale at $y = m/\tan\theta$, which is pricked off on the right-hand scale and marks the pole there. Successive lines N of the network chart are brought to pass through 0 by sliding S along a T-square. Since all poles are shifted to the right by the same amount, the dual diagram is preserved and is the desired alignment diagram.

These have a certain sameness for ordinary network charts. Each set of coordinate lines has constant slope and hence constant y and will pass to a straight scale of points lying in the prime direction, as in Figure 8–8. A diagonal of the network chart should preferably be made to lie along line 0X of slider S, for then these two scales of points will be sides of a rectangle equal in length to the diagonal of the network chart and the breadth of this rectangle = $2m/\sin 2\theta$. Lines being dualized which have slopes lying between the positive coordinate directions will give rise to points lying within this rectangle, *otherwise outside it.* This latter fact is a characteristic of nomography and not a weakness of the method. Placement of the *other* diagonal of the network chart along 0X reverses the direction of one of the side scales and brings inside the rectangular shape of the diagram points (poles) which formerly were outside. The equivalent analytical steps and their significance are discussed in the text under dependent variable, determinant changes, substitutions, central projection, etc.

The process of sliding S over the network chart until each line has been caused to pass through point 0 can be called "scanning" the network chart. Some of the lines of the figure being dualized may not cut the line 0X, so prick a hole through the plastic at a point on 0X suitable for 0 and slightly countersink the under side of it. Thread the hole from the under side and knot the thread, clipping the end short at the knot. Hammer a two-inch length of lead square and wind up or unwind thread on it, using it for a weight to keep the thread taut. When the thread is brought into careful coincidence with any line N of the network chart, N will be known to pass through point 0 and a reading can be made where the thread cuts y. Scanning is more automatic and much faster if done with the device described in Problem 8–7.

8–8. *Dual Curves.* In finding dual figures up to now, it has always been assumed that the original figure consisted of *points* and *lines.* Thus the network chart for which an alignment diagram was sought was assumed to have a family (referred to as the W-family) consisting only of *straight lines,* each one of which carried over into a *point* of the W-curve of the alignment diagram. If a network chart has a family of curves, it is always advisable to straighten or rectify these curves so that this process can be used, there being a number of standard ways to try to do this (See Chapter 9).

Assume the nomographer has a network chart that he wishes to dualize which has curves that either cannot be straightened or else he has been unable to find a way to straighten them. Can these *curves* be dualized? A hint is found from the self-dual curves of Figures 8–3(c), 8–6 where the conic used to create the dual relationship consisted of lines or points which were self-dual (the tangent points and tangent lines of the curve). Either the tangent points or the tangent lines define the curve. In the latter case, the curve is the *envelope* of the tangent lines. Thus the hint is that any curve can be said to be defined either by its points or by its tangent lines and hence can be immediately dualized. The duals of its points will give a set of lines whose envelope will be the dual curve; in a perfect duality, the duals of its tangent lines will give a set of points defining the same dual curve. The "scanning" method described above permits such dual curves to be found quickly. *The tangents to the curve in* the network chart can be taken anywhere along the curve in that chart, their chosen closeness depending upon whether their dual points are close enough to give a good dual curve.

Figure 9–1 shows a congested network chart and the corresponding alignment diagram derived by the method of Figure 8–7. Curves which could not be rectified in the network chart have been dualized into curves of the alignment diagram. The use of such curves is described in Chapter 9.

8–9. *Dual Curvature.* A point has a radius of curvature zero; a line has an infinite radius of curvature. Since point and line are dual figures, one naturally wonders if dual curves always have reciprocal curvature at corresponding points. If a parabola or scanner is used for the dualizing, this fact is *almost* true. *For A SET OF CURVES HAVING A COMMON TANGENT LINE AND TANGENT POINT,* the dual curvature of a set member *is* proportional to the reciprocal of original curvature. (See Appendix for proof.)

8–10. *The General Straight Line Network Chart.* The scanner converts an ordinary network chart into an alignment diagram having two parallel straight line scales. An alignment diagram with three curved scales could never come this way from dualizing a conventional network chart, for each family of points of the alignment diagram would have had to have come from a family of lines in the network chart *whose envelope was the curve dual to the curve of the family of points.* Thus the most general straight line network chart comparable to a *three-curve* alignment diagram would appear schematically as in Figure 8–10, a form infrequent in practice. The conventional network chart can be said to be a special case of this form where the envelopes of two of the families of straight lines are *points which lie at infinity 90° apart.* The envelopes are included in Figure 8–10 because they have been mentioned above.

PROBLEMS

PROBLEM 8–1. Plot carefully the curves $y = 1/x$ and $y = 1/2x$. Using the near-coordinate duality of Figure 8–3 graphically, draw a line L and find its dual point P. Verify by several such cases that P and L are pole and polar with respect to the curve $y = 1/2x$.

PROBLEM 8–2. Show that the tangent to the curve $y = 1/2x$ at the point (x_1, y_1) can be put in the form $y_1x + y_1y = 1$. Using this as a formula for pole and polar, plot several of these and show that pole and polar are related graphically by near-coordinate duality.

PROBLEM 8–3. Using near-coordinate duality, derive either graphically or analytically (or both if possible) the duals of the following points and lines:

Points	*Lines*
(2, 3)	$3x - 4y = 12$
(0, 5)	$x + 4y = 4$
(7, 0)	$x + 8y = 8$
(5, 1/10)	$x - 7y = 0$
(5, 1/5)	$x = 2$
(0, 0)	$y = 3$
	$x = 0$
Point at infinity in	$y = 0$
direction $\frac{Y}{X} = \frac{3}{2}$	Line at infinity

PROBLEM 8–4. Using near-coordinate duality, show that the alignment diagram dual to every conventional network chart will contain two straight scales of points at right angles through the origin.

PROBLEM 8–5. Using near-coordinate duality, derive graphically the figure dual to the ellipse $x^2/9 + y^2/4 = 1$. Assume first that the ellipse is a line curve, second that it is a point curve.

PROBLEM 8–6. In Problem 8–5, draw in the curve $y = 1/2x$ and a tangent T to it where it is cut by the ellipse. Why does T turn out also to be tangent to the dual of the ellipse?

PROBLEM 8–7. A second type of scanner, Figure 8–11, more automatic and much faster than Figure 8–9, consists of slider S and disk D attached to S at point 0 but free to turn about point 0. Line M in D is perpendicular to line L drawn on D. Scale Y′ in S is perpendicular to line 0X drawn on S. The intersection of line M and scale Y′, marked at the same reading on scale Y, yields *point P dual to a line covered by L.* Sliding S and turning D permits L to move so as to cover any line N.

1) Explain why and how the scanner works.

2) Describe all the effect of using a second scale Y'_2 further from point 0.

PROBLEM 8–8. Using the method of Figure 8–7(a), draw a parabola $y^2 = 2mx$, a line cutting it, and its pole P. Verify that lines tangent to the parabola from pole P meet it where it is cut by L.

PROBLEM 8–9. Prepare a quick, empirical sketch of a network chart rather similar to that of Figure 9–1(a). Using the method of Figure 8–9(b), arrange the network chart in such positions that it will yield alignment diagrams similar to those of Figure 9–1(b). Derive these.

PROBLEM 8–10. In the near-coordinate duality of Section 8–4, the relation between point coordinates and line coordinates was given by $x = B$, $y = A$. The roles of x and y are interchanged if the coordinate duality is made to be $x = A$, $y = B$. This can be done most easily by a *reflection* of x and y in the 45° ray in Figure 8–3(a). Carry thru this figure with this modification and verify the new type of conic resulting from the change. Derive the new equations (8–6)′ and (8–7)′. and show that they check with these results.

PROBLEM 8–11. The equation in canonical form

$$\begin{vmatrix} U & U^2 & 1 \\ V^2 & V & 1 \\ W & W^3 & 1 \end{vmatrix} = 0$$

defines an alignment diagram with three curves. Construct the dual network chart by the duality method of Figure 8–7(b) and show that it has the form of Figure 8–10. Verify by using both charts to get the same numerical results.

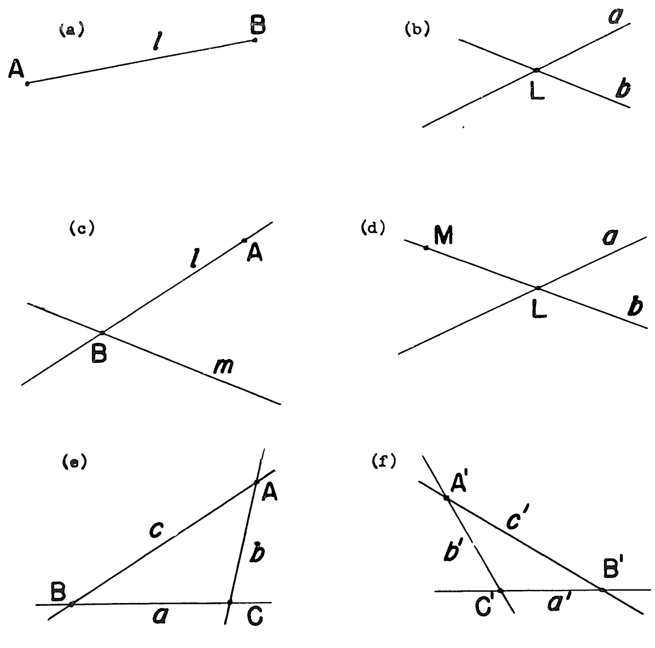


Figure 8–1.

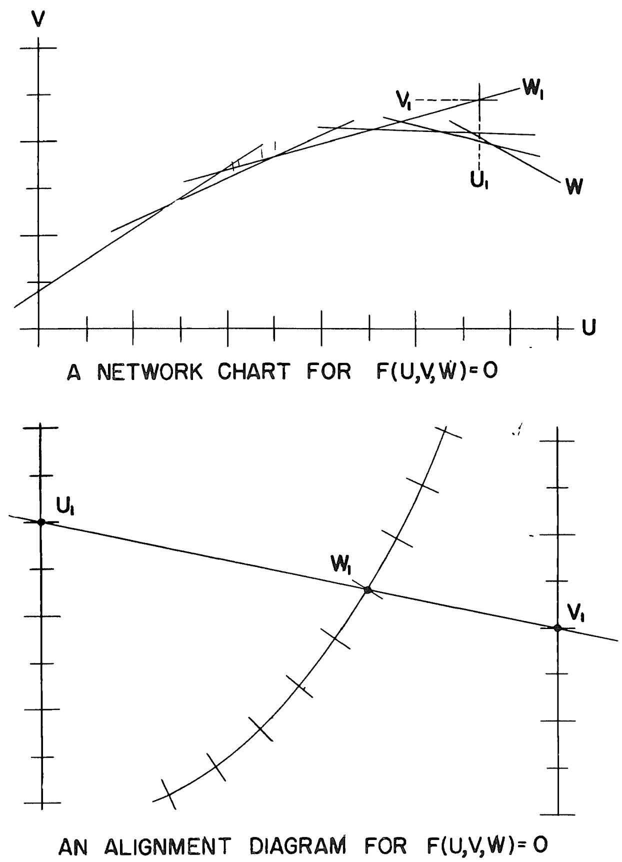


Figure 8–2.

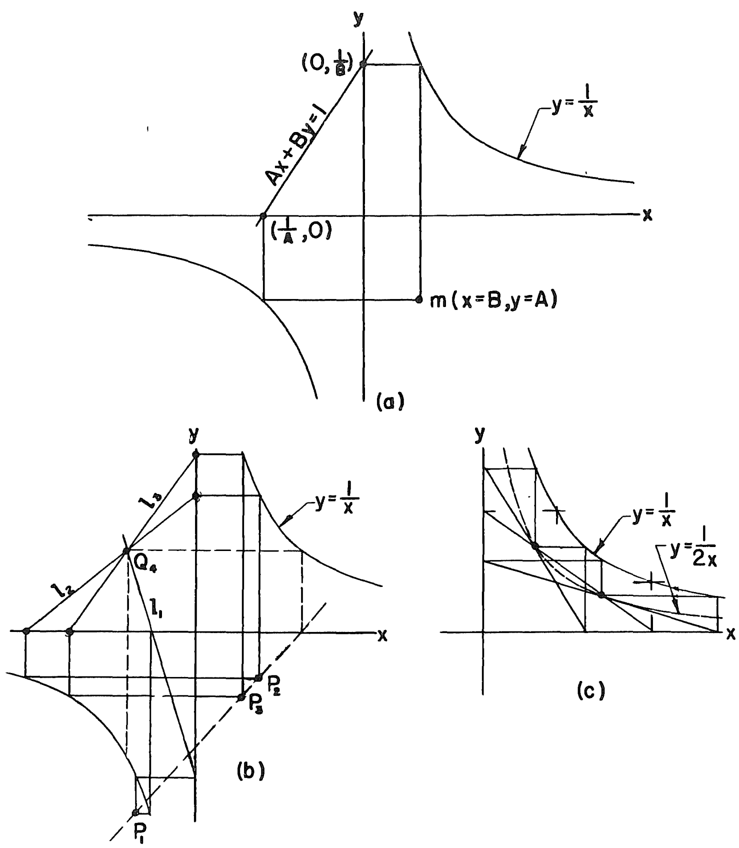


Figure 8–3.

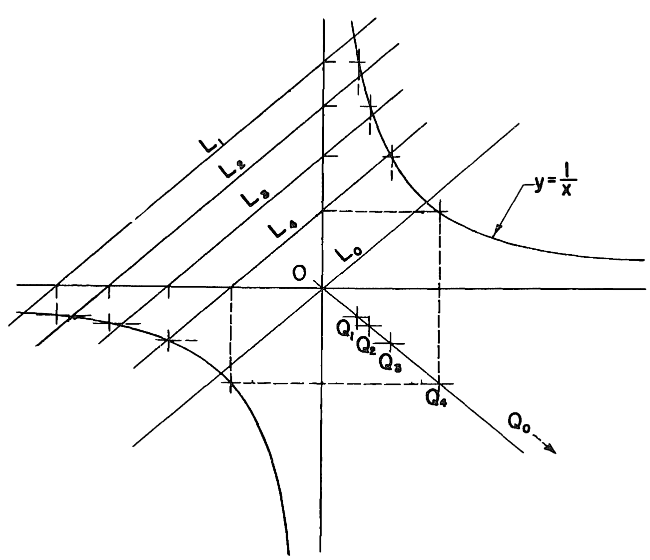


Figure 8–4.

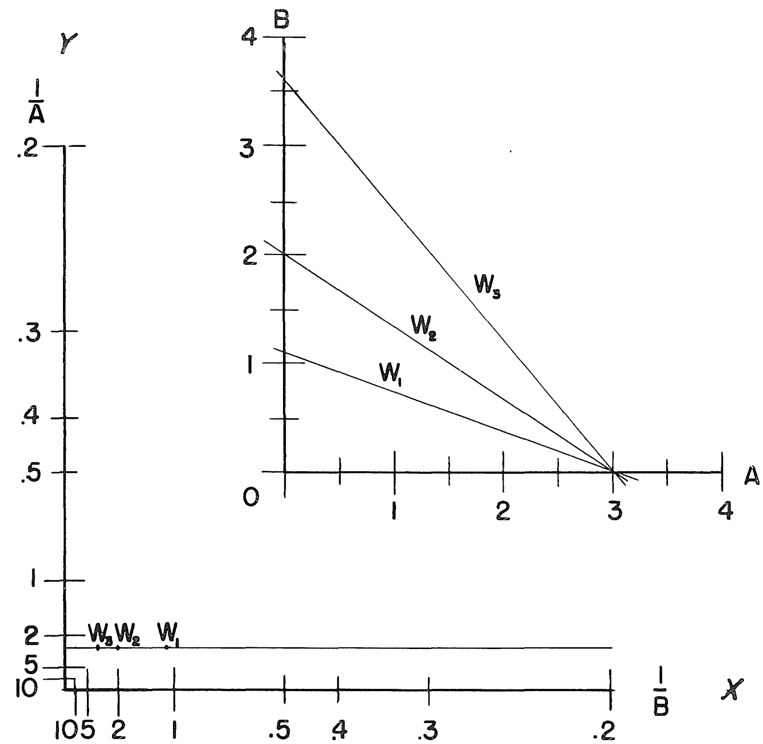


Figure 8–5.

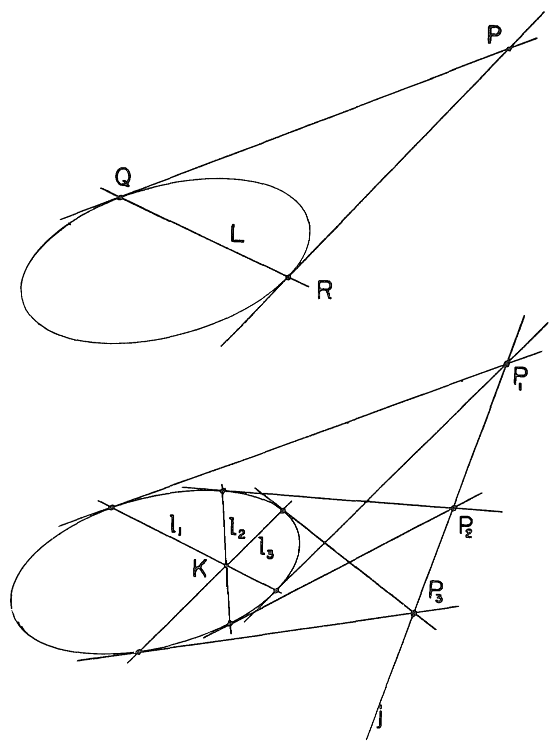


Figure 8–6.

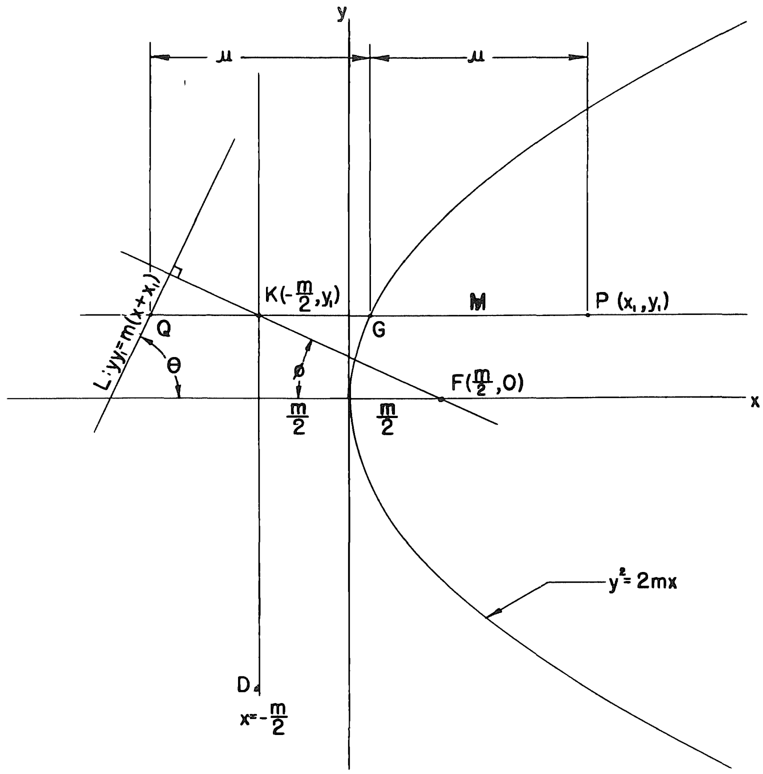


Figure 8–7a.

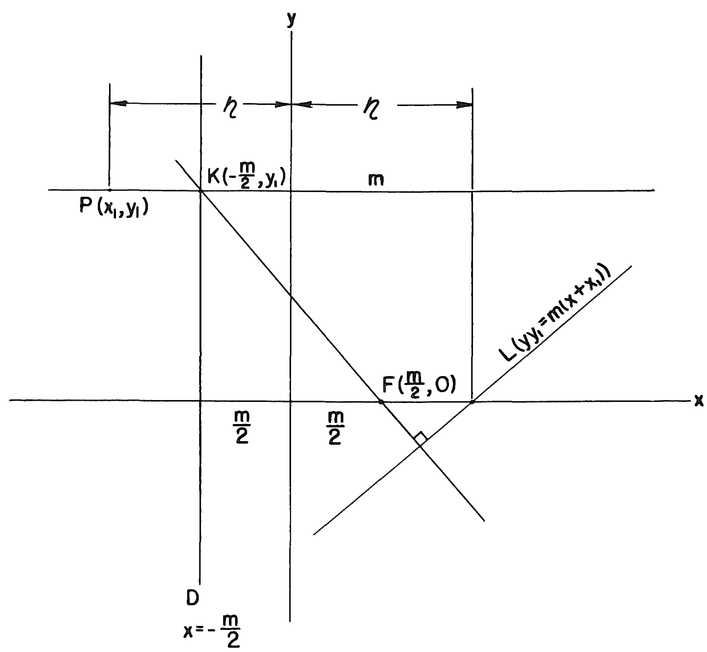
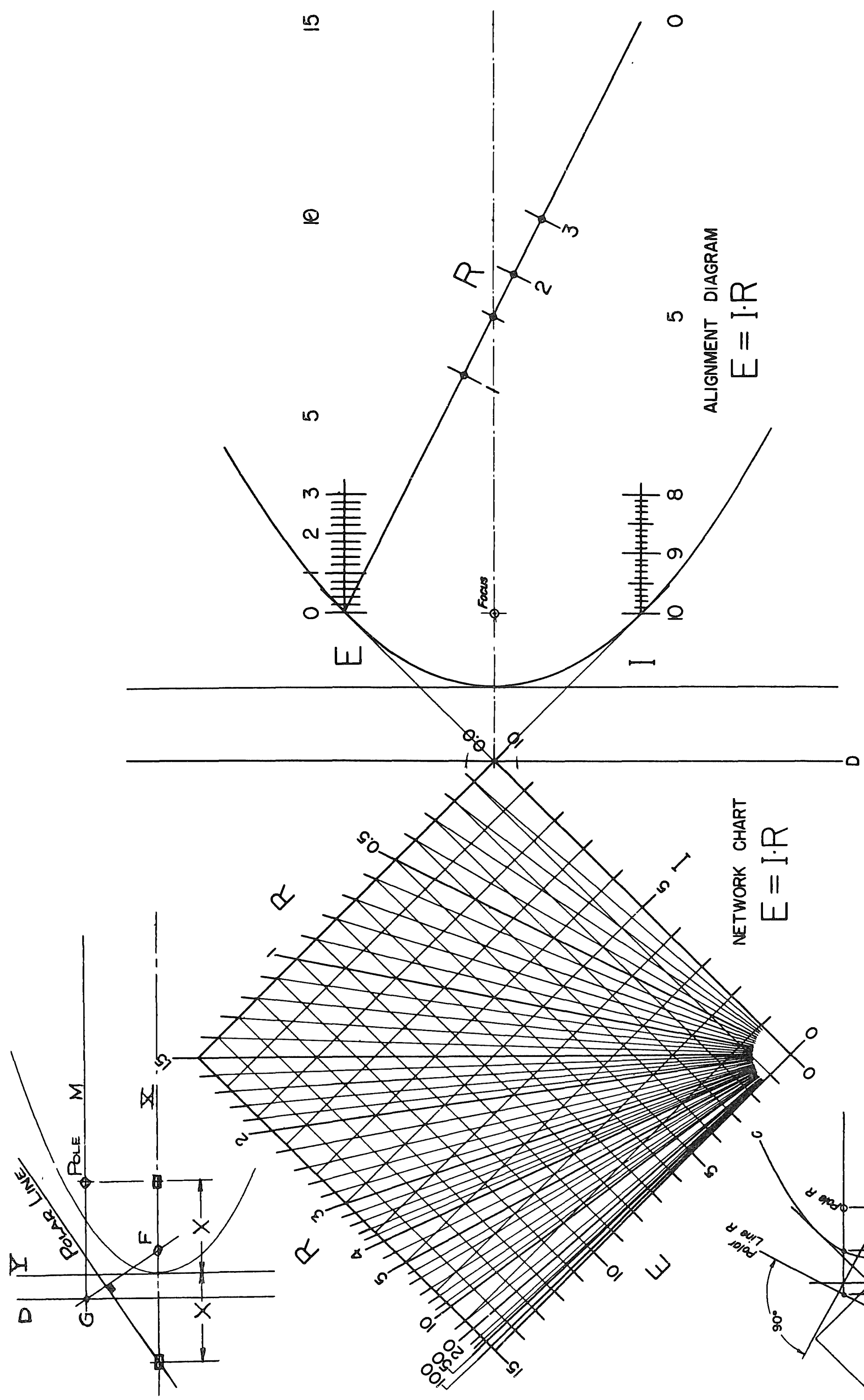


Figure 8–7b.


ALIGNMENT DIAGRAM
E = I·R
NETWORK CHART
E = I·R
POLAR LINE
POLE
Focus
Pole R
Polar Line R
90°

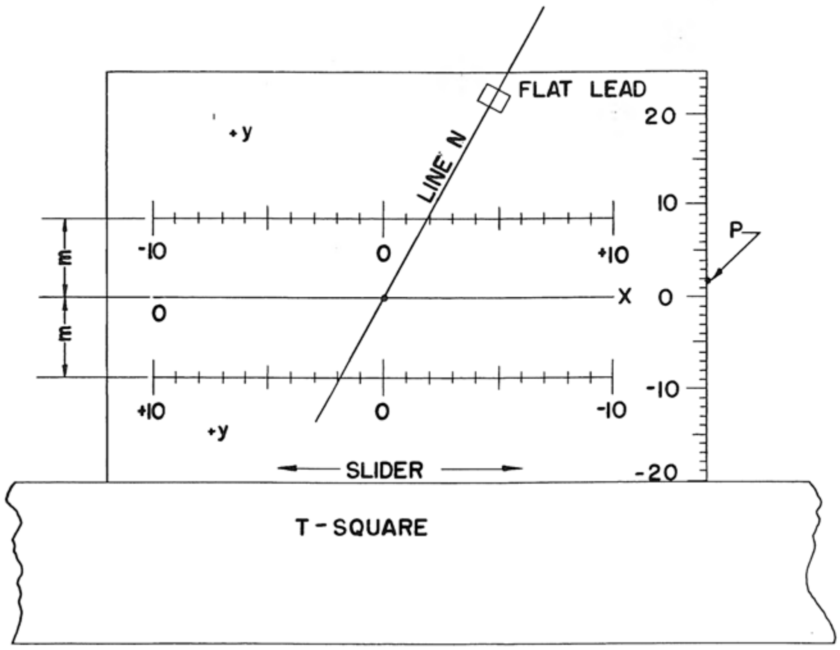


Figure 8–9.

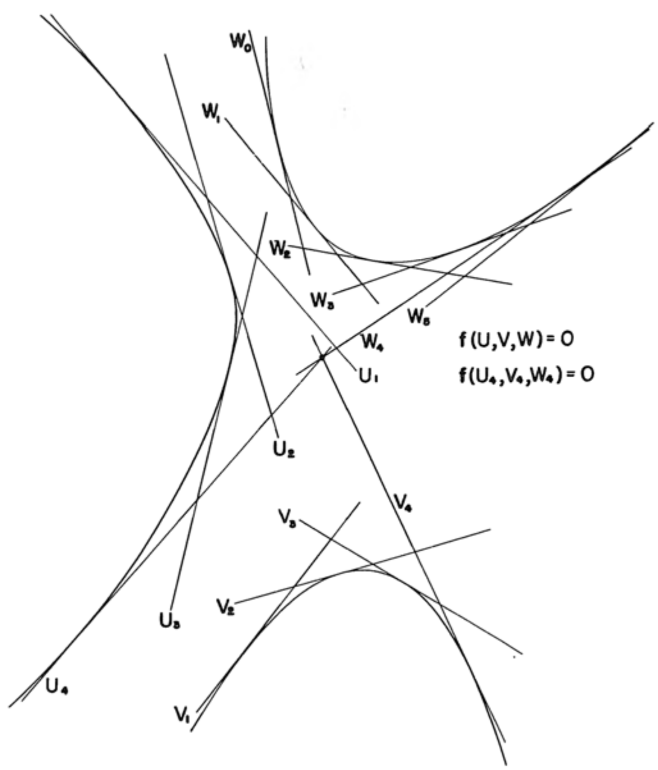


Figure 8–10.

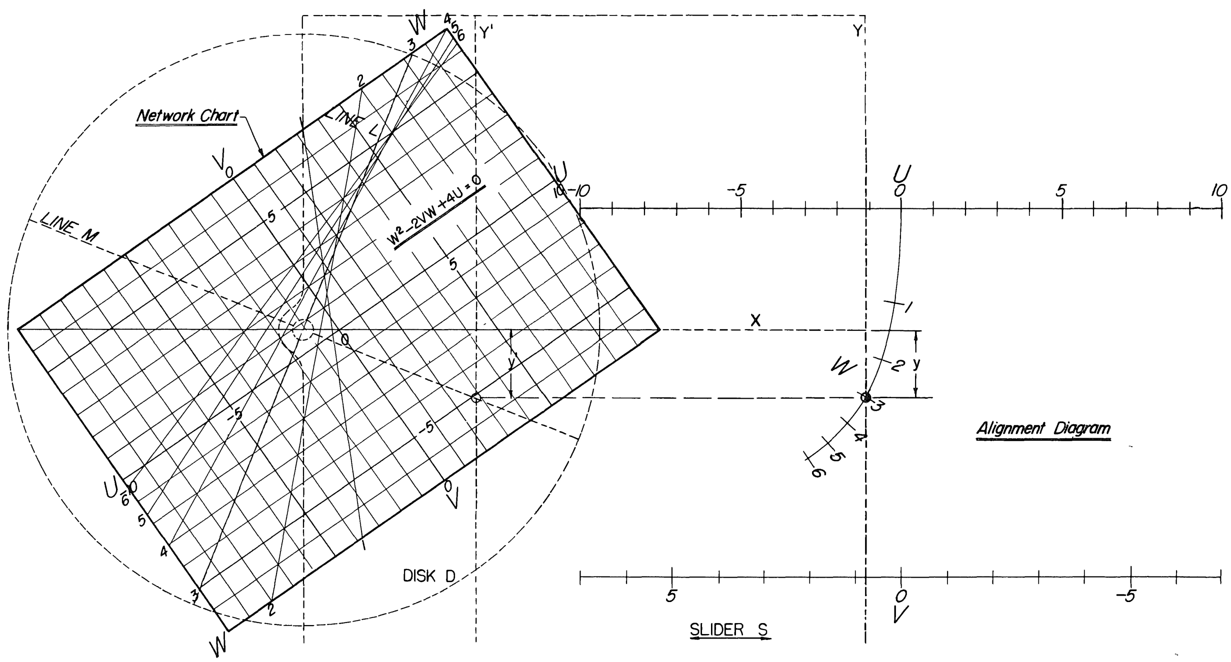

Network Chart
LINE L
LINE M
W²-2VW+4U=0
DISK D
Y'
Y
X
y
Alignment Diagram
SLIDER S

Figure 8–11.

CHAPTER 9

QUASI-ALIGNMENT DIAGRAMS

RECTIFICATION OF NETWORK CHARTS

9–1. *The Quasi-Alignment Diagram.* Frequently the nomographer is unable to rectify a network chart which he wishes to dualize by any of the devices described earlier or in this chapter. The dual alignment diagram will then consist of sets of points corresponding to the sets of lines of the network chart plus a set of curves, one for each of the curves of the network chart. Figure 9–1 shows a network chart with such unrectifiable curves on it and the corresponding alignment diagram derived similarly to that of Figure 8–9. For this particular scanning technique which is related very closely to that of Figure 8–9, see "Alignment Diagrams from Network Charts by Graphics," *Mechanical Engineering,* Volume 78, November 1956, pp. 1013-1015. Figure 9–2 shows a problem arising in practice. In each diagram, any line of collineation will pass through one point of each of the two scales of *points* and will be TANGENT to that particular *curve* in the family of curves which is the answer. *The line of collineation is a defining line, a tangent, of that curve; just as the point to which it was dual was a defining point of the corresponding curve in the network chart.* An interpolation of tangency will have to be made whenever the line of collineation is not tangent to one of the curves actually drawn.

9–2. *The Most General Network Chart.* Extending the notion of quasi-alignment, one imagines three distinct sets of numbered curves, Figure 9–3(a), with the line of collineation tangent to one member of each set. The network chart to which this is dual would have three sets of curves, with a member of each set through any point representing a solution. This is the most general type of network chart. Figure 9–3(b). See also Figure 8–10.

9–3. *Rectification of a Network Chart by Changing the Dependent Variable, by a Change of Scale Equation, etc.* See Section 4–4. The nomographer should do his best to rectify the network chart which has curves in it, so that dualizing will yield an alignment diagram consisting only of *points.* Sometimes changing the family variable will do this trick, and it should be tried early. Occasionally a chart will be completely rectified by adopting a different scale equation for one of the axes. In one example of Section 4–4, changing the coordinate scale from linear to reciprocal form rectified the family of curves. By the change of scale $Y = \log y$, the entire family of exponential curves $y = ae^{bx}$ is rectified. By the *double* change of scale $Y = \log y$, $X = \log x$, the entire family of *power* curves $y = ax^b$ is rectified.

9–4. *Rectification of a Single Curve of a Network Chart. Anamorphosis.* If unsuccessful in rectifying all the curves of a family of curves in a network chart, one can *always* rectify a single curve of the family graphically by a change in the scale of one of the variables. It is usually advisable to apply the rectification to one of the curves which is centrally located within the group of curves because this tends to spread partial rectification throughout the set of curves evenly. This method is commonly referred to as anamorphosis (Figure 9–4). It consists simply of assuming that the curve in question has been carried into a desired line and noting the new positions of the x- or y-coordinates required to bring this about. Any slanting line can be chosen for this purpose but a device like Figure 9–4 is customary.

9–5. *Rectification of Two Curves of a Network Chart.* An ingenious system for doing this was devised by Colonel Lafay, in *Le Genie Civil,* Volume 40, March 1, 1902, pages 298-299, and referred to in d'Ocagne's historic treatment "Traite de Nomographie," 1921, pages 449-454, Figure 9–5. The paper is divided into quadrants, the original network chart occupying the upper left of these, the rectified chart the lower right and conversion curves the other two. The x, y axes are rotated clockwise through 90° in the rectified chart. Two straight lines are drawn in that diagram which represent the new, rectified positions of the two curves chosen for rectification in the network chart. The positions of these are very arbitrary. The "step" diagram shown there is constructed next with points numbered consecutively inward from an arbitrary starting point. A corresponding diagram is constructed similarly for the original chart. Line segments with the same number pairs in the two charts are now extended to "fix" points on the two curves of correspondence in the other two quadrants. By using these, all other curves of the original network chart can be transferred to the rectified chart.

9–6. *Closure.* It is conceivable that when two curves of a set are rectified, the entire set will be rectified. Example 9–1. An interesting criterion for this outcome exists and can be applied to the original set of curves. Figure 9–6. It uses a six-sided test-figure with sides parallel alternately to the x and y axes. Starting this figure at a point A on one curve of the network chart, if the final vertex V turns out to lie on the same curve, the figure is said to have "closure." If *every such test figure* throughout the network chart turns out to have closure, then rectifying any two curves of the network chart will rectify all the curves at the same time. In practice, a reasonable number of tests for closure suffice to indicate such general rectification.

Example 9–1. *The Chi² Function.* This function is shown in Figures 9–6 and 9–7, and the closure test applied in Figure 9–6 by drawing the test figure in a fair number of places on the curves of the chart. Two such test figures are shown in place and most others would show good closure. Good rectification for all curves is accordingly expected when two are rectified. The eventual nomogram dual to the rectified diagram is shown in the next figure. The rectification appears to have been quite successful causing most of the curves to straighten out and thus give rise to points in the alignment diagram.

9–7. *The Dual to a Point of Inflection.* A network chart whose curves have no points of inflection may acquire such points during the process of partial rectification. Under parabola, or "scanner" duality, the dual curve to one with a point of inflection will have a cusp at the point dual to the tangent at any finite point of inflection. In words, this is because the x, y coordinates of the dual point are proportional respectively to the *x-intercept* of a tangent to the network curve and to the *slope* of this tangent. At a point of inflection, the x-intercept of the tangent to the network curve starts to backtrack and so does the slope of the tangent. Since coordinates of the dual point backtrack correspondingly, a cusp results. The cusp is sharp — both sides of the curve approach the same tangent at the point, namely the *line* dual to the *point* of inflection of the network curve.

PROBLEMS

PROBLEM 9–1. (a) In Figure 9–8 apply the closure test. (b) Rectify two curves and derive the rest of the rectified diagram. (c) Derive the dual diagram.

PROBLEM 9–2. In Problem 9–1, rectify one curve. Compare the resulting rectification of the others with that of Problem 9–1.

PROBLEM 9–3. (a) In Problem 9–1, compare the curves with members of a power series. (b) Rectify them on log paper as a result. (c) Can you find any relation between this rectified set and that of Problem 9–1? (Consider a projective relation, for instance?)

PROBLEM 9–4. The scales of an alignment diagram have equations.

$$X = U;\ Y = U^2$$

$$X = R \sin V;\ Y = R \cos V$$

$$X = W;\ Y = W - (3/2)\,a$$

(a) Sketch these scales.

(b) Find the equation represented by the diagram.

(c) Derive the dual *network chart.* Show that it works. Use "parabola" duality.

PROBLEM 9–5. What can you state about the general problem of rectifying three curves of a conventional network chart? On what basis would you extend the theory of rectification already presented? What significance do you attach to the degrees of freedom in Figure 9–5 for the location of the two newly rectified lines? This problem has not been fully solved.

PROBLEM 9–6. Remarks were made in Section 9–7 about a point of inflection of the network curve giving rise to a cusp in the dual curve if the latter was derived from the former through a parabola. Make the corresponding remarks when the dual is derived through coordinate duality. Illustrate graphically.

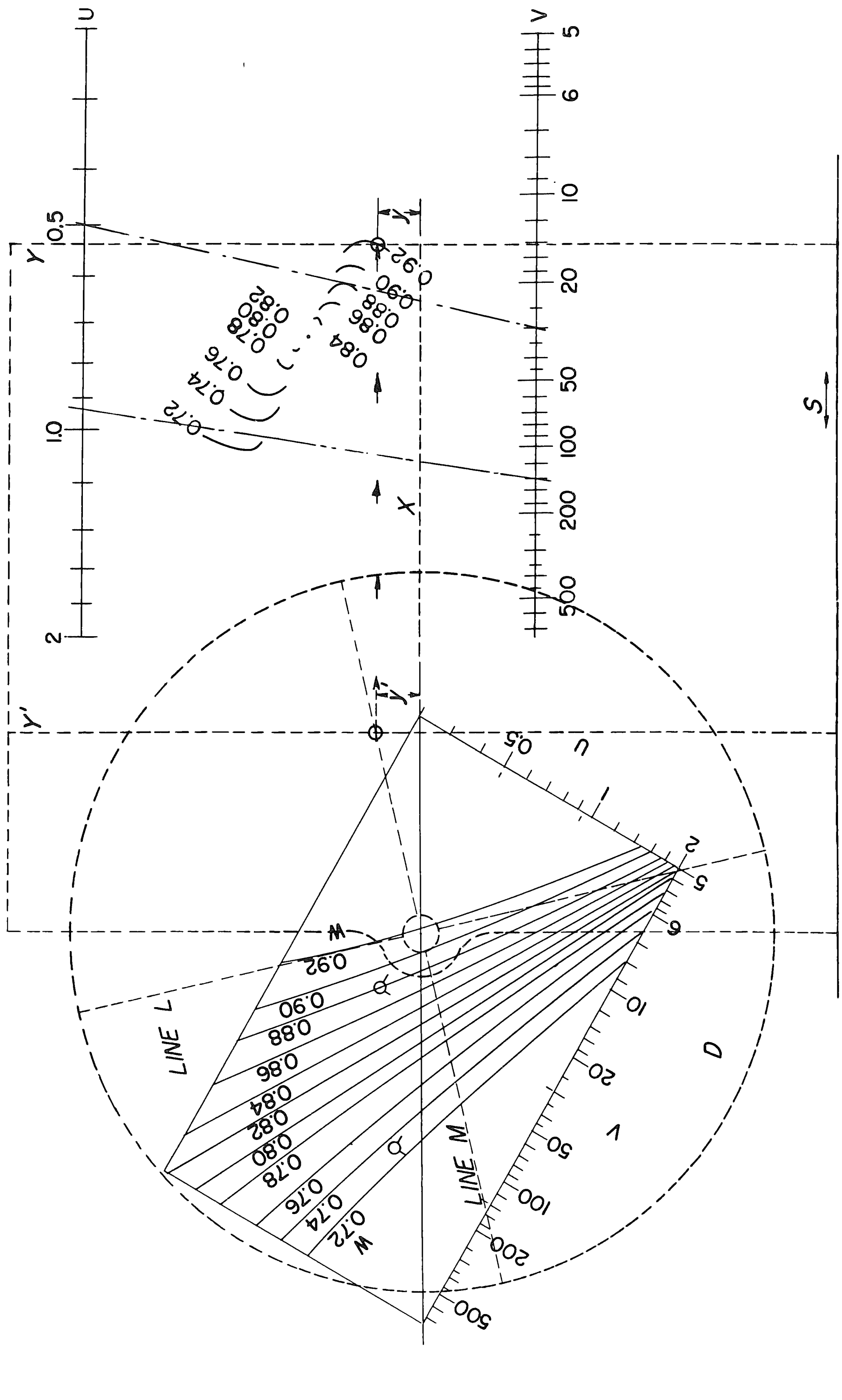

U
V
0.5
1.0
2
5
6
10
20
50
100
200
500
Y
Y'
X
S
D
W
LINE L
LINE M
0.72
0.74
0.76
0.78
0.80
0.82
0.84
0.86
0.88
0.90
0.92

Figure 9–1.

GEOMETRICAL RELATIONS OF THE PLAIN WEAVE

BASED ON EQUATIONS GIVEN BY F. T. PEIRCE

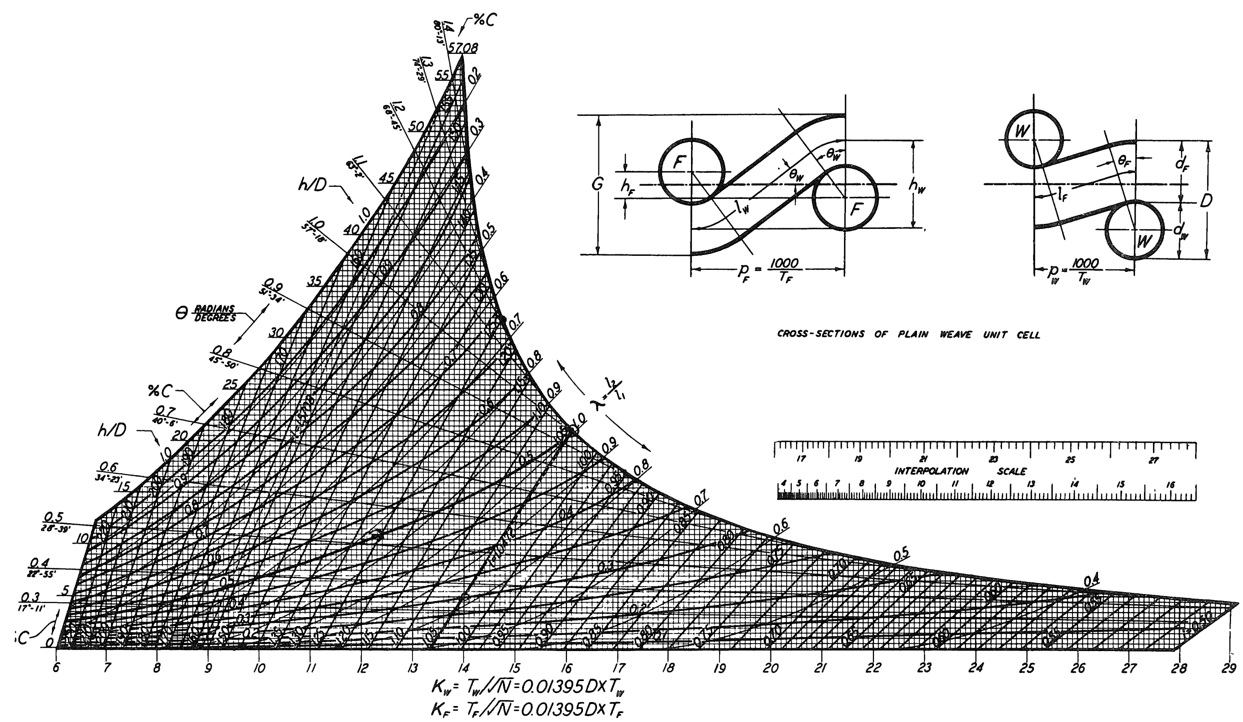


Figure 9–2a.

GEOMETRICAL RELATIONS OF THE PLAIN WEAVE

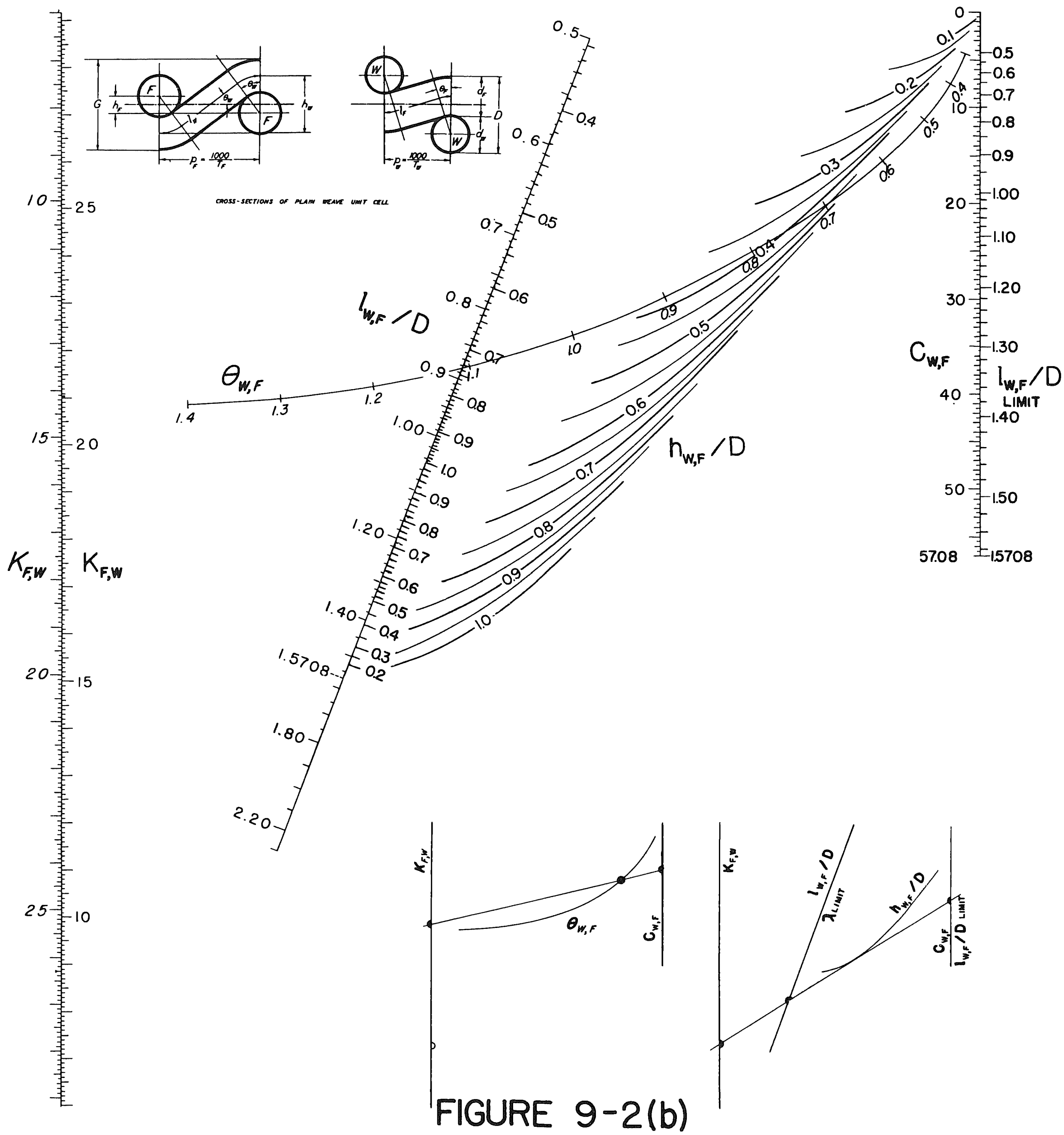


FIGURE 9-2(b)

Figure 9–2b.

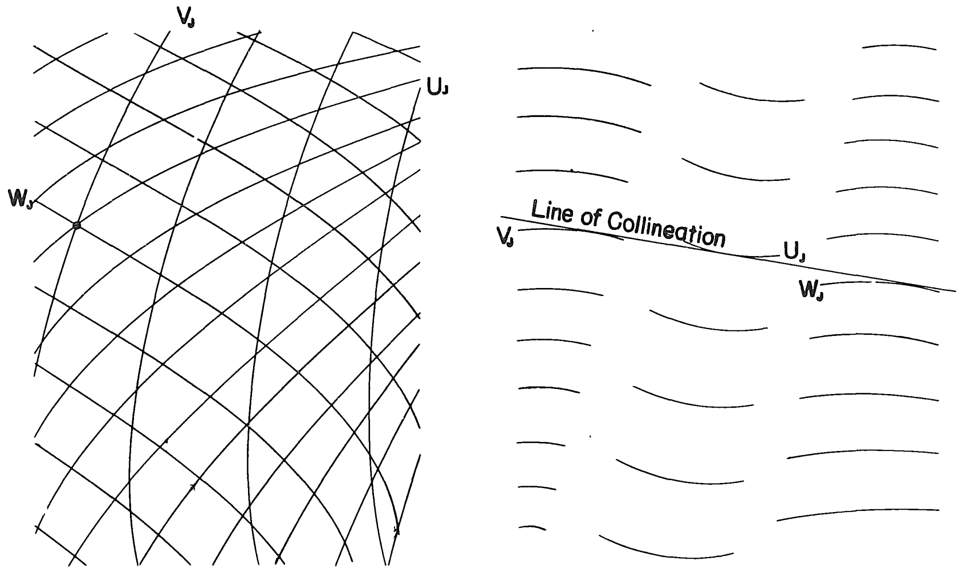


Figure 9–3.

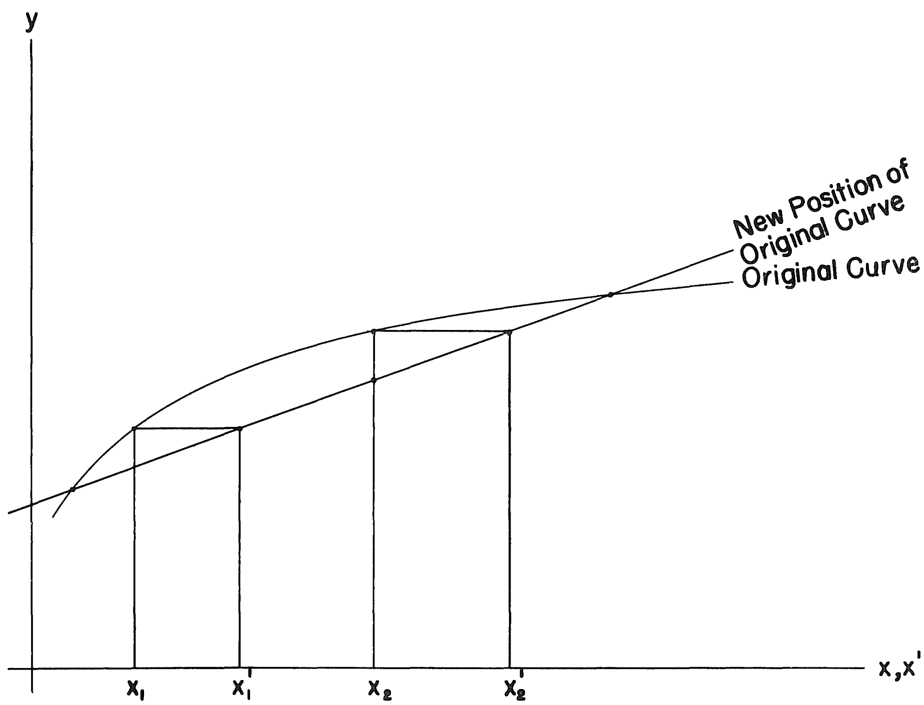


Figure 9–4.

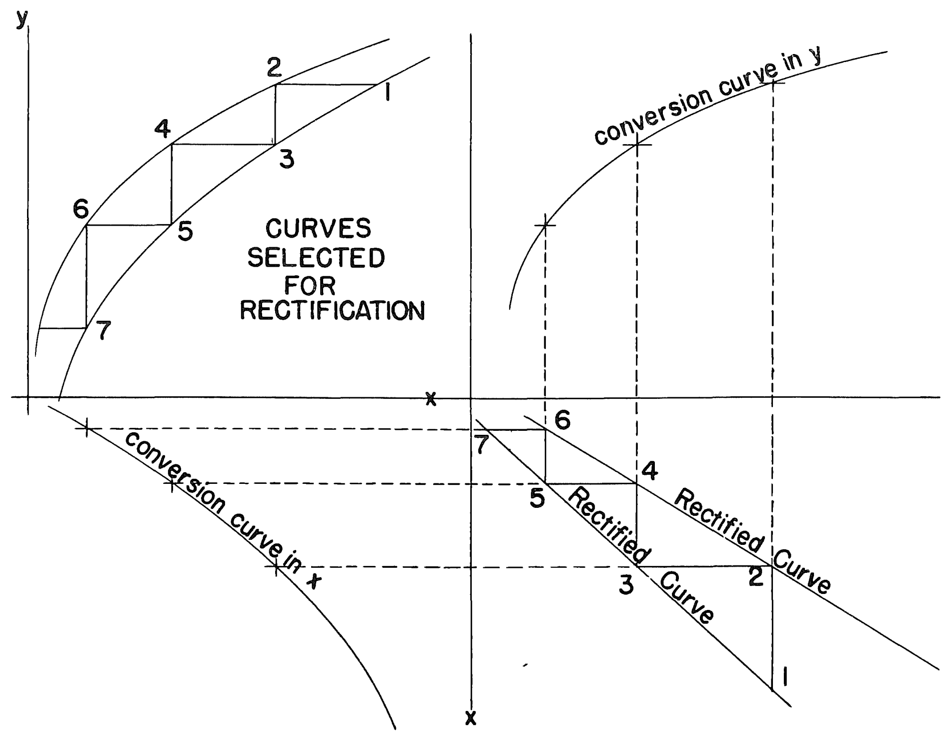
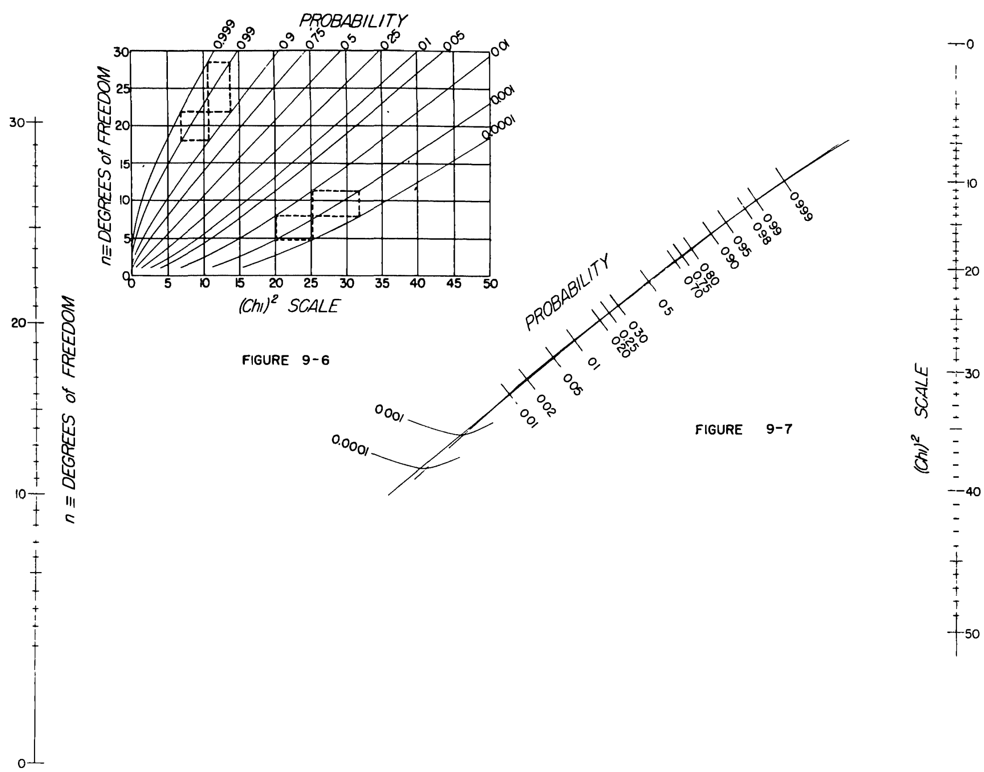


Figure 9–5.



Figures 9–6 & 9–7.

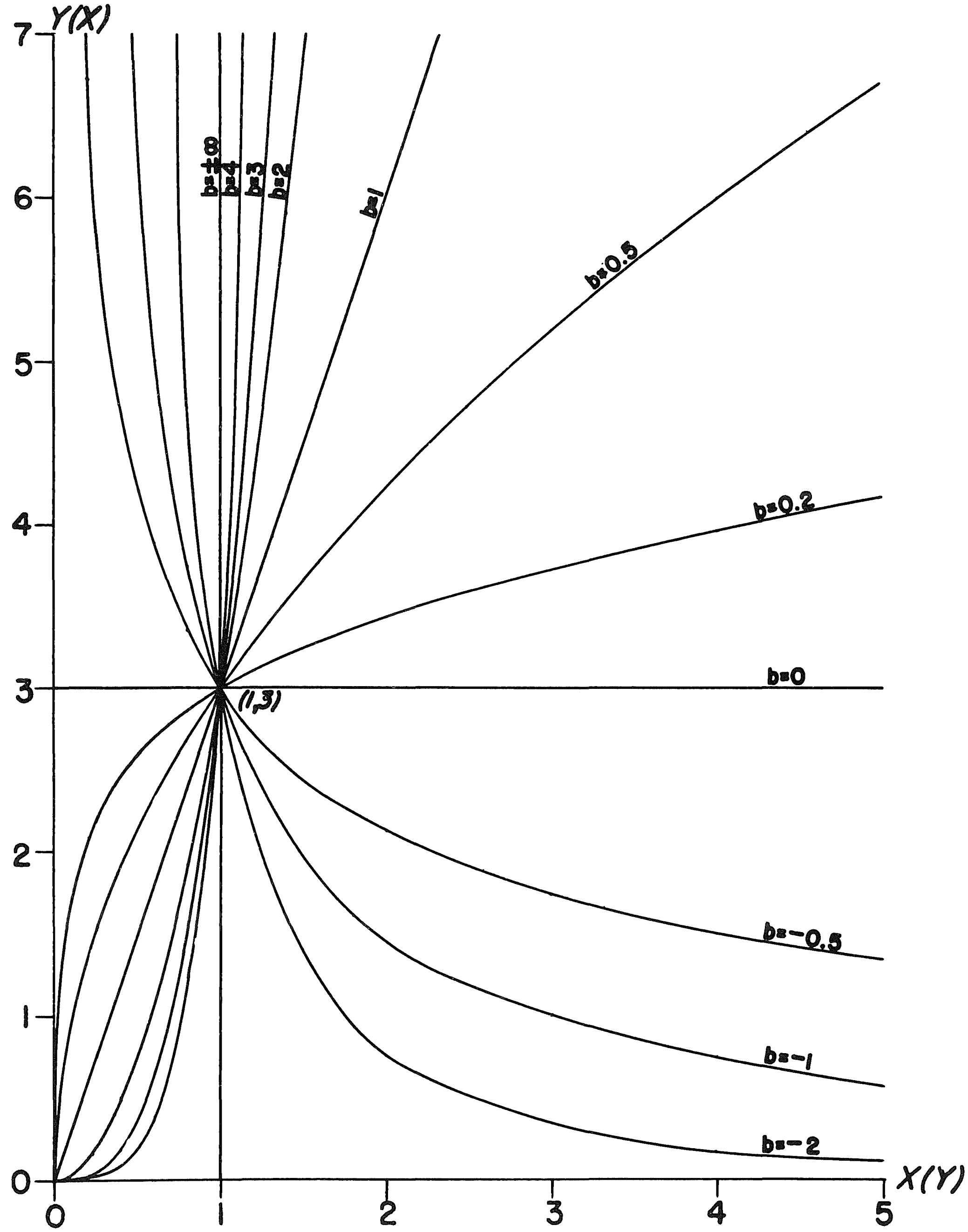


Figure 9–8.

CHAPTER 10

CURVED JOINS

(On an alignment diagram, answers lie upon a straight line. Couldn't they in some cases be arranged to lie upon something else—say a circle?)

10–1. *The Notion of a Curved Join.* A more direct way of looking at the work already done on the straight line will show that results of this kind can actually be arranged sometimes.

10–2. *Determinant Form of the Straight Line from its Linear Form.* The equation of the straight line was linear. It was shown it could be put in the determinant form

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0 \qquad (10\text{–}1)$$

by expanding this determinant and identifying it after expansion with the conventional equation of the straight line. Thus, the condition for collineation of three points was derived (1–4), (8–3),

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0. \qquad (10\text{–}2)$$

Using the linear form of the equation of the straight line, the same result could have been found in the following way:

The coordinates of these three points satisfy the equation of the straight line

$$Ax + By + 1 = 0 \qquad (10\text{–}3)$$

from which it follows:

$$\begin{aligned} Ax_1 + By_1 + 1 &= 0 \\ Ax_2 + By_2 + 1 &= 0 \\ Ax_3 + By_3 + 1 &= 0. \end{aligned} \qquad (10\text{–}4)$$

These are three non-homogeneous, linear equations in the two unknowns, A, B. They are compatible by hypothesis. Since some $|\ |_2 \neq 0$ in the determinant of the coefficients, this determinant vanishes (10–2). If x_3, y_3 are thought of as being the coordinates of a general point, the subscripts can be dropped, (10–1). This derivation has the advantage that it is direct and general. *It suggests handling other equations in linear form the same way.*

10–3. *The Circle in Determinant Form.* The equation

$$x^2 + y^2 + Ax + By + C = 0 \qquad (10\text{–}5)$$

is that of a circle (if it has any *real* locus). The x-coordinate of the center, y-coordinate of the center and the radius will vary with the three parameters A, B, C. Three points will determine such a circle, permitting solution for A, B, C. If four points lie on such a curve, the four equations result:

$$\begin{aligned} x_1^2 + y_1^2 + Ax_1 + By_1 + C &= 0 \\ x_2^2 + y_2^2 + Ax_2 + By_2 + C &= 0 \\ x_3^2 + y_3^2 + Ax_3 + By_3 + C &= 0 \\ x_4^2 + y_4^2 + Ax_4 + By_4 + C &= 0 \end{aligned} \qquad (10\text{–}6)$$

These are four linear, non-homogeneous, compatible equations in three unknowns A, B, C, so

$$|\ |_4 \equiv \begin{vmatrix} x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \\ x_4^2 + y_4^2 & x_4 & y_4 & 1 \end{vmatrix} = 0 \text{ (some } |\ |_3 \neq 0) \qquad (10\text{–}7)$$

If one of the points is regarded as a general or running point, the equation of the circle through the three points is

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0. \qquad (10\text{–}8)$$

10–4. *The Circle Sometimes Joins Solution Values.* Just as the straight line joined solution values in earlier work, the circle can do so for appropriate equations.

The early steps for the use of the straight line in the "alignment" diagram have just been repeated for the circle. The determinant (10–7) has four rows and columns, instead of the three rows and columns for the straight line. Continuing the kind of interpretation applied to the straight line in Section 1–4, assume that

x_1, y_1 are functions of U only

x_2, y_2 are functions of V only

x_3, y_3 are functions of W only

x_4, y_4 are functions of T only. (10–9)

Then $\triangle$ in (10–7) becomes a function of four variables, $\triangle(U, V, W, T) = 0$. In the plane, four parametric scales in U, V, W and T are defined by (10–7), (10–9). If the values of three of these variables are known, say U_1, V_1, W_1, then a circle through the three points bearing these values will cut the scale of the fourth, T, at the value T_1 that satisfies the equation, so that

$$\triangle (U_1, V_1, W_1, T_1) = 0. \qquad (10\text{–}10)$$

The analogy with the straight line treatment is complete.

Example 10–1. Figure 10–1. Consider an equation in four variables which has been placed in the form

$$\begin{vmatrix} 2U^2 & U & U & 1 \\ V^2 & 0 & V & 1 \\ W^2 & W & 0 & 1 \\ 2T^2 & -T & T & 1 \end{vmatrix} = 0 \qquad (10\text{–}11)$$

This has the form (10–7), the parametric equations of the scales being

$$x = U, y = U; \; x = 0, y = V;$$
$$x = W, y = 0; \; x = -T, y = T. \qquad (10\text{–}12)$$

To find the proper circle to act as a join, use numbered concentric circles on a transparent sheet. Make adjustment of one of these (drawn or interpolated) to contain the three given values, which quickly permits the value of the answer to be read at the point where this circle cuts the fourth scale. The diagram shows that if $T = 1.50$, $U = 2.80$, $V = 2.80$, then $W = -2.03$ or 4.78.

10–5. *Parameters. Degrees of Freedom in the Join. Variables in the Equation.* Equation (10–5) is said to have three parameters, A, B, C. The circle it represents is said to have three "degrees of freedom," three different ways in which one member of the family of circles given by the equation can differ from another. Parameters and degrees of freedom are equivalent. Thus, if the x-coordinate of the center of a circle is specified and the y-coordinate and radius, the circle is definite. One circle of the family will differ from another if one of these three things is different between them. Each of these can be regarded as a parameter, as a way in which one of them can differ from another, as a degree of freedom. A different three could be found, such as three points through which the circle passes or as already noted, A, B, C. When one set of degrees of freedom or parameters has been fixed, the values of all other sets are determined.

A diagram for an equation generally contains one scale for each of the variables in the equation. The number of variables exceeds by *one* the number of parameters of degrees of freedom in the type of join used. Thus a three variable equation can use a straight line join which has two parameters or degrees of freedom. These can be constants A, B of the line, or the slope and intercept of the line, or two intercepts of the line, etc. An equation like (10–11) with four variables, uses a three-parameter join like the circle. The case of a two-parameter circle with variable center and fixed radius lacks sufficient practical value to be taken up here.

A parabola with fixed axis, variable latus rectum and variable vertex has two degrees of freedom and could be a join for some equation in three variables. If the parabola's axis remains parallel to the prime direction but can shift parallel to itself, a third degree of freedom is given the parabola which could now be a join for a four-variable equation. The first parabola could have been written $y^2 = 2m(x - a)$, m and a parameters; the second, $(y - b)^2 = 2m(x - a)$, m, a, and b parameters. If the angle of the axis can vary, the equation can be written

$$y^2 + Ax + By + Cxy + D = 0,$$

with A, B, C and D as convenient parameters, and can be a join for an appropriate five-variable equation. The parameters used need not always be identical with the set of geometric parameters which most easily describe the freedom present.

The general equation of the second degree is a conic (real or imaginary).

$$x^2 + Ay^2 + Bx + Cy + Dxy + E = 0 \quad (10\text{–}13)$$

This is a five-parameter linear form which lends itself to the sort of treatment given the circle in Section 10–3. To use such a curve for a join, one would have to have a six-variable equation with a scale for each variable. One would know the values of five of the variables in the equation and wish to find the sixth. The five known values would provide five points and it would be necessary to set up the conic determined uniquely by them. This conic would cut the sixth scale in the answer. Methods for handling conics this way are discussed in Chapter 13, Problem 13–18, Figure 13–18.

10–6. *More General Joins.* Generalizing can be carried further. Any linear equation with three (for the moment) parameters

$$f(x, y) + Ag(x, y) + Bh(x, y) + Cj(x, y) = 0 \quad (10\text{–}14)$$

can have parameters A, B, C determined specifically if three points are known which satisfy it ($\triangle_3 \neq 0$). If a fourth point lies on the curve,

$$\begin{aligned} f(x_1, y_1) + Ag(x_1, y_1) + Bh(x_1, y_1) + Cj(x_1, y_1) &= 0 \\ f(x_2, y_2) + Ag(x_2, y_2) + Bh(x_2, y_2) + Cj(x_2, y_2) &= 0 \\ f(x_3, y_3) + Ag(x_3, y_3) + Bh(x_3, y_3) + Cj(x_3, y_3) &= 0 \\ f(x_4, y_4) + Ag(x_4, y_4) + Bh(x_4, y_4) + Cj(x_4, y_4) &= 0 \end{aligned} \quad (10\text{–}15)$$

and the compatibility of these equations gives rise to (10–16)

$$\begin{vmatrix} f(x_1, y_1) & g(x_1, y_1) & h(x_1, y_1) & j(x_1, y_1) \\ f(x_2, y_2) & g(x_2, y_2) & h(x_2, y_2) & j(x_2, y_2) \\ f(x_3, y_3) & g(x_3, y_3) & h(x_3, y_3) & j(x_3, y_3) \\ f(x_4, y_4) & g(x_4, y_4) & h(x_4, y_4) & j(x_4, y_4) \end{vmatrix} \begin{matrix} \text{some} \\ \triangle_3 \neq 0 \\ = 0. \\ \\ \end{matrix} \quad (10\text{–}16)$$

Assume for the moment that we have equations

$$\begin{aligned} x_1 &= x_1(U); & y_1 &= y_1(U) \\ x_2 &= x_2(V); & y_2 &= y_2(V) \\ x_3 &= x_3(W); & y_3 &= y_3(W) \\ x_4 &= x_4(T); & y_4 &= y_4(T). \end{aligned} \quad (10\text{–}17)$$

If these are substituted into (10–16), one obtains

$$\begin{vmatrix} f_U(U) & g_U(U) & h_U(U) & j_U(U) \\ f_V(V) & g_V(V) & h_V(V) & j_V(V) \\ f_W(W) & g_W(W) & h_W(W) & j_W(W) \\ f_T(T) & g_T(T) & h_T(T) & j_T(T) \end{vmatrix}$$

$$= M(U, V, W, T) = 0. \quad (10\text{–}18)$$

For the nomographer, this sequence of equations means the following: Assume (a) that an equation such as the right hand side of (10–18) can be put into the form on the left hand side of (10–18) *i.e.,* it can be *disjoined.* The question then arises (b) whether equations such as (10–17) exist such that (10–18) takes on the form (10–16). If such equations can be found there, since they are parametric equations, (c) a scale can be plotted from (10–17) for each variable U, V, W, T. Then (d) a curve of the form (10–14) will invariably cut all these scales at calibration values which satisfy (10–18). Or, if a member of the family (10–14) is selected which meets all but one of the scales at specified values, it will cut the last scale at the value of its variable consistent with those specified values and (10–18).

When an equation has more than three variables, it is seldom easy to find equations (10–17) even after disjunction has been successfully carried through. More often the process turns out to be reversed, the results are first noted and then used wherever seen to be applicable.

PROBLEMS

PROBLEM 10–1. Given the canonical form

$$\begin{vmatrix} U^2 & U & 0 & 1 \\ V^2 & -V & 0 & 1 \\ R^2 & 0 & R & 1 \\ S^2 & 0 & -S & 1 \end{vmatrix} = 0. \quad (10\text{–}19)$$

1) Interpret (10–19) in the form of (10–7).

2) Sketch the resulting diagram.

3) Expand (10–19) to show that $U \cdot V = R \cdot S$. Check that the diagram works for this equation, provided $U \neq -V$, $R \neq -S$. Why is this exception to be expected?

4) Start with the original equation and place it in the canonical form (10–19).

5) Give a quick plane geometry derivation of this diagram.

PROBLEM 10–2. The equation

$$3\,(\sin U - \sqrt{1-T})(\cos V - \sqrt{1+T}) - (\sin V - \sqrt{1-T})(\cos U - \sqrt{1+T})$$
$$= (\sin U - \sqrt{2+W})(\cos V - \sqrt{2-W}) - (\sin V - \sqrt{2+W})(\cos U - \sqrt{2-W}) \qquad (10\text{–}20)$$

can be placed in the canonical form

$$\begin{vmatrix} 1 & \sin U & \cos U & 1 \\ 1 & \sin V & \cos V & 1 \\ 4 & \sqrt{2+W} & \sqrt{2-W} & 1 \\ 2 & \sqrt{1-T} & \sqrt{1+T} & 1 \end{vmatrix} = 0 \qquad (10\text{–}21)$$

Carry through the first four steps of the preceding problem applied to this one. (Except for $R \neq -S$ in 3)

PROBLEM 10–3. This is a problem in which the type of join is specified at the outset. Given the equation

$$-VW - U^2W + UV = 0 \qquad (10\text{–}22)$$

Assume that for mechanical reasons it is desirable to use a parabola for a join which has to remain fixed while the plane of the nomogram slides freely beneath it but cannot rotate. By hypothesis, the parabola is required to have the form

$$(y - b)^2 = 2m(x - a) \qquad (10\text{–}23)$$

where m is fixed, a and b are parameters. Such a curve can act as a join for an equation in three variables like (10–22).
Then an equation of the form

$$y^2 - 2mx + b^2 + 2ma - 2by = 0$$

or

$$y^2 - 2mx + k + ly = 0 \qquad (10\text{–}24)$$

is the basis for three compatible equations in two unknowns k, l

$$y_1^2 - 2mx_1 + k + ly_1 = 0$$
$$y_2^2 - 2mx_2 + k + ly_2 = 0$$
$$y_3^2 - 2mx_3 + k + ly_3 = 0. \qquad (10\text{–}25)$$

These equations stating that three points of the scales of the diagram now lie on such a join. Then

$$\begin{vmatrix} y_1^2 - 2mx_1 & y_1 & 1 \\ y_2^2 - 2mx_2 & y_2 & 1 \\ y_3^2 - 2mx_3 & y_3 & 1 \end{vmatrix} = 0. \qquad (10\text{–}26)$$

It now remains to put (10–22) if possible, in the form (10–26). One obtains (10–22) disjoined as

$$\begin{vmatrix} U^2 & U & 1 \\ -V & 0 & 1 \\ 0 & W & 1 \end{vmatrix} = 0 \qquad (10\text{–}27)$$

and one can write further

$$x_1 = 0; \qquad y_1 = U$$
$$x_2 = V/2m; \qquad y_2 = 0$$
$$x_3 = W^2/2m; \qquad y_3 = W \qquad (10\text{–}28)$$

transforming (10–27) into (10–26).

1) Sketch this chart and show that it works with a parabola (10–23) as a join.

2) Note that (10–27) also permits interpretation for a straight line join, namely

$$X_1 = U^2; \qquad Y_1 = U$$
$$X_2 = -V; \qquad Y_2 = 0$$
$$X_3 = 0; \qquad Y_3 = W \qquad (10\text{–}29)$$

Sketch this chart and show that it works for a straight line join and gives the same answers as 1 above.

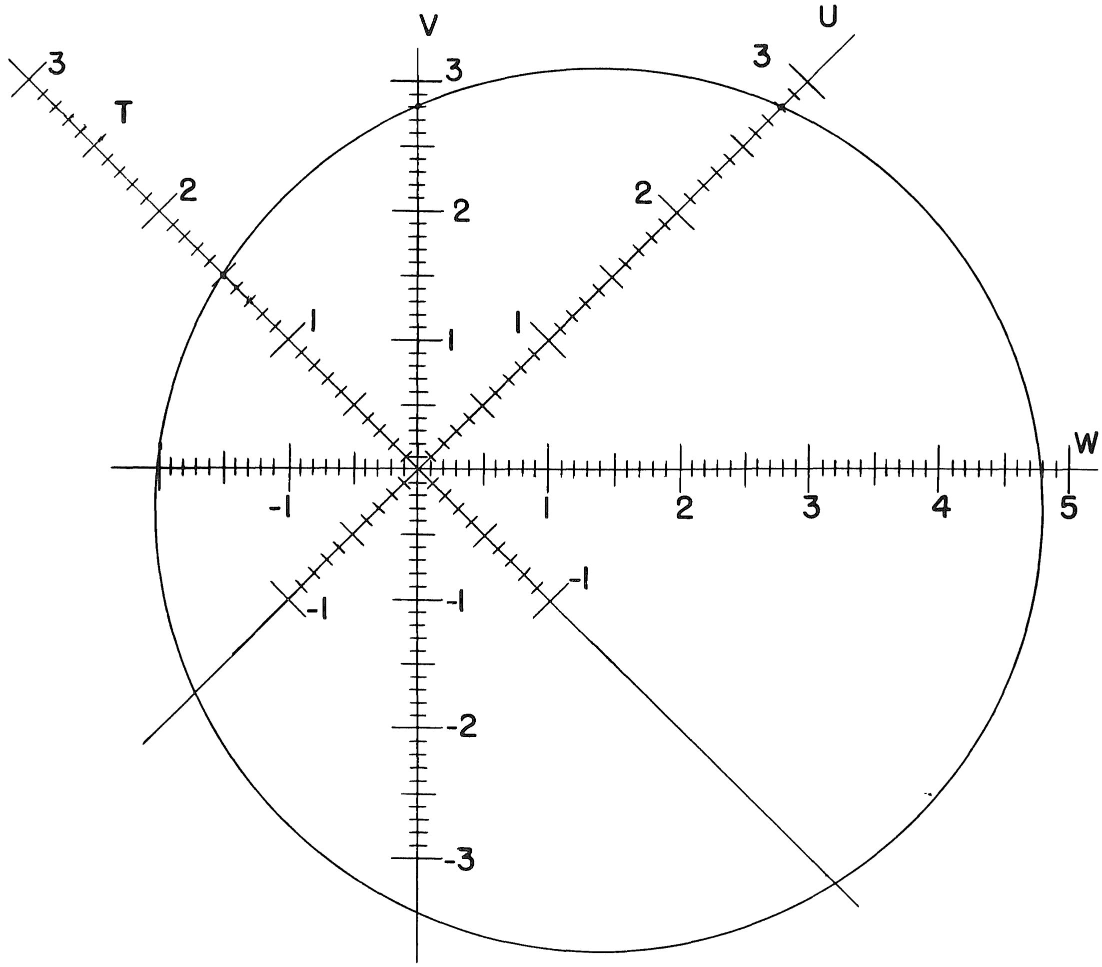


Figure 10–1.

PART III: APPENDICES

The three appendices A-C (Chapters 11-13) of basic material present reference details for persons who do not happen to have training in parametric representations, determinants, conics, central projection, and similar fields. Finally, in Appendix D (Chapter 14), we have so broad an application that it seems logical to give it this position in the book. In addition to drawing expressly or by implication upon almost every chapter of the text, the important developments of Appendix D refer to other processes, such as electronic counting and properties of differential equations, where an intuitive grasp is sufficient to convey enough of an idea of what is going on to effect an introduction to the general area of the application. The Bibliography will be useful to those who wish to study the application further.

CHAPTER 11 (APPENDIX A)

PARAMETRIC REPRESENTATION

SCALE DESIGN

11–1. *Parametric Equations.* An equation or empirical graph describes the way variables behave with respect to one another, as

$$y^2 = 2x \quad \text{or} \quad x^2 + y^2 = r^2 \qquad (11\text{–}1)$$

respectively a parabola and a circle. A slightly different way of giving this behavior would be to describe how x and y both behave with respect to a third variable.

$x = a^2/2$; $y = a$, showing the behavior of x and y with respect to a, and

$x = r \cos \theta$; $y = r \sin \theta$, showing the behavior of x and y with respect to θ. (11–2)

Figure 11–2. The two original equations have been given here parametrically in terms of intermediate variables, or parameters, a and θ respectively. Eliminating the parameter produces the original equation. It may be better to leave the equation in its parametric form if it remains simpler and the parameter allows the behavior of the variables to be interpreted more easily.

Example 11–1. Figure 11–1. The equation of the cycloid in parametric form is

$$x = a(\theta - \sin \theta); \quad y = a(1 - \cos \theta) \qquad (11\text{–}3)$$

In this form, θ is readily seen to be the angular displacement of *the initial tangent point* of a circle rolling on a straight line. Many properties of the cycloid follow in simple terms of this single variable. Nothing would be gained by eliminating it to give y in terms of x, for this would yield a difficult expression not easily interpreted.

The curve appears below drafted directly from the rolling circle and the parametric angle θ. The center of rolling is always the point of contact, but the motion of this center causes the center of curvature to lie elsewhere (on the same radius). A tangent to the curve is always perpendicular to a line thru the contact point, as shown. These facts appear below for $\theta = \frac{2}{3}\pi$ and $\theta = \frac{5}{3}\pi$.

If the left half of the curve is reproduced directly below the right half, the tangent for the first is seen to be the continuation of the normal of the second in corresponding positions. Hence the normals to the upper curve are tangents to the lower curve and the latter is their envelope. The lower cycloid is the evolute of the upper curve. Hence the cycloid is itself its own evolute and involute. *The parametric representation has made it easy to plot this curve, observe these properties and understand them.*

11–2. *Calibration in Terms of the Parameter. Scale Design. Scale Equations.* Each value of the parameter gives rise to some point (sometimes several points) on the curve. If marks are placed along the curves at points defined by round values of the parameter, a scale results in the parameter.

Example 11–2. The scales for parameter *a* and parameter θ come from (11–2). Every scale should be specially *tailored for its job*. It should usually be divided decimally, that is, into halves, fifths, tenths, etc., rather than halves, fourths, eighths, etc., unless the units being recorded are naturally given some other way. Subdivision into the days of the week, pipe sizes, inches, are typical exceptions.

A uniform scale is one where regularly changing graduation values are regularly spaced. This can happen on curved scales as well as on straight ones, as Figure 11–2(b) indicates for (11–2). Such scales are designed rather easily because the scale remains the same everywhere.

A non-uniform scale will need a pattern or design that continually changes, sometimes slowly, sometimes rapidly, and this should be worked out so that readings can be made with maximum speed and accuracy and minimum fatigue. Careful attention to these details frequently spells the difference between impressions of practicality or impracticality of a chart. Many, if not most, non-uniform scales in practice are sufficiently difficult to merit such care. The size of a scale can be adjusted by introducing a scale factor, or scale multiplier, into its equation. Knowing the desired range of the scale variable and the amount of space available for the scale makes it possible to evaluate this scale factor. Use of this value in the scale equation yields the desired results.

Example 11–3. A scale for the function 3 log U for U between 2.70 and 11,750 is to fill a stretch of the Y axis 5.67 inches in length. Denoting distance along the scale stem for U by Y_U, one has

$$X_U = 0; \qquad Y_U = u \cdot 3 \log U \qquad (11\text{–}4)$$

where u is a factor, the scale factor which will enlarge or reduce the scale until it fits the available length exactly. (11–4) is a scale equation for the desired scale. Y_U is zero when U is 1, so scale distances are measured from that point, which lies below the desired range.

$$Y_{U=11,750} = u \cdot 3 \log 11{,}750$$

$$Y_{U=2.70} = u \cdot \log 2.70 \qquad (11\text{–}5)$$

$$Y_{U=11,750} - Y_{U=2.70} = 5.67$$
$$= u\, 3(\log 11{,}750 - \log 2.70)$$

$$u = \frac{5.67}{3\,\{4.07004 - 0.43136\}} \qquad (11\text{–}6)$$

$$= 0.519$$

$$Y_U = (0.519)\, 3 \log U$$

$$X_U = 0; \quad Y = 1.557 \log U \quad \text{Answer} \qquad (11\text{–}7)$$

Each cycle of the logarithmic function here is sufficiently spread out so that all the integral values can be shown. If they were shorter, it might be necessary to include only even ones, and if shorter still, only the value $5 \cdot 10^k$, etc.

Three types of graduation have been used here—long, intermediate, and short,—of lengths 3/8, 1/4, and 1/8. A scale stem can be graduated differently on its two sides for two different equations, an exact half of such a scale being almost as legible as the whole scale. It should never be necessary to use more than three types of graduation; if more seem desirable, it would probably be better to redesign the scale.

Calibrations should seldom be made at any but round values of the variable. The number of calibrations should be small—an excess of them hampers reading and looks poor. A calibration should probably occur whenever the scale pattern changes and sometimes at regular intervals in between. The subject of scale design could cover many pages and use many diagrams. The same attention to reciprocal, projective, squared, cubic, etc., scales could be given as for the logarithmic above. See *Elements of Nomography,* R. D. Douglass and D. P. Adams, McGraw-Hill, (1947) Chapter VI, especially Figure 13, for many examples of scale design.

11–3. *Dependent or Independent Variables as Parameter.* Every equation y = f(x) can be thought of as being in parametric form where one of the variables themselves has been taken as the parameter. In Section 11–1, (11–2) the parabola has been expressed parametrically in terms of *a,* which is identical with the dependent variable. In such a case, the scale along the curve can be constructed extra easily from the scale of the variable itself, or this may be handy to use instead of the curve. (See Figure 1–10.) Any curve y = f(x) can likewise be regarded as parametric in x.

11–4. *Differentiation of an Equation in Parametric Form. Curvature.* The practicing nomographer may wish to differentiate an equation that occurs in parametric form. Let the curve be given as

$$y = f(x) \qquad (11\text{–}9)$$

and parametrically as

$$y = y(u); \quad x = x(u) \quad \text{also } u = u(x). \quad (11\text{–}10)$$

Then

$$dy/du = y_u; \; dx/du = x_u;$$

$$dy/dx = y_u/x_u; \; u_x = 1/x_u$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\frac{dy}{dx} = \left\{\frac{d}{du}\frac{dy}{dx}\right\} \cdot \frac{du}{dx} = \frac{d}{du}\frac{y_u}{x_u} \cdot u_x \qquad (11\text{–}11)$$

$$= \frac{x_u y_{uu} - y_u x_{uu}}{x_u^2} \cdot u_x = \frac{x_u y_{uu} - y_u x_{uu}}{x_u^3}.$$

The formula for curvature is

$$k = \frac{d^2y}{dx^2} \Big/ (1 + (dy/dx)^2)^{3/2} \qquad (11\text{–}12)$$

It is handy sometimes to have it in parametric form:

$$k = \frac{x_u y_{uu} - y_u x_{uu}}{(x_u^2 - y_u^2)^{3/2}} \qquad (11\text{–}13)$$

Example 11–4. Show that the curvature of the curve

$$x = \frac{a}{(1 - u^2)}; \quad y = \frac{au}{(1 - u^2)} \qquad (11\text{–}14)$$

remains constant. Then

$$x_u = \frac{a(-2u)}{(1-u^2)^2} \qquad x_{uu} = \frac{a(-2(1-3u^2))}{(1-u^2)^3}$$

$$y_u = \frac{a(1-u^2)}{(1-u^2)^2} \qquad y_{uu} = \frac{a(-6u-2u^3)}{(1-u^2)^3}$$

$$k = \frac{a^2}{a^3}\ \frac{(1-u^2)^3}{(1-u^2)^5}(-2u(-6u-2u^3) -$$

$$-2(1-3u^2)(1-u^2))$$

$$k = \frac{2}{a}$$

Then this is a circle of diameter a.

By Example 1–7, eliminating the parameter u confirmed that (11–14) was a circle of diameter a.

11–5. *Other Uses of Term Parameter.* There are other ways in which the word parameter may come to the student's attention. Consider the equation

$$(x-a)^2 + y^2 = r^2. \qquad (11\text{–}15)$$

This is a circle with center at (a, 0). As *a* increases the circle moves to the right. If one considers all possible circles of this form, they can be described as being a one-parameter family of circles. Such a use of the term was discussed in Chapter 10.

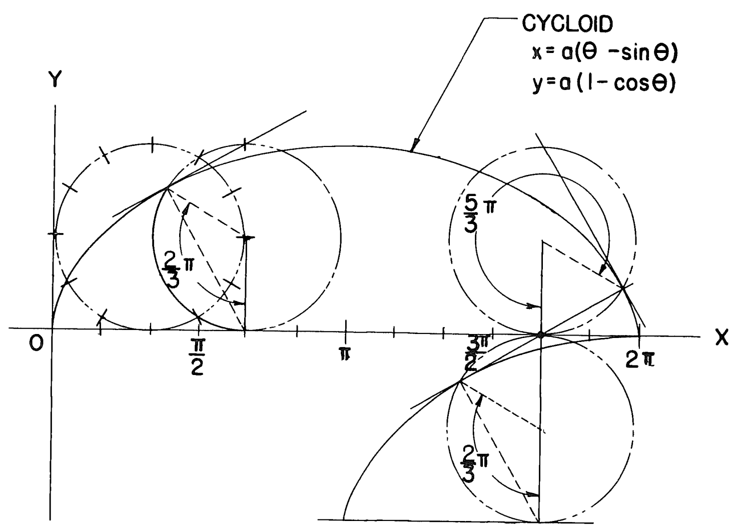


Figure 11–1.

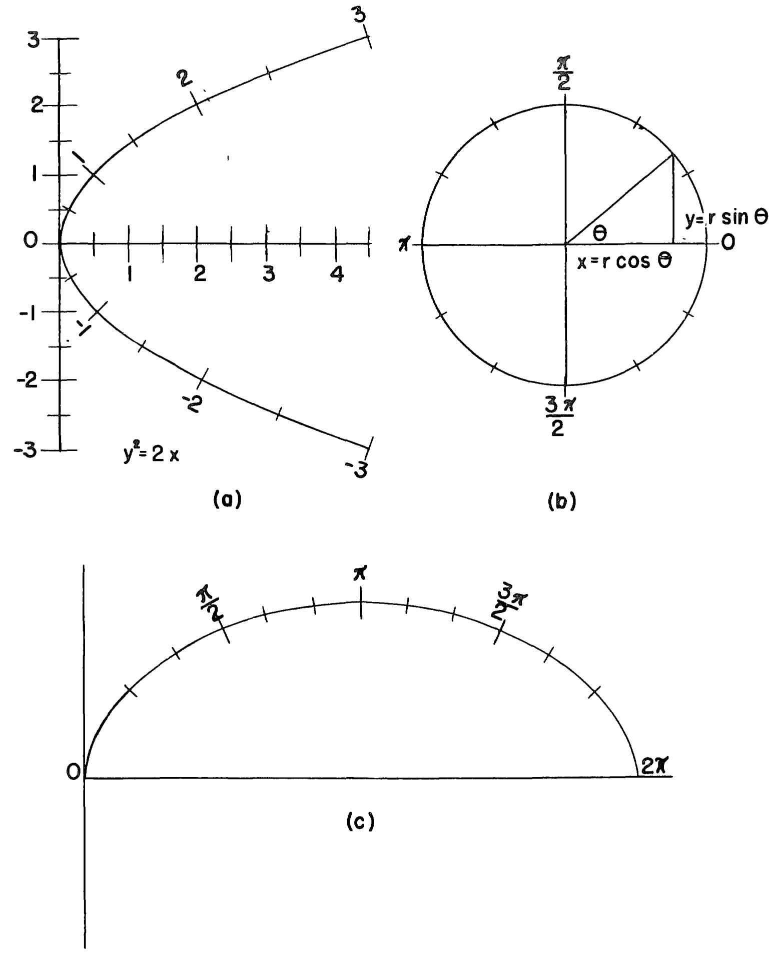
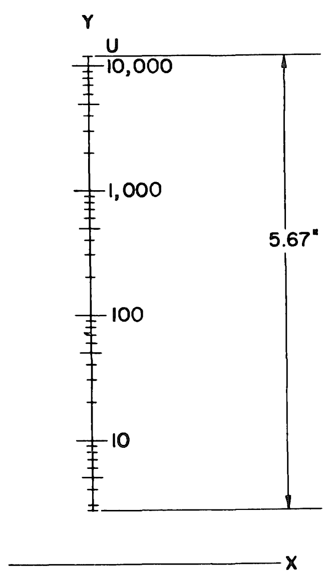


Figure 11–2.



$y = 1.557 \log U$

Figure 11–3.

CHAPTER 12 (APPENDIX B)

DETERMINANTS AND LINEAR EQUATIONS

12–1. *A Determinant is a Square Set of Terms Implying a Sum of Products.* The lines

$$A_1x + B_1y = C_1$$
$$A_2x + B_2y = C_2 \qquad (12\text{–}1)$$

are parallel if and only if their slopes are equal, that is

$$\frac{-A_1}{B_1} = \frac{-A_2}{B_2}$$

or, if and only if

$$A_1B_2 - A_2B_1 = 0. \qquad (12\text{–}2)$$

The left hand side here is a criterion for parallelism. It is the sum of two products and is called the determinant of the left side of the equations (12–1). It is often written in *notation form* between vertical bars:

$$\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} \equiv A_1B_2 - A_2B_1 \qquad (12\text{–}3)$$

It is easily remembered by this barred, square arrangement since *the products summed contain in every case one element from each row and each column.*

This turns out to be a convenient rule for writing similar expressions from larger determinants. Going on to the three-rowed determinant, one has by analogy,

$$\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$$

$$\equiv A_1B_2C_3 + A_2B_3C_1 + A_3B_1C_2 - A_3B_2C_1 - A_2B_1C_3 - A_1B_3C_2. \qquad (12\text{–}4)$$

The barred, square array can be further abbreviated by the notation $| A_1B_2C_3 |$ or even to $| ABC |$ both of which mean (12–4). The expansion of (12–4) is the sum of six products; each product consists of three terms; each term has a letter A, B, and C in it and a subscript 1, 2, 3. This is another way of saying, as mentioned above, that each product in the sum has in it one and only one element from each row and from each column, there being six ways of choosing such products if they are all included. See also Problem 12–1.

12–2. *Several Properties of a Determinant.* The algebraic sign given a term can be found *by arranging its letters in normal order and then counting the number of "jumps" or "moves" (inversions) to get the subscripts into normal order.* An odd number of moves means a minus sign, an even number, a plus sign. The same result will obtain if the subscripts are arranged in normal order and the *letters* "moved" into normal order. Hence, moving an entire row or column over one place changes the sign of the determinant. There are n! ways of arranging n numbers at a time. *Hence there are always n! terms in the expansion of a determinant with n rows and n columns.*

If the elements of a row or column are all zeros, the value of the determinant is zero, (because some one of the zeros will be in each product being summed).

Multiplying the terms of a row or of a column by a factor multiplies the value of the determinant by that factor. If the value were zero before this was done, it remains zero—a fact frequently used in nomography.

A minor of a determinant is a smaller determinant lying within the larger one. It is formed by excluding one row and one column from the original determinant. The element common to the excluded row and column is called the multiplier of the minor. The product of the minor and its multiplier is often formed and given an algebraic sign. If the sum of the row and column of the multiplier is odd, the sign is negative; if the sum is even, the sign is positive. The definition of a determinant shows very quickly that *if the elements of a row or column are taken as multipliers, the sum of the products of them and their minors is the expansion of the determinant.* This is called the expansion of the determinant by the minors of that row or column.

Example 12–1. Write down the six products of the following determinant and show that they add to zero. Hence the value of the determinant is zero.

$$\triangle \equiv \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} =$$

$$1 \cdot 5 \cdot 9 + 4 \cdot 8 \cdot 3 + 7 \cdot 2 \cdot 6 - 3 \cdot 5 \cdot 7 - 4 \cdot 2 \cdot 9 - 1 \cdot 6 \cdot 8 = 225 - 225 = 0$$

Example 12–2. The determinant

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0 \qquad (12\text{–}5)$$

when expanded by the minors of the first row reads

$$x \begin{vmatrix} y_1 & 1 \\ y_2 & 1 \end{vmatrix} - y \begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix} + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = 0.$$

Hence the equation of a line through two points can be written immediately

$$x \frac{\begin{vmatrix} 1 & y_1 \\ 1 & y_2 \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}} + y \frac{\begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}} = 1 \qquad (12\text{–}6)$$

12–3. *A Determinant with Two Proportional Rows or Columns Equals Zero.* In the case of the two-rowed determinant, called $\triangle_2$, if the columns are proportional,

$$\triangle_2 = \begin{vmatrix} a_1 & Ka_1 \\ a_2 & Ka_2 \end{vmatrix} = K(a_1a_2 - a_1a_2) = 0$$

In the case of the three-rowed determinant, $\triangle_3$, if two columns are proportional

$$\triangle_3 = \begin{vmatrix} a_1 & Ka_1 & c_1 \\ a_2 & Ka_2 & c_2 \\ a_3 & Ka_3 & c_3 \end{vmatrix}$$

If $\triangle_3$ is expanded by the minors of the third column, each minor has proportional columns and is zero. By induction one covers all cases.

This property makes it possible to show that *if the elements of a row or column are multiplied by a constant and added to those of another row (or column, respectively) the determinant's value does not change.* One observes first the expansion of the determinant on the left by the minors of the first row and reassembling of terms:

$$\begin{vmatrix} A_1 + A'_1 & B_1 + B'_1 & C_1 + C'_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$$

$$= \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} + \begin{vmatrix} A'_1 & B'_1 & C'_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$$

Then for the case at hand, assume that

$$A'_1 = KA_2,\ B'_1 = KB_2,\ C'_1 = KC_2$$

which makes the second determinant on the right zero and leaves the value of the original determinant unchanged. This property makes it possible to make numerous elements zero and thus simplify expansion of the determinant.

Example 12–3. Evaluate the determinant

$$\triangle_4 = \begin{vmatrix} 4 & 2 & 3 & 0 \\ 3 & 0 & 4 & 2 \\ 2 & 2 & 0 & 1 \\ 1 & 2 & 2 & 2 \end{vmatrix} = ?$$

Two methods are given; the first adds multiples of rows to rows or multiples of columns to columns in order to get as many zeros into some row or some column as possible, because then expansion of the determinant by the minors of that row or column is easy. The second method expands by the minors of the top row and then uses the first method of getting zeros into a row or column.

Method 1.

$$\triangle_4 = \begin{vmatrix} 4 & 2 & 3 & 0 \\ -1 & -4 & 4 & 0 \\ 2 & 2 & 0 & 1 \\ -3 & -2 & 2 & 0 \end{vmatrix} = - \begin{vmatrix} +4 & 2 & 3 \\ -1 & -4 & 4 \\ -3 & -2 & 2 \end{vmatrix} = \begin{vmatrix} -4 & 2 & 3 \\ 1 & -4 & 4 \\ 3 & -2 & 2 \end{vmatrix} = \begin{vmatrix} 0 & -14 & 19 \\ 1 & -4 & 4 \\ 0 & 10 & -10 \end{vmatrix}$$

$$= - \begin{vmatrix} -14 & 19 \\ 10 & -10 \end{vmatrix} = (-140 + 190) = 50.$$

First, the third row was doubled and subtracted from the second and fourth. Finally, in the three-rowed determinant, multiples of the second row were subtracted from the first and third rows.

Method 2.

$$\triangle_4 = 4 \begin{vmatrix} 0 & 4 & 2 \\ 2 & 0 & 1 \\ 2 & 2 & 2 \end{vmatrix} -2 \begin{vmatrix} 3 & 4 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 2 \end{vmatrix} +3 \begin{vmatrix} 3 & 0 & 2 \\ 2 & 2 & 1 \\ 1 & 2 & 2 \end{vmatrix}$$

$$= 4 \begin{vmatrix} 0 & 4 & 2 \\ 2 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix} -2 \begin{vmatrix} -1 & 4 & 2 \\ 0 & 0 & 1 \\ -3 & 2 & 2 \end{vmatrix} +3 \begin{vmatrix} 3 & 0 & 2 \\ 1 & 0 & -1 \\ 1 & 2 & 2 \end{vmatrix}$$

$$= 4(-2) \begin{vmatrix} 4 & 2 \\ 2 & 1 \end{vmatrix} +2 \begin{vmatrix} -1 & 4 \\ -3 & 2 \end{vmatrix} +3(-2) \begin{vmatrix} 3 & 2 \\ 1 & -1 \end{vmatrix}$$

$$= 4(-2)(0) + 2(+10) + 3(-2)(-5) = 20 + 30 = 50.$$

Example 12–4. Expand determinants A and B to show that they are both the same equation in U, V, W. Then derive B from A by determinant properties studied above.

$$A \equiv \begin{vmatrix} 1 & 0 & \frac{-1}{uU} \\ 0 & 1 & \frac{-1}{vV} \\ u & v & \frac{-1}{W} \end{vmatrix} = 0$$

$$A \equiv \begin{vmatrix} 1 & \frac{-1}{vV} \\ v & \frac{-1}{W} \end{vmatrix} + u \cdot \begin{vmatrix} 0 & \frac{-1}{uU} \\ 1 & \frac{-1}{vV} \end{vmatrix} = \frac{-1}{W} + \frac{1}{V} + \frac{1}{U} = 0; \frac{1}{U} + \frac{1}{V} = \frac{1}{W}$$

$$B \equiv \begin{vmatrix} \frac{-U}{2} & \frac{\sqrt{3}U}{2} & 1 \\ V & 0 & 1 \\ \frac{W}{2} & \frac{\sqrt{3}W}{2} & 1 \end{vmatrix} = 0$$

$$B = \frac{V\sqrt{3}W}{2} + \frac{W\sqrt{3}U}{4} - \frac{V\sqrt{3}U}{2} + \frac{U\sqrt{3}W}{4} = \frac{VW}{2} + \frac{WU}{2} - \frac{UV}{2} = 0; \frac{1}{U} + \frac{1}{V} = \frac{1}{W}$$

$$0 = A \equiv \begin{vmatrix} 1 & 0 & \frac{-1}{uU} \\ 0 & 1 & \frac{-1}{vV} \\ u & v & \frac{-1}{W} \end{vmatrix} = \begin{vmatrix} uU & 0 & -1 \\ 0 & vV & -1 \\ uW & vW & -1 \end{vmatrix} = \begin{vmatrix} U & 0 & 1 \\ 0 & V & 1 \\ W & W & 1 \end{vmatrix} = \begin{vmatrix} U/2 & 0 & 1 \\ 0 & V & 1 \\ W/2 & W & 1 \end{vmatrix}$$

$$\begin{vmatrix} U/2 & -U/2 & 1 \\ 0 & V & 1 \\ W/2 & W/2 & 1 \end{vmatrix} = \begin{vmatrix} \frac{\sqrt{3}}{2}U & -U/2 & 1 \\ 0 & V & 1 \\ \frac{\sqrt{3}}{2}W & W/2 & 1 \end{vmatrix} = \begin{vmatrix} \frac{U}{-2} & \frac{\sqrt{3}}{2}U & 1 \\ V & 0 & 1 \\ \frac{W}{2} & \frac{\sqrt{3}}{2}W & 1 \end{vmatrix} \equiv B = 0$$

12–4. *Cramer's Rule.* Given three linear, simultaneous, non-homogeneous equations in x, y, z

$$\begin{aligned} a_1x + b_1y + c_1z &= k_1 \\ a_2x + b_2y + c_2z &= k_2 \qquad |abc| \neq 0. \\ a_3x + b_3y + c_3z &= k_3 \qquad (12\text{–}7) \end{aligned}$$

Let A_1, A_2, A_3 be the signed minors of the elements a_1, a_2, a_3 in the determinant of the coefficients

$$\triangle_3 \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \equiv |abc|.$$

We assume for a moment that there is a solution in existence. We multiply the equations above by A_1, A_2, A_3 respectively and add:

$$(a_1A_1 + a_2A_2 + a_3A_3)x + (b_1A_1 + b_2A_2 + b_3A_3)y + (c_1A_1 + c_2A_2 + c_3A_3)z = k_1A_1 + k_2A_2 + k_3A_3. \qquad (12\text{–}8)$$

The coefficient of x is $\triangle_3$ evaluated by the minors of the A's. The right hand side of the equality is $|kbc|$ evaluated by the minors of the k's. The coefficients of y and z are the expansions of determinants having identical columns and hence are zero. y and z are found similarly, thence

$$x = \frac{|kbc|}{|abc|}; \; y = \frac{|akc|}{|abc|}; \; z = \frac{|abk|}{|abc|}; \; |abc| \neq 0. \qquad (12\text{–}9)$$

It was assumed above that a solution existed. Now it must be shown that (12–9) *is* a solution. Substituting the values of x, y, z into (12–7), one finds, from the first equation

$$\begin{aligned} &a_1|kbc| + b_1|akc| + c_1|abk| - k_1|abc| \\ &= a_1|bck| - b_1|ack| + c_1|abk| - k_1|abc| = 0. \end{aligned}$$

The last expression is the expansion by minors of its first row of the four-rowed determinant $\triangle_4$.

$$\triangle_4 = \begin{vmatrix} a_1b_1c_1k_1 \\ a_1b_1c_1k_1 \\ a_2b_2c_2k_2 \\ a_3b_3c_3k_3 \end{vmatrix} = 0$$

which is zero because the first two rows are identical. Corresponding substitutions show that the other two equations are satisfied and complete the proof of Cramer's rule. The solution can be shown to be unique.

Example 12–5. Show by Cramer's Rule that the coordinates of a point P determined by lines

$$A_1x + B_1y + C_1 = 0$$

$$A_2x + B_2y + C_2 = 0 \qquad (12\text{–}10)$$

can be written in the form

$$x = \frac{-|C_1B_2|}{|A_1B_2|} \qquad y = \frac{-|A_1C_2|}{|A_1B_2|} \qquad |A_1\,B_2| \neq 0. \qquad (12\text{–}11)$$

This result comes immediately from (12–8).

12–5. *Homogeneous and Non-Homogeneous Linear Equations.* The three equations which follow are linear and non-homogeneous. If the values of all the k's were zero, they would have been homogeneous.

$$a_1x + b_1y + c_1z = k_1$$

$$a_2x + b_2y + c_2z = k_2$$

$$a_3x + b_3y + c_3z = k_3 \qquad (12\text{–}12)$$

If x_o, y_o, z_o are solutions of three simultaneous linear *homogeneous* equations in x, y, z then jx_o, jy_o, jz_o are also solutions.

12–6. *Solutions of Homogeneous Equations.* By Cramer's rule the solution is

$$x = \frac{|kbc|}{|abc|}\,;\ y = \frac{|akc|}{|abc|}\,;\ z = \frac{|abk|}{|abc|}\,;\ |abc| \neq 0. \qquad (12\text{–}13)$$

The solution is unique.

Applying this result to a set of homogeneous equations (12–12), $k_{j=0}$ the fact that the k's are zero would make x, y, z solution values all zero. Hence, only this trivial solution is possible for the equations.

$$a_1x + b_1y + c_1z = 0$$

$$a_2x + b_2y + c_2z = 0$$

$$a_3x + b_3y + c_3z = 0 \qquad (12\text{–}14)$$

unless $|abc| = 0$. Acceptance of this restriction removes the case from Cramer's Rule and does provide a non-trivial (that is, non-zero) solution. Assume for the moment that $z = 1$. Then Cramer's Rule can be applied to the first two equations, yielding

$$x = \frac{|c_1b_2|}{|a_1b_2|}\,; \quad y = \frac{|a_1c_2|}{|a_1b_2|}\,; \quad z = -1; \quad |a_1b_2| \neq 0 \qquad (12\text{–}15)$$

and the proportional solution, by Section 12–5,

$$x = |c_1b_2|\,; \quad y = |a_1c_2|\,; \quad z = |a_1b_2|. \qquad (12\text{–}16)$$

If substituted into the third equation of (12–12) this gives

$$a_3\,|c_1b_2| + b_3\,|a_1c_2| - c_3\,|a_1b_2| = 0 \text{ or}$$

$$a_3\,|b_1c_2| - b_3\,|a_1c_2| + c_3\,|a_1b_2| = 0.$$

Since this is the expansion of

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

by the minors of the bottom row, (12–16) is a solution of (12–14).

12–7. *Three Linear Non-Homogeneous Equations in Two Unknowns. Compatibility.* (12–17) gives three linear, non-homogeneous equations in two unknowns. By Cramer's rule, one has, assuming that $|a_1b_2| \neq 0$, from the first two of these the solution (12–18).

$$a_1x + b_1y = c_1$$

$$a_2x + b_2y = c_2$$

$$a_3x + b_3y = c_3 \qquad (12\text{–}17)$$

$$x = |c_1b_2|/|a_1b_2|\,;$$

$$y = |a_1c_2|/|a_1b_2| \qquad (12\text{–}18)$$

Now the question that remains is whether or not (12–18) is a sensible answer as far as the third equation of (12–17) is concerned, that is, whether or not (12–18) is *COMPATIBLE* with that third equation. If so,

$$a_3\frac{|c_1b_2|}{|a_1b_2|} + b_3\frac{|a_1c_2|}{|a_1b_2|} = c_3. \qquad (12\text{–}19)$$

This is the expansion of

$$-\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0 \qquad |a_1b_2| \neq 0 \qquad (12\text{–}20)$$

so (12–20) is a *necessary* condition for compatibility of equations (12–19). It can be shown to be sufficient also.

Example 12–6. Are the following equations compatible?

$$x + y + 1 = 0$$
$$x + 2y + 3 = 0$$
$$3x + 2y + 1 = 0. \qquad (12\text{–}21)$$

The determinant of the equations

$$\triangle_3 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 3 & -1 & -2 \end{vmatrix}$$

$$= -2 + 2 = 0.$$

Since

$$|a_1b_2| = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \neq 0,$$

the equations are compatible. A solution obtained from the first two, namely $x = +1$; $y = -2$, satisfies all three.

12–8. *Concurrency of Three Lines. Collineation of Three Points.* If lines L_1, L_2, L_3 pass through point $P(x_o, y_o)$ then

$$A_1x_o + B_1y_o = C_1$$
$$A_2x_o + B_2y_o = C_2$$
$$A_3x_o + B_3y_o = C_3 \qquad (12\text{–}22)$$

The necessary and sufficient condition that these three equations be compatible is

$$\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = 0. \qquad (12\text{–}23)$$

The particular point is given by

$$x_o = \frac{-|B_1C_2|}{|A_1B_2|} \qquad y_o = \frac{+|A_1C_2|}{|A_1B_2|} \qquad |A_1B_2| \neq 0. \qquad (12\text{–}24)$$

Similarly, if points (x_1, y_1), (x_2, y_2), (x_3, y_3) are collinear, then

$$Ax_1 + By_1 = 1$$
$$Ax_2 + By_2 = 1$$
$$Ax_3 + By_3 = 1 \qquad (12\text{–}25)$$

and it will be necessary that

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0. \qquad (12\text{–}26)$$

The particular line is given by

$$A = \frac{|1\ y_2|}{|x_1y_2|} \qquad B = \frac{|x_1\ 1|}{|x_1y_2|} \qquad |x_1y_2| \neq 0. \qquad (12\text{–}27)$$

See (12–6) and (12–11).

Example 12–7. The Scanner, Problem 8–7, Figure 8–11, was explained as an outgrowth of the parabola method of dualizing lines into points. The parabola duality correspondence is now verified. Figure 12–1.

Let the network chart be stationary and Cartesian equations of three of its lines, L_1, L_2, L_3, be

$$A_1x + B_1y = C_1$$
$$A_2x + B_2y = C_2$$
$$A_3x + B_3y = C_3.$$

If three such lines are concurrent, then by (12–23)

$$|ABC| = 0. \qquad (12\text{–}28)$$

Let the x-intercept, $+C/A$ of each line, with sign reversed become the x-coordinate of the real point (pole) of the alignment design and the y-coordinate of each such point be given by $-mB/A$ of the line, as acquired by the figure.

Then

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} -C_1/A_1 & -mB_1/A_1 & 1 \\ -C_2/A_2 & -mB_2/A_2 & 1 \\ -C_3/A_3 & -mB_3/A_3 & 1 \end{vmatrix}$$

$$= \left[\frac{-m}{A_1 \cdot A_2 \cdot A_3}\right] |ABC| = 0$$

and the three points corresponding to the three concurrent lines are collinear.

Example 12–8. Show that the line of collineation just derived for the points in alignment diagram A corresponds in the same way to the point of concurrency P of the three lines in network chart N.

By (12–27) the coefficients of the line will be

$$A = \frac{|1\ y_2|}{|x_1y_2|} \qquad B = \frac{|x_1\ 1|}{|x_1y_2|} \qquad C = 1 \qquad |x_1y_2| \neq 0.$$

The x-intercept, with sign reversed, and the value $-mB/A$ should now give us the x- and y-coordinates of the point common to two of the original three lines, L_1, L_2, L_3 of the network chart of Example 12–7. Using (12–24),

---

$$(-)\text{x-intercept} \equiv -C/A = \frac{|-x_1y_2|}{|\ 1\ y_2|} = -\frac{\begin{vmatrix} -C_1/A_1 & -mB_1/A_1 \\ -C_2/A_2 & -mB_2/A_2 \end{vmatrix}}{\begin{vmatrix} -mB_1/A_1 & 1 \\ -mB_2/A_2 & 1 \end{vmatrix}} = \frac{|C_1B_2|}{|A_1B_2|} = x_p$$

$$-mB/A = \frac{-m\ |x_1\ 1|}{|1\ y_2|} = \frac{-m\begin{vmatrix} -C_1/A_1 & 1 \\ -C_2/A_2 & 1 \end{vmatrix}}{\begin{vmatrix} 1-mB_1/A_1 \\ 1-mB_2/A_2 \end{vmatrix}} = \frac{|A_1C_2|}{|A_1B_2|} = y_p$$

See (12–24)

PROBLEMS

PROBLEM 12–1. Verify the popular rule for expanding a three-rowed determinant illustrated by the diagram of Figure 12–2. Each solid line passes through three terms of a positive product, each dotted line through three terms of a negative product. Check that all the terms of (12–4) are present.

PROBLEM 12–2. The scheme of Figure 12–2 for expanding a three-rowed determinant does NOT work for determinants of higher order. Using the definition for determinants show clearly why it does not, indicating any terms not indicated by it.

PROBLEM 12–3. Show quickly that the following points are collinear:

$x_1 = 4$ $\quad y_1 = 7$

$x_2 = 9$ $\quad y_2 = 11$

$x_3 = 19$ $\quad y_3 = 19.$

PROBLEM 12–4. Using the coordinates of Figure 12–3, write the medians of a triangle in the form

$$bx - (a + c)y = 0$$
$$2bx + (c - 2a)y - 2bc = 0$$
$$bx + (2c - a)y - 2bc = 0.$$

Show in a step or two that these are concurrent.

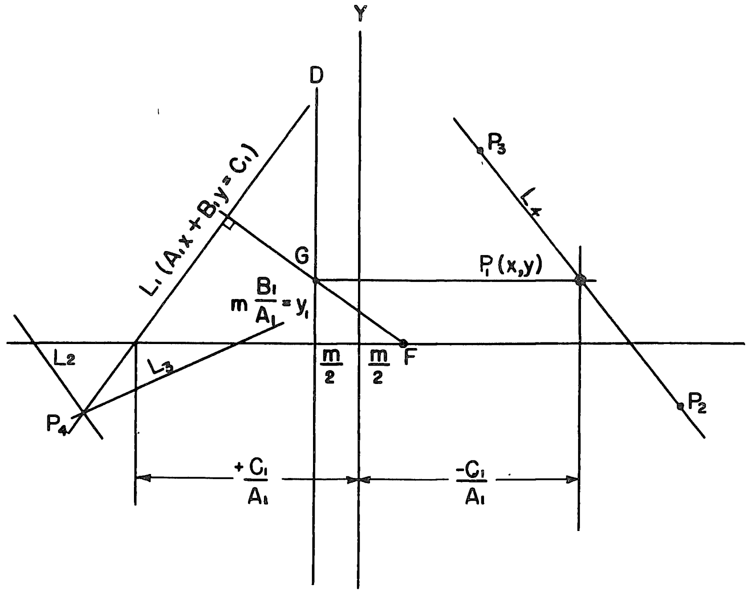


Figure 12–1.

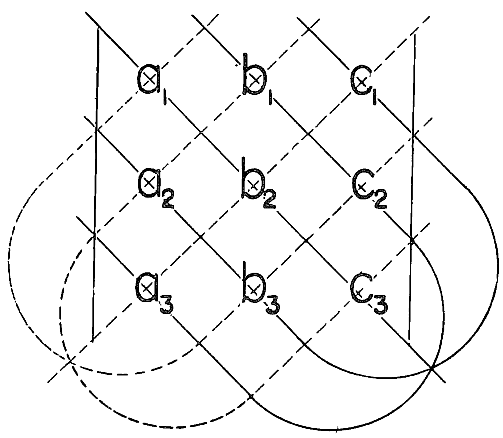


Figure 12–2.

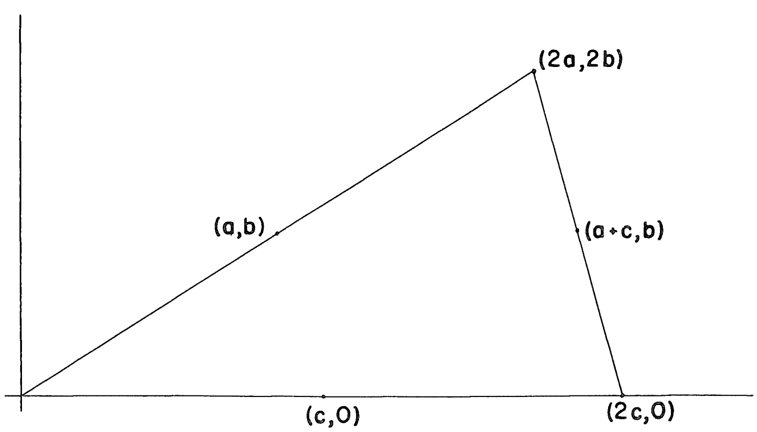


Figure 12–3.

CHAPTER 13 (APPENDIX C)

SOME METRIC AND PROJECTIVE PROPERTIES OF CONICS

CENTRAL PROJECTION • DUALITIES

13–1. *The General Equation of the Second Degree.* The equation (13–1) represents a conic, one straight line, two straight lines and a point, or else has no real locus.

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \qquad (13\text{–}1)$$

Only the real conic form of this equation will interest us most of the time. One often wishes to know what type of conic is present, whether it is a circle, ellipse, parabola or hyperbola. Criteria for this are as follows:

$$\text{Let } \triangle = 4ACF - B^2F - AE^2 - CD^2 + BDE \qquad (13\text{–}2)$$

1. Assume $B^2 - 4AC = 0$. If $A \neq 0$, then the equation represents a parabola. If $A = 0$ it represents two parallel lines, a single line, or no locus.

2. Assume $B^2 - 4AC > 0$, it represents a hyperbola if $\triangle \neq 0$; if $\triangle = 0$, it represents two intersecting lines.

3. Assume $B^2 4AC < 0$, it represents an ellipse or has no locus according as $A \cdot \triangle$ (or $C \cdot \triangle$) is negative or positive. If $A = 0$ the equation represents a single point.

4. A further useful piece of information is the location of the center of symmetry, given by

$$x_o = (2CD - BE) / (B^2 - 4AC)$$
$$y_o = (2AE - BD) / (B^2 - 4AC)$$
$$B^2 - 4AC \neq 0. \qquad (13\text{–}3)$$

The asymptotes of a hyperbola can often be found (Problem 13–1) and the entire curve plotted from such properties. If the stem of a nomographic scale is such a curve, it can be helpful to have this independent means of fixing it.

5. A useful graphical criterion for type of conic appears in Figure 13–1. Here T_1 and T_2 are any two points on the curve. Tangents have been drawn here, determining point P. Let Q be the midpoint of T_1, T_2 and M the midpoint of PQ. If point x falls at M, the curve is a parabola. If it lies nearer to Q, the curve is a circle or an ellipse, if nearer to P a hyperbola. See Problem 13–1.

13–2. *Some Analytic Properties of the Conic.* We consider these for the ellipse; the corresponding ones for the other conics will follow closely. The tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is given by

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1. \qquad (13\text{–}4)$$

If from any point P outside the ellipse, Figure 13–2(a), two lines are drawn tangent to the ellipse at points Q and R, then line QR is known as the polar line L to pole P. Pole and polar are related as follows:

Pole	*Polar*
Point P: (x_o, y_o)	Line L: $\frac{xx_o}{a^2} + \frac{yy_o}{b^2} = 1$

(13–5)

Thus tangent point and tangent line are seen to be a special case of pole and polar. This is consistent with the definition of pole and polar, for if points Q and R are allowed to approach one another, P draws toward them and lies eventually on the tangent line at the point of tangency. This can be shown analytically as follows:

Let points Q and R have coordinates x_1, y_1 and x_2, y_2 respectively, then line L joining them will have the equation, by (12–6)

Line L:

$$\frac{x(y_2 - y_1)}{x_1y_2 - x_2y_1} + \frac{y(x_1 - x_2)}{x_1y_2 - x_2y_1} = 1. \qquad (13\text{–}6)$$

The two tangents at Q and R will have the equations.

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1; \frac{xx_2}{a^2} + \frac{yy_2}{b^2} = 1. \qquad (13\text{–}7)$$

Point P at their intersection will have coordinates, by (12–11)

Point P:

$$x_o = \frac{+a^2(y_2 - y_1)}{x_1y_2 - x_2y_1}; \; y_o = \frac{b^2(x_1 - x_2)}{x_1y_2 - x_2y_1}. \qquad (13\text{–}8)$$

So the equation of L can be written, in terms of the coordinates of P,

$$\frac{xx_o}{a^2} + \frac{yy_o}{b^2} = 1$$

which is (13–5).

13–3. *Further Pole and Polar Properties.* Assume for the moment that the two points Q (x_1, y_1) and R (x_2, y_2) (which were tangent points above) are now points exterior to the ellipse and serving as poles. Then their polar lines by (13–5) have equations identical with those for the tangents (13–7) above. The intersection of their polar lines, P, will have the same coordinates as P had before in (13–8) and line L will have the same equation it had before. Then once again point P and the line L joining points Q and R have the pole and polar relationship (13–5) *but now point P is interior to the ellipse.* There is thus a unique one-to-one correspondence between all the points in the plane and all the lines of the plane—that of pole and polar. In the particular cases where the line contains the point (or the point has the line passing through it) they are tangent line and tangent point to the ellipse itself. Figure 13–2(b) achieves the same results as follows: Two lines from P cut the conic at points x_1, y_1 and x_2, y_2. Q_2 and R_2 are the intersections of lines crossed thru these points and determine the polar line L. Starting with point Q_2 instead of point P, one ends with polar line M if the preceding procedure is followed. Thus there is no distinction between poles inside and outside the conic. Points Q and R have been drawn in as a graphical check to show that P is indeed the pole of line L.

A line K joining two poles Q and R will be the polar line of the point P common to the polars L and M of points Q and R. Figure 13–3. A third pole S on line K will have a third polar N which will pass through point P.

Analytically this relation is shown as follows:
Since the three poles are collinear by (12–26)

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

If the three polars are concurrent, we should have, by (12–23)

$$\begin{vmatrix} \frac{x_1}{a^2} & \frac{y_1}{b^2} & 1 \\ \frac{x_2}{a^2} & \frac{y_2}{b^2} & 1 \\ \frac{x_3}{a^2} & \frac{y_3}{b^2} & 1 \end{vmatrix} = 0$$

which follows from the preceding determinant. These properties can be checked graphically using the circle (Problem 13–2). By central projection they hold for all conics. Numerous properties of pole and polar follow from the form of the equation (13–5). A typical property says that a set of poles collinear on a line through the center of the conic will have parallel polars. Since the ratio y_o/x_o remains constant for such a set of poles, the slopes of the polars given by (13–5) also remain constant and they are all parallel. Figure 13–4. Another property shown by (13–6) is that the slope of a line through an external pole and the center of a conic is $b^2(x_1 - x_2)/a^2(y_1 - y_2)$. The midpoint of the polar will be $(y_1 + y_2)/2, (x_1 + x_2)/2$ and lies on this line from the center to the pole, for if these two slopes are set equal, a true equation results

$$\frac{-b^2(x_1 - x_2)}{a^2(y_1 - y_2)} = \frac{y_1 + y_2}{x_1 + x_2}$$

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = \frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} = 1.$$

The corresponding property can be shown for all conics and yields the theorem that the locus of the midpoints of a set of parallel chords of a conic is a line through the center that contains the poles of all those chords.

13–4. *Projective Properties of Conics.* Because central projection is very useful in nomography, it is well to visualize and understand a number of the projective properties of conics. If a right circular cone is cut by a plane, the section is a conic. Figure 13–5(a). If the plane is perpendicular to the cone's axis, a circle results. If the plane is parallel to an element, the curve cannot cut that element, such a curve has one branch only, cannot close and hence is a parabola. As the plane passes from the circular position to the parabolic one, it cuts a set of ellipses. As it passes beyond the parabola position, it will be cutting both nappes of the cone and will yield a set of hyperbolas. Thus, as the angle of the cutting plane with the axis varies away from 90°, the eccentricity *e* of the resulting conic varies and one has:

a circle	$e = 0$
ellipses	$e < 1$
a parabola	$e = 1$
hyperbolas	$1 < e \leqq \sec \frac{\alpha}{2}$

$$ \tag{13-9} $$

Two parallel planes will cut similar conics, that is, having the same eccentricity and differing only in size. Figure 13–5(b) is a construction to find the angle of inclination of the resulting plane to the axis to obtain a conic of specified eccentricity. Thus any conic whatsoever, $\varepsilon \leqq \sec \frac{\alpha}{2}$, can be "fitted onto" the surface of any right circular cone. $V_1'' V_2''$ is a similar ellipse of larger size. The vertex angle of the cone is immaterial; "there will be a place for each conic, $\varepsilon \leqq \sec \frac{\alpha}{2}$, upon it." Any one conic is a central projection of any other conic projected through the vertex of the cone with the elements of the cone acting as the projecting rays. A tangent to a conic in one plane will project as a tangent to the projected conic in a second plane. Hence pole and polar of one conic project into pole and polar of the second. A line and a point dual to it with respect to a conic before projection will be dual after projection with respect to the projected conic.

13–5. *Cross-Ratio and Harmonic Division.* Figure 13–6. A point C on line segment AB can be said to divide it in the ratio CA/CB. Point D similarly would divide it in the ratio DA/DB. Either or both of these points can be interior or exterior to segment AB, the ratio being negative and positive respectively if the line segments are considered as "directed." The *ratio of ratios*

$$\frac{CA}{CB} \Big/ \frac{DA}{DB} = \text{Cross Ratio (AB, DC)} \tag{13-10}$$

is called a *cross-ratio of the four points A, B, C, D.* It describes by means of this ratio how two of the points divide the other two. *Cross-ratio remains invariant under a central projection.* Any four concurrent lines through four collinear points take on the cross-ratio of the four points which is also given by the cross-ratio of the sines of their corresponding angles. Figure 13–7(a).

There are twenty-four ways of forming the cross-ratio of four points, that is, twenty-four ways in which it can be said that any two of four points divide the remaining two. Problem 13–6. There are only six *distinct* ways, however, for the twenty-four cross-ratios turn out in general to have only six distinct values. If α is one of these values, then the six have the formulas:

$$\alpha, \frac{1}{\alpha}, 1 - \alpha, \frac{1}{1-\alpha}, \frac{\alpha}{\alpha - 1}, \frac{\alpha - 1}{\alpha}. \tag{13-11}$$

When the value of a cross-ratio is (-1), the four points are said to be in *harmonic ratio.* Figure 13–7(b) shows how the fourth of four points in harmonic ratio can be obtained graphically where three are known by using a "complete quadrilateral." An important special case of harmonic ratio occurs when C is the midpoint of AB, for then D must be at infinity on the line segment. Figure 13–8 shows that for the conic, points X_1 and Y_1 divide points P and Q_1 harmonically by using the properties of Figures 13–2(b), 13–7(b). Hence the theorem that if a line K cuts a conic in points X_1 and Y_1 and contains point P, the polar line L of point P cuts the line K in point Q_1 such that the points X_1 and Y_1 separate points P and Q_1 harmonically, and conversely.

13–6. *Results of the Harmonic Properties of a Conic.* The harmonic ratio

$$\frac{QX}{PX} \Big/ \frac{QY}{PY} = -1 \tag{13-12}$$

can be written

$$\frac{QX}{XP} = \frac{YQ}{YP}. \tag{13-13}$$

In Figure 13–9, line M passes thru the center of the conic. If the conic is a parabola, Y and Q are infinitely far out. (13–12) becomes

$$\frac{QX}{XP} = 1 \tag{13-14}$$

QX = XP. Here M is parallel to the parabola axis. By earlier work, the tangent at X is parallel to line L.

13–7. *A Practical Conic Duality.* A chart consisting of lines, like a network chart, can be taken into a diagram consisting of points, like an alignment diagram, by treating each line of the former as a polar and deriving each point of the latter as the corresponding pole. (Chapter 8) Then if three of the lines of the network chart are concurrent, the three poles of the alignment diagram will be collinear. By general use of the principle, the network chart will have been transformed into an alignment diagram. Section 13–3. The polar of the center of symmetry of a conic will be the line at infinity. The poles of the lines through the center all lie on the line at infinity. Hence the center of symmetry of the conic is undesirable for lines or points to be near. Pole and polar lie on opposite sides of the conic. Tangent point and tangent line are dual elements and the conic is a self-dual curve. It would be nice to have all the network chart "just inside" the conic so to derive all the alignment diagram "just outside." This would require a conic that was large with respect to the size of the network chart and hence awkward to work with in the dualizing, for unless the conic is large, some portion of the network chart will be near the undesirable area of the center of symmetry.

The parabola can be thought of as an ellipse whose "other vertex" has been dragged off to infinity. The center of symmetry is half-way to that vertex and hence also at infinity, so the parabola should not be awkward to work with in dualizing..

With intentional repetition, we now present the results of this relation. Figure 13–10. Since the line PX0Y of Figures 13–8, 13–9 passed through the center of the conic, it now lies parallel to the axis of the parabola. Let us call this line M. Equation (13–14) states that, for the parabola, the harmonic property takes the following forms: The point Q (where the polar line crosses M) and the point P are equidistant from point X where M cuts the parabola (and on opposite sides of it). Now see Section 13–3, Figure 13–4. The polars of poles collinear on a line M through the center of a conic are all parallel, including the tangent where this line M cuts the conic (Figure 13–4). Hence the direction of the polar through point Q is that of the tangent to the parabola at point X. This tangent is easily found since it cuts the primary axis as far behind the vertex as the point of tangency is in front of it. This property turns out to be true for the pole and polar also (of which tangent point and tangent line are a special instance). Hence the polar cuts the primary axis as far behind the vertex as the pole is in front of it. A further (non-projective) relationship is that the polar line is perpendicular to a line from the focus, F, to the point G where line M through P (parallel to the axis) cuts the directrix of the parabola. These simplifications permit dualization with respect to the parabola without drawing the curve by merely using its focus and directrix and the most elementary drafting techniques. This dualization turns out to be the same dual relationship that was used in the scanner based on identity of point and line coordinates. (See Section 13–8, also Examples 12–7, 12–8.) It motivated that approach. See Chapter 8.

These results come readily analytically from the equations for the parabola:

Pole	*Polar*
$P(x_1, y_1)$	$L\ yy_1 = m(x + x_1)$ (same as tangent)

$$(13\text{–}15)$$

$$\text{Then L cuts the prime axis at } x_a = -x_1 \text{ Q.E.D} \qquad (13\text{–}16)$$

and the coordinates of point G, where line L cuts the directrix $(-m/2, y_1)$.
The coordinates of the focus F are $(m/2, 0)$. The slope of the line GF is $-y_1/m$. The negative reciprocal of this, the slope of the tangent at P, and the slope of the polar line L are m/y_1.

13–8. *The Meaning of Perfect Duality.* Only conic duality has been used thus far, inasmuch as the coordinate dualities of Section 8–4 and Problem 8–10 both turned out to be conic dualities. Every conic duality has self-dual elements comprising the points and tangents of the conic itself. In conic duality, if points P_1 and P_2 *determined* line L_3, then L_1 and L_2 *corresponding* to points P_1 and P_2 *determine* point P_3 which *corresponded* to line L_3. This was a "perfect duality" which was very useful. Among other things, it permitted passing from a curve to its dual by passing either from lines of the first to points of the second, *or* from points of the first to lines of the second. There is implied here the condition that three collinear points will correspond to three concurrent lines. If a duality is set up on the last condition, namely that three collinear points correspond to three concurrent lines, will this imply a perfect duality? Will every perfect duality have to be the same as some conic duality? Such questions warrant looking closely at dualities. Perhaps some dualities can be discovered which will be more useful than those already tried.

13–9. *One-Way, Imperfect Duality.* We now experiment with "coordinate dualities" beyond earlier work. Let there be a correspondence between points and lines of the plane thru their coordinates.

$$A = A(x, y) \qquad B = B(x, y) \tag{13–17}$$

with inverse

$$x = x(A, B) \qquad y = y(A, B)$$

and assume that under this correspondence three collinear points give rise to three concurrent times.

Then

$$|x \;\; y \;\; 1| = 0 \text{ implies}$$

$$|A \;\; B \;\; 1| = 0. \tag{13–18}$$

But also

$$|x \;\; y \;\; 1| = 0 \text{ implies}$$

$$|x(A,B) \;\; y(A,B) \;\; 1| = 0$$

and conversely.

For the right hand sides of (13–18) to be equal, which must now be true, it suffices if x(A, B), y (A, B) are linear in form because linear changes inside the determinant can then bring about the equality.

Hence, we try correspondences based upon linear forms:

$$x = fA + gB + h \qquad y = jA + kB + l$$

$$A = f'x + g'y + h' \qquad B = j'x + k'y + l'$$

$$|fk - gj| \neq 0 \tag{13–19}$$

h and l can be zero without the slightest loss of generality since they merely translate the plane. Then h′ and l' are zero also. On seeking the point coordinate locus of self-dual elements, one assumes that dual elements coincide—a point lies on its dual line and conversely.

Then, in general, $A\,x + B\,y = 1$.

Using (13–19), we have, for self-dual elements

$$(f'x_1 + g'y_1)x_1 + (j'x_1 + k'y_1)y_1 = 1 \tag{13–20}$$

which is the equation of a conic. *If it is a real conic, it is the physical locus of the self-dual elements and the duality is the perfect one based upon that conic.* If it is not a real conic, there will be no self-dual elements available, thus the condition of (13–18) that collinear points imply concurrent lines may or may not bring about a perfect duality with a real conic.

Example 13–1. Let the correspondence be

$$A = -x + 2y \qquad B = -2x - 10y$$

$$x = \frac{-5}{7}A - \frac{B}{7} \qquad y = \frac{A}{7} - \frac{B}{14}.$$

The locus of self-dual elements, $x^2 + 10y^2 = -1$, is an *imaginary* conic. Figure 13–11 shows three collinear points P_1, P_2, P_3 on line L_5 and the three lines L_1, L_2, L_3 corresponding to them. It also shows point P_4 common to these three lines and the line L_4 corresponding to it. Since L_4 and L_5 do not coincide, we have here *an imperfect duality, with no self-dual elements available.* Such a duality may be useful for limited purposes. This example proves that something more than correspondence between collinear points and concurrent lines is required for a perfect duality.

13–10. *Guaranteeing Perfect Duality.* More explicitly (13–19) can be written

$$x = fA + gB \qquad y = jA + kB$$

$$A = \frac{\begin{vmatrix} x & g \\ y & k \end{vmatrix}}{\begin{vmatrix} f & g \\ j & k \end{vmatrix}} \qquad B = \frac{\begin{vmatrix} f & x \\ j & y \end{vmatrix}}{\begin{vmatrix} f & g \\ j & k \end{vmatrix}} \qquad |f \cdot k - jg| \neq 0. \tag{13–21}$$

We now require the condition for perfect duality, namely that when P_1 and P_2 determine L_3, then L_1 and L_2 *corresponding* to P_1 and P_2 shall determine P_3 which shall *correspond* to L_3. What is the correspondence?

We have line L_3 determined by P_1, P_2, of the form, from (12–6)

$$L_3: \quad x \cdot \frac{\begin{vmatrix} 1 & jA_1 + kB_1 \\ 1 & jA_2 + kB_2 \end{vmatrix}}{\begin{vmatrix} fA_1 + gB_1 & jA_1 + kB_1 \\ fA_2 + gB_2 & jA_2 + kB_2 \end{vmatrix}} +$$

$$+ y \cdot \frac{\begin{vmatrix} fA_1 + gB_1 & 1 \\ fA_2 + gB_2 & 1 \end{vmatrix}}{\begin{vmatrix} fA_1 + gB_1 & jA_1 + kB_1 \\ fA_2 + gB_2 & jA_2 + kB_2 \end{vmatrix}} = 1 \qquad |\text{denom.}| \neq 0$$

$$x \cdot A_3 + y \cdot B_3 = 1 \tag{13–22}$$

We have point P_3 determined by lines L_1 and L_2.

$$P_3: \quad x_3 = \frac{\begin{vmatrix} 1 & B_1 \\ 1 & B_2 \end{vmatrix}}{\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}} \qquad y_3 = \frac{\begin{vmatrix} A_1 & 1 \\ A_2 & 1 \end{vmatrix}}{\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}}$$

$$|AB| \neq 0 \tag{13–23}$$

Then from the required correspondence of L_3 and P_3, using (13–23) and (13–21) for A_3:

$$L_3: \qquad A_3x + B_3y = 1 \tag{13–22}$$

$$\frac{k \cdot \begin{vmatrix} 1 & B_1 \\ 1 & B_2 \end{vmatrix}}{\begin{vmatrix} f & g \\ j & k \end{vmatrix} \cdot \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}} - \frac{g \cdot \begin{vmatrix} A_1 & 1 \\ A_2 & 1 \end{vmatrix}}{\begin{vmatrix} f & g \\ j & k \end{vmatrix} \cdot \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}} = A_3. \tag{13–24}$$

On comparing coefficient A_3 in (13–24) and (13–22), the two denominators are equal and one obtains

$$k(B_2-B_1)+g(A_2-A_1)=k(B_2-B_1)+j(A_2-A_1) \tag{13–25}$$

$$g \equiv j.$$

A similar comparison of B_3 brings the same result.

This condition on the correspondence (13–21) *guarantees a perfect duality* but leaves the question: Does every such duality have to be the same as some conic duality? Can a perfect duality exist without the presence of a *real conic?* The conic in question would be the locus of self-dual elements. The condition (13–25) gives the *involution correspondence:*

$$x = fA + gB \qquad y = gA + kB$$

$$A = \frac{(kx - gy)}{\begin{vmatrix} f & g \\ g & k \end{vmatrix}} \qquad B = \frac{-gx + fy}{\begin{vmatrix} f & g \\ g & k \end{vmatrix}} \tag{13–26}$$

the line corresponding to $P_1(x_1, y_1)$ will read

$$(kx_1 - gy_1)x + (-gx_1 + fy_1)y = fk - g^2$$

The locus of points where the point lies on the line (the locus of self-dual elements) is

$$(kx_1 - gy_1)x_1 + (-gx_1 + fy_1)y_1 - fk + g^2 = 0$$
$$(f \cdot k - g^2) \neq 0$$

or the conic

$$kx_1^2 - 2gx_1y_1 + fy_1^2 + g^2 - fk = 0. \tag{13–27}$$

The question is "Must this conic always be real?" The discriminant for distinguishing here between *types* of conic represented is $g^2 - fk$, which is identical with the constant term of (13–27). Since this determinant can never be zero, degenerate cases of the conic are ruled out. Section 13–1. In fact, a real conic will always result except possibly when $g^2 - fk < 0$, that is unless *f and k have the same sign.* Moreover, the discriminant for (13–27) as a quadratic in y is

$$D = (g^2 - fk)(x_1^2 - f). \tag{13–28}$$

The conic will be imaginary if $D < 0$. The first bracket is already negative and the second can be guaranteed positive if $f < 0$. Then (13–26) gives a perfect duality not based upon a real conic if

$$f < 0, k < 0.$$

13–11. *Non-Conic Perfect Dualities.* Thus, perfect or imperfect duality of point and line appears to be best described as a function of point and line relationships with the conic entering graphically only as a secondary phenomenon.

Example 13–2. Figure 13–12. The correspondence between point and line coordinates

$$x = -A + B \qquad y = A - 2B$$
$$A = -(2x + y) \qquad B = -(x + y)$$

is of the form (13–26) where k and f are both negative. Derive the locus of self-dual points and show that it is an imaginary conic. Show by a single example that three collinear points give rise to three concurrent lines and the point of concurrency *does* correspond to the line of collineation.

Locus:

$$(-2x_1 - y_1)x_1 - (x_1 + y_1)(y) = 1;$$

$$y_1^2 + 2x_1y_1 + 2x_1^2 + 1 = 0$$

$$B^2 - 4A \cdot C = -4(x_1^2 + 1) < 0.$$

No real y_1, no real locus. Such a duality would seem to avoid several of the major troubles of conic dualities. See Figure 13–12 for example of perfect dual correspondence not based upon a real conic.

13–12. *Quadrilateral Projected Graphically into a Rectangle or Parallelogram of Given Dimensions.* Four points in plane I have been carried into four points of plane II analytically by the projective transformation in Section 7–4. The central projection required to take an irregular quadrilateral into a rectangle of given dimensions can be determined graphically and even usefully in a pictorial. Figure 13–13 shows the pictorial derivation. Engineering drawing methods using conventional orthographic views can follow along these lines if desired. The quadrilateral is set up in plane I in pictorial form and plane III passed perpendicular to it through the line joining points P and Q, the intersections of the opposite sides of the quadrilateral. Plane II carrying the final projection will be made parallel to plane III. The desired center of projection 0 will lie somewhere on a circle with PQ as diameter in plane III because the projection must carry points P and Q to infinity in plane II and in directions at right angles to one another. Points R and S, where the diagonals of the quadrilateral cut line PQ, must also be projected to infinity from 0 in such a way that the angles α and β they make with 0P and 0Q are the same as in the eventual rectangle, thereby using its proportions. Then angles PQR′ and R′PQ are equal to α and β and triangle PQR′ has the proportions of one half of the rectangle. Hence PR′ and QR′ have the same proportions as PK and QK, namely those of the rectangle, the line R′R determines point 0 properly. *If the rectangle needs enlargement or reduction, this is done by changing the distance of plane II from plane III* proportionally. Establishing point R properly establishes point S also, since points PQRS are in harmonic division and so are the directions of the sides and diagonals of the rectangle. Point S is not shown in the figure. Figure 6–13 shows that once point 0 is found, only its affine coordinates need to be preserved and the angle between planes I and II need not remain at 90°.

If the final figure is a parallelogram instead of a rectangle, the same direct method for finding the center of projection is possible. Problem 13–12.

13–13. *Quadrilateral Projected Graphically Into a Quadrilateral.* To find graphically the center of projection between two *general* quadrilaterals, it is easiest to draw them both on transparent paper, extending rather fully the sides and diagonal lines. A long straight line L is then drawn on a third paper and the quadrilaterals maneuvered on top of it until corresponding lines cross in every case at points on this line L. It is frequently possible to work this out fairly rapidly. It will then be found that lines joining corresponding points are concurrent thru a point Z′. Figure 13–14 shows that this figure is the result of folding the space projection into a single plane about the line of intersection of the planes of the quadrilaterals, which become line L. If the views are now folded back about line L into separate planes, the space projection is recreated. Lines joining corresponding points will be concurrent in space at a point Z′ related to the original point Z′ by affine coordinates. Figure 6–13 and 13–13 also illustrate the underlying relations here. (See also Problems 13–12 thru 13–16.)

PROBLEMS

PROBLEM 13–1. If the conic

$$Ax^2 + Bxy + Cy^2 + F' = 0$$

is a hyperbola, show that the asymptotes are defined by the equation

$$Ax^2 + Bxy + Cy^2 = 0$$

Note: The conic is centered at the origin.

PROBLEM 13–2. Using the hyperbola as conic, verify graphically the propositions of Section 13–2, 13–3.

PROBLEM 13–3. A secant from point P cutting an ellipse at points x and y has the polar meeting it at point Q. Using the principle of Figure 13–7(b) verify graphically that points P and Q divide points X and Y harmonically.

PROBLEM 13–4. Using Figures 13–5(c) check by inspection as many elementary properties of conic as you can. For instance, show that there is a conic the sum of whose distances from two points remains constant, that this conic has eccentricity less than one, and that hence it is an ellipse with these points as foci.

PROBLEM 13–5. Explain the useful property of Figure 13–1, using the material of this chapter, especially Section 13–6.

PROBLEM 13–6. (a) Indicate the twenty-four ways of arranging the letters A, B, C, D and hence the existence of twenty-four cross-ratios of four points. (b) Check that the latter three cross-ratios of (13–11) must follow from the first three. (c) If one of the separations is harmonic, what happens to the values of the other five?

PROBLEM 13–7. Using principles of cross-ratio explain the property of Figure 6–13 that a change in angle of ϕ will not shift the positions of the projected points in plane II if the affine coordinates of point 0 remain constant.

PROBLEM 13–8. Derive by Euclidean geometry the property of harmonic division of PQ, XY for a line through the center of a circle. (Compare Figure 13–9.) Relate this to the corresponding general property for the general conic.

PROBLEM 13–9. The equations (13–17) were allowed in (13–19) to be linear. Explain how this linear relation could be generalized further. A first step would be to indicate how one can bring about the change

$$|xyl| = |fA+gB+h \quad jA+kB+l \quad pA+qB+r|$$

$$|ABl| = 0$$

This then implies that a more general form of (13–19) could be

$$x = \frac{fA + gB + h}{pA + qB + r} \qquad y = \frac{jA + kB + l}{pA + qB + r}.$$

PROBLEM 13–10. Make a general observation about

(a) the group of perfect dualities each of which is associated with a real conic,

(b) the group of perfect dualities none of which is associated with a real conic,

(c) the group of imperfect dualities.

PROBLEM 13–11. Reproduce the irregular quadrilateral of Figure 13–13 and carry it into a rectangle $\frac{3}{4} \times 2$ by a *pictorial.* Write down the coordinates of the center of projection that will do this.

PROBLEM 13–12. Carry through the construction of Figure 13–13 for a parallelogram instead of a rectangle. PQ will now be a chord of the circle rather than a diameter. First find R′, then the circle, then PQR′.

PROBLEM 13–13. In Figure 13–15, points ABCD have the cross-ratio $\dfrac{AD/BD}{AC/BC} = AD \cdot BC.$ (Because AC = 5/4, BD = 4/5.) (a) Check that if the primed distances are measured, the cross-ratio remains the same. (b) Check that the cross-ratio remains the same for A″B″C″D″ on a transversal across a second set of four lines through 0′. (Note that computation of the cross-ratio can be greatly simplified by strategic choice of two of the values on the straight edge.) Thus the cross-ratio "belongs" to the points, and to the lines, and is a projective property.

PROBLEM 13–14. In Figure 13–16, quadrilateral (a) is supposed to be projected centrally from quadrilateral (b). Point x is the midpoint of the side a and lies on line X through point Q. A card edge E carries the cross-ratio of the lines DXCL and has reproduced them in (b). (See Problem 13–13.) Check that x′ will not in general be the midpoint of side a′.

PROBLEM 13–15. Prove that the figure formed by lines thru all four midpoints in (a) is a parallelogram. Having established the parallelogram in (a) find the four corresponding points in (b).

PROBLEM 13–16. The problem of projecting quadrilateral (b) into quadrilateral (a) is now in the form of Problem 13–12, carrying a quadrilateral into a parallelogram. Do not use an axis of homology. Carry this through for your own figures in pictorial form.

PROBLEM 13–17. Figure 13–17 shows a set of lines and points. Write down in words the projective relations between these with respect to centers of projection S_1, S_2 and 0. Discuss the cross-ratios present in the diagram.

PROBLEM 13–18. If four points on a conic are given, the locus condition for a fifth is that lines

from it through the given four maintain a constant cross-ratio. Figure 13–18 shows an application of this principle to the development of a conic when five points have been given. This is the method that is referred to in Section 10–5. Explain the method, and use it on your own five points to get the conic determined by them. Identify the type of conic.
Hint: The five given points are S_1, S_2, A, B, C.

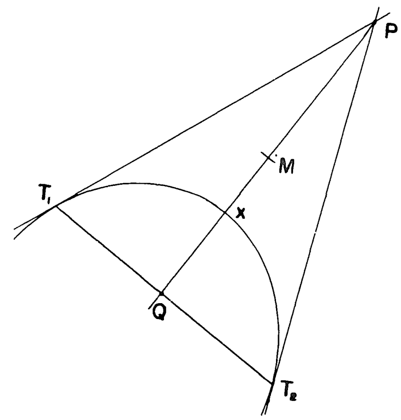


Figure 13–1.

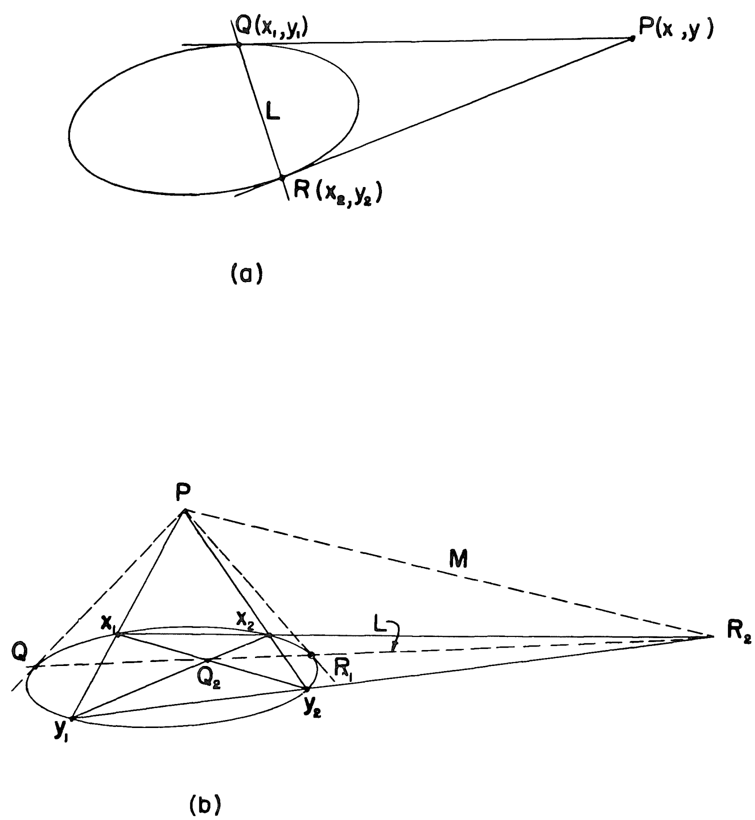


Figure 13–2.

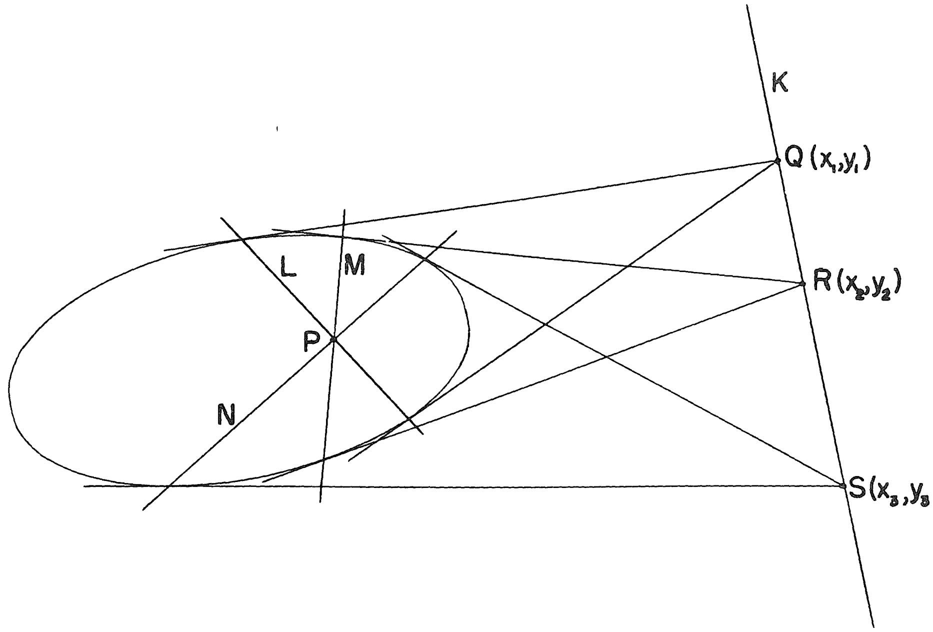


Figure 13–3.

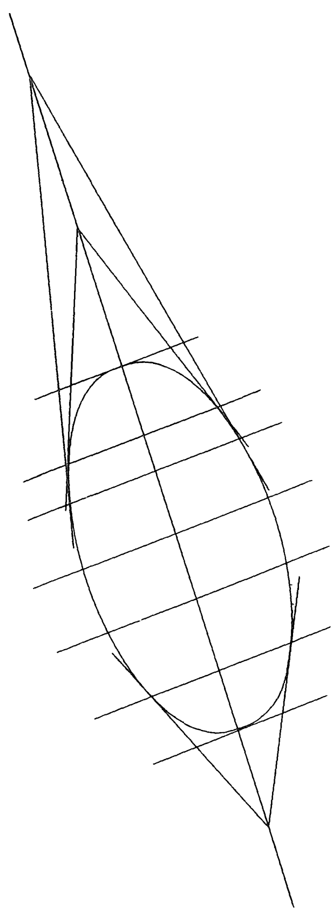


Figure 13–4.

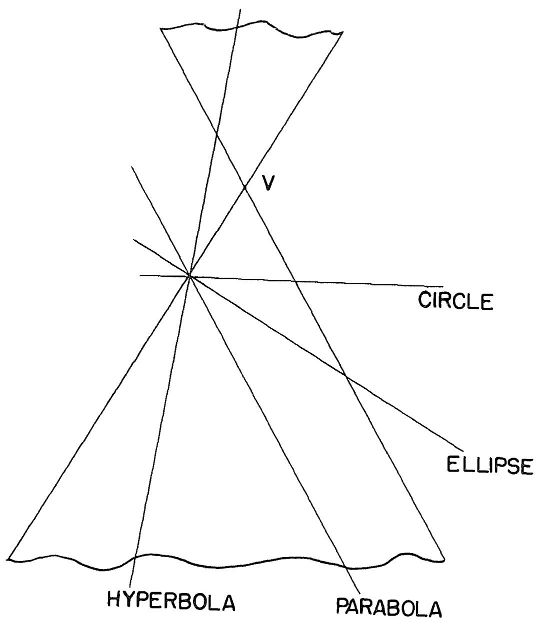


Figure 13–5a.

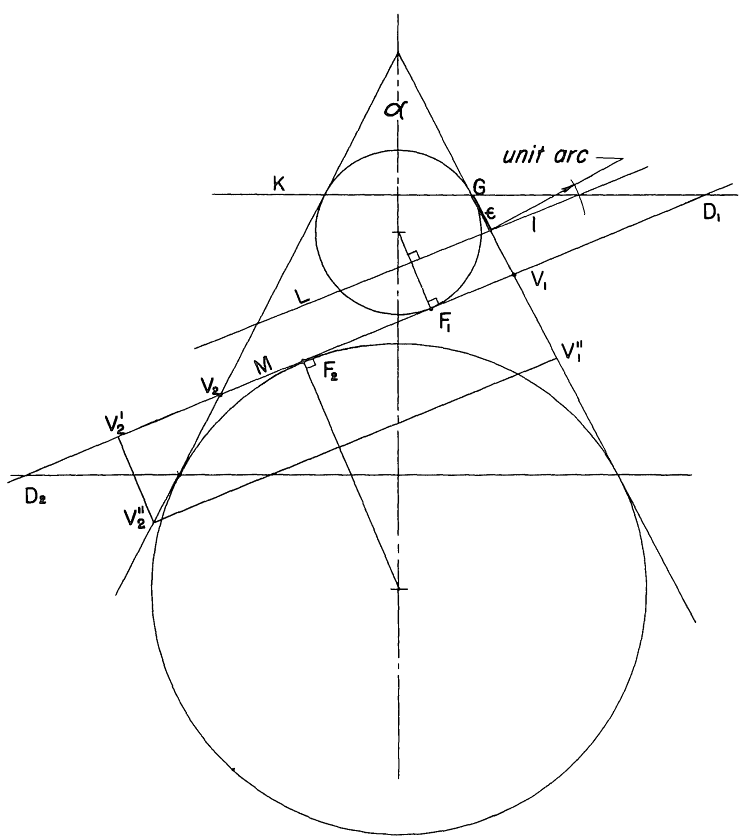


Figure 13–5b.

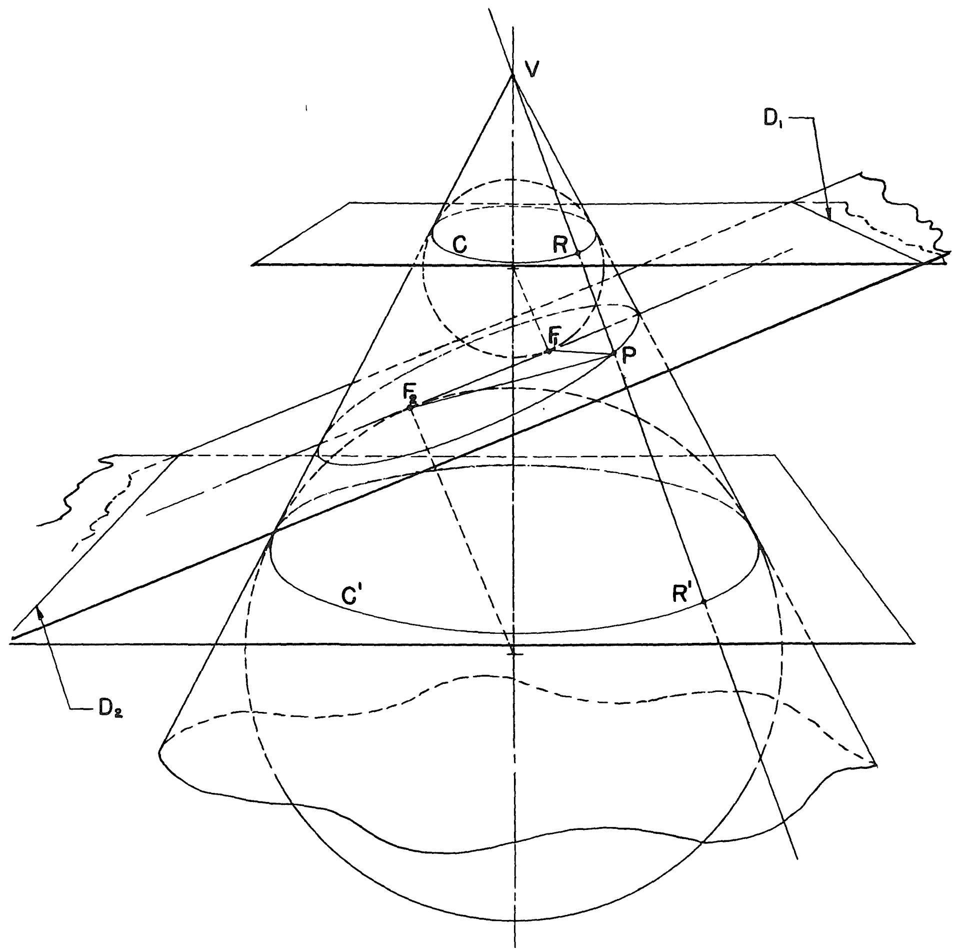


Figure 13–5c.

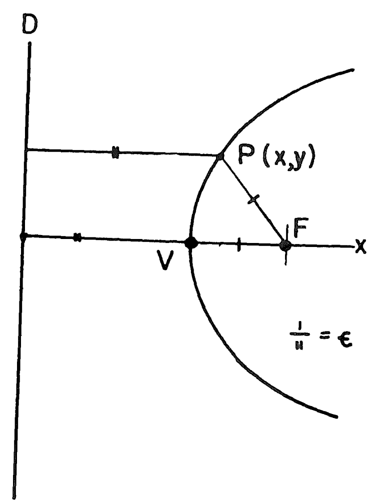


Figure 13–5d.

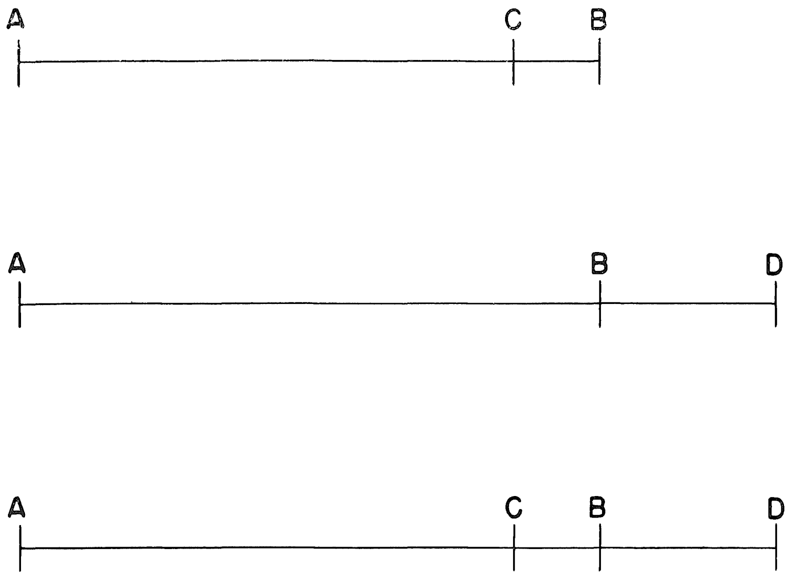


Figure 13–6.

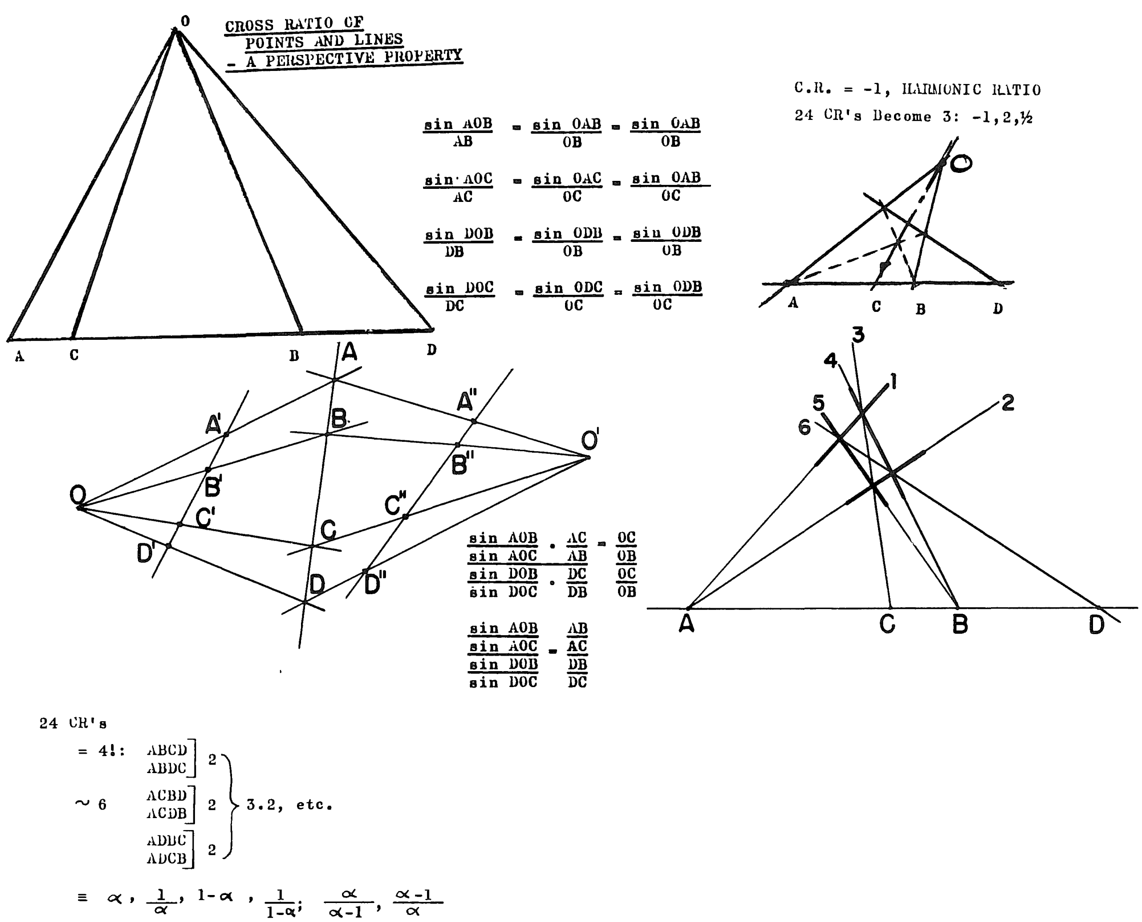
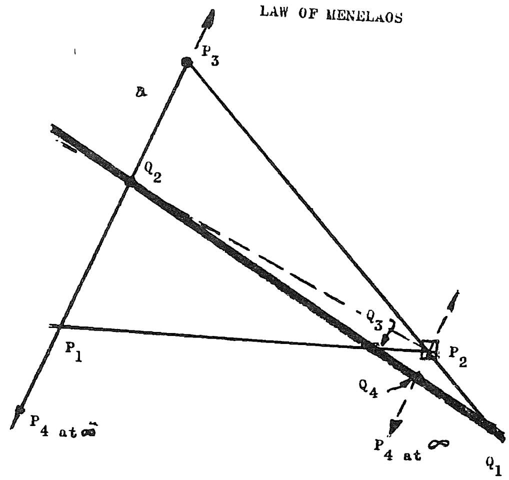


Figure 13–7a.



$$\frac{\dfrac{P_3P_4}{P_3Q_2}}{\dfrac{P_1P_4}{P_1Q_2}} = \frac{\dfrac{Q_1Q_4}{Q_1Q_2}}{\dfrac{Q_3Q_4}{Q_3Q_2}}$$ by central projection from P_2

$$\frac{P_1Q_2}{P_3Q_2} \cdot \left[\frac{P_3P_4}{P_1P_4}\right] = \frac{Q_1P_2}{Q_1P_3} \cdot \frac{Q_3P_1}{Q_3P_2}$$

$$P_1Q_2 \cdot P_3Q_1 \cdot P_2Q_3 = P_3Q_2 \cdot P_2Q_1 \cdot P_1Q_3$$

Figure 13–7b.

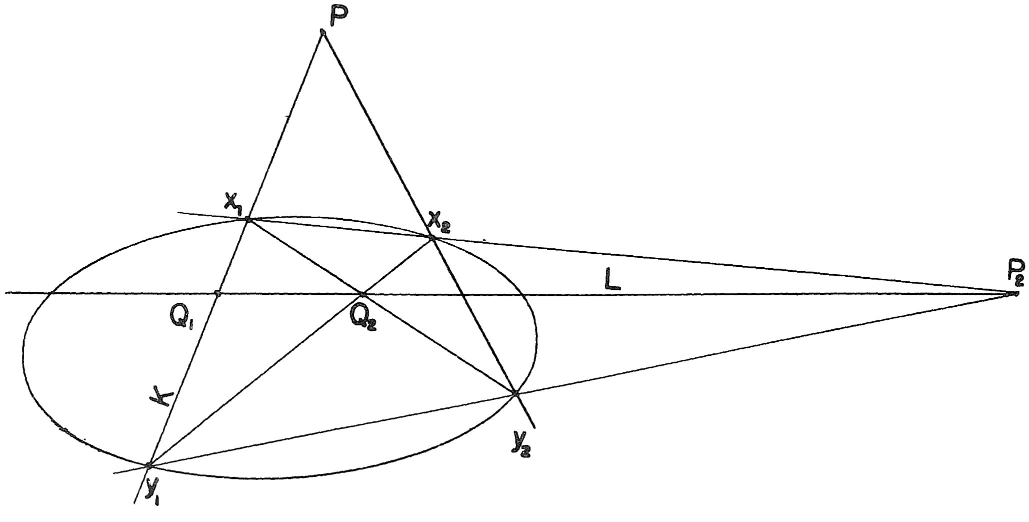


Figure 13–8.

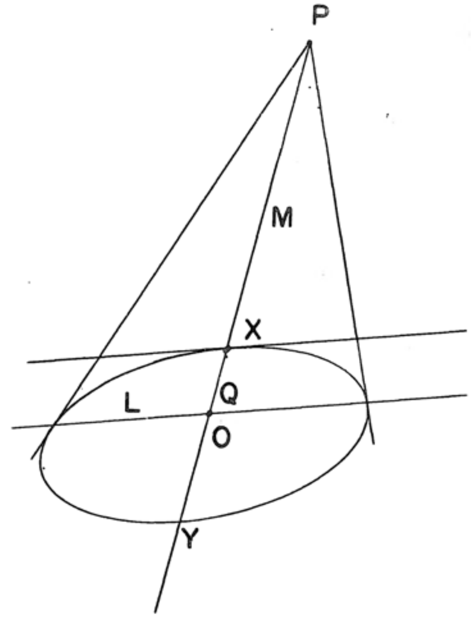


Figure 13–9.

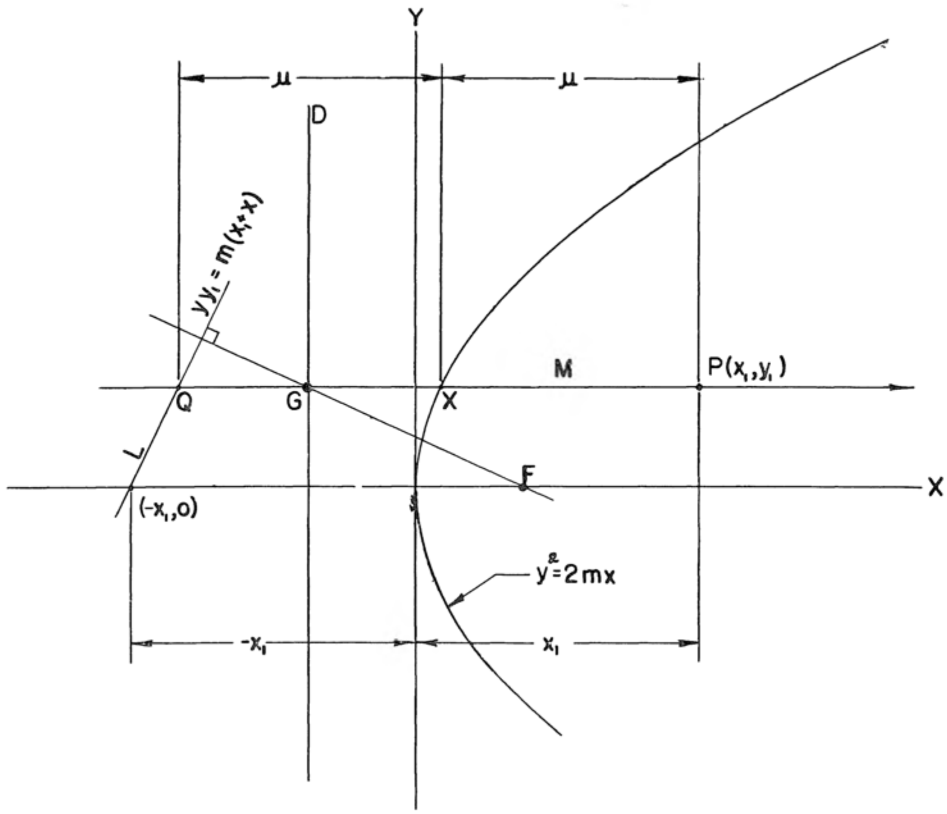
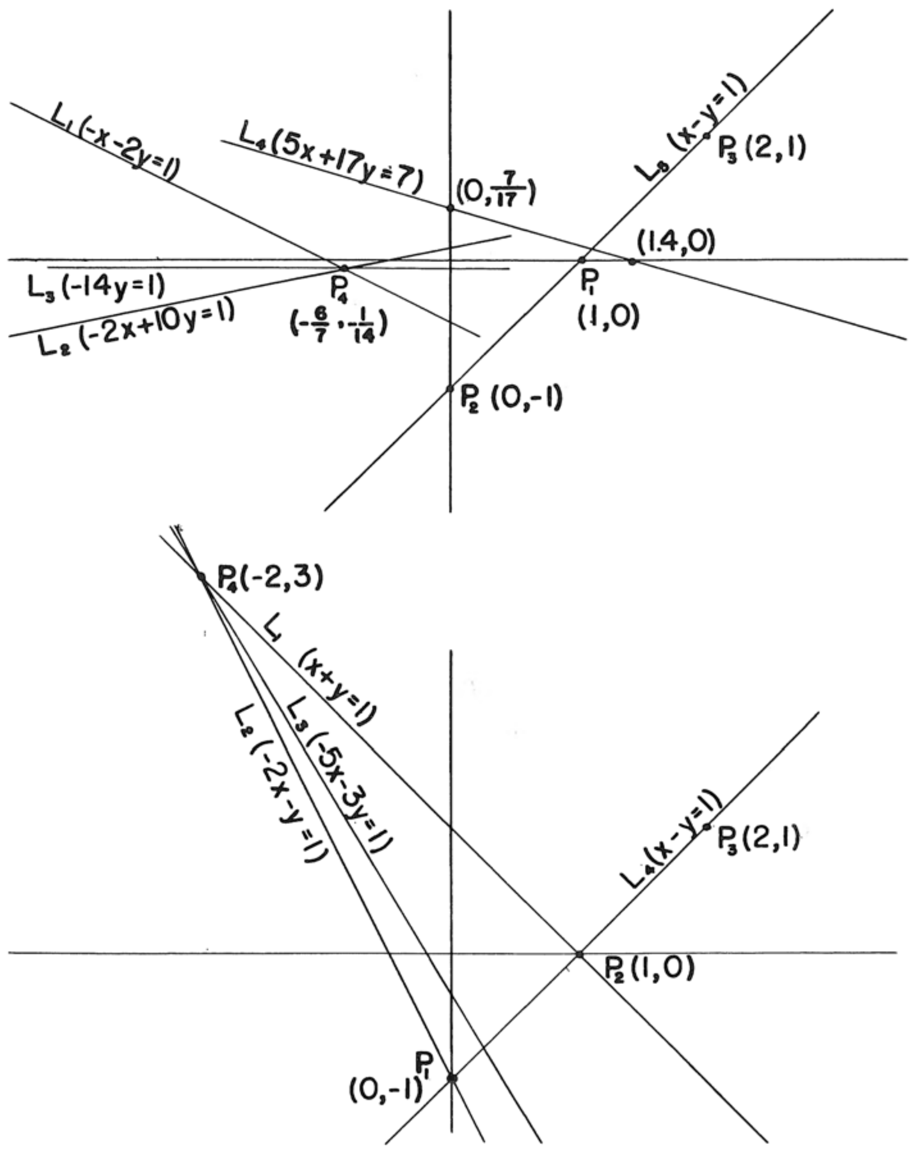
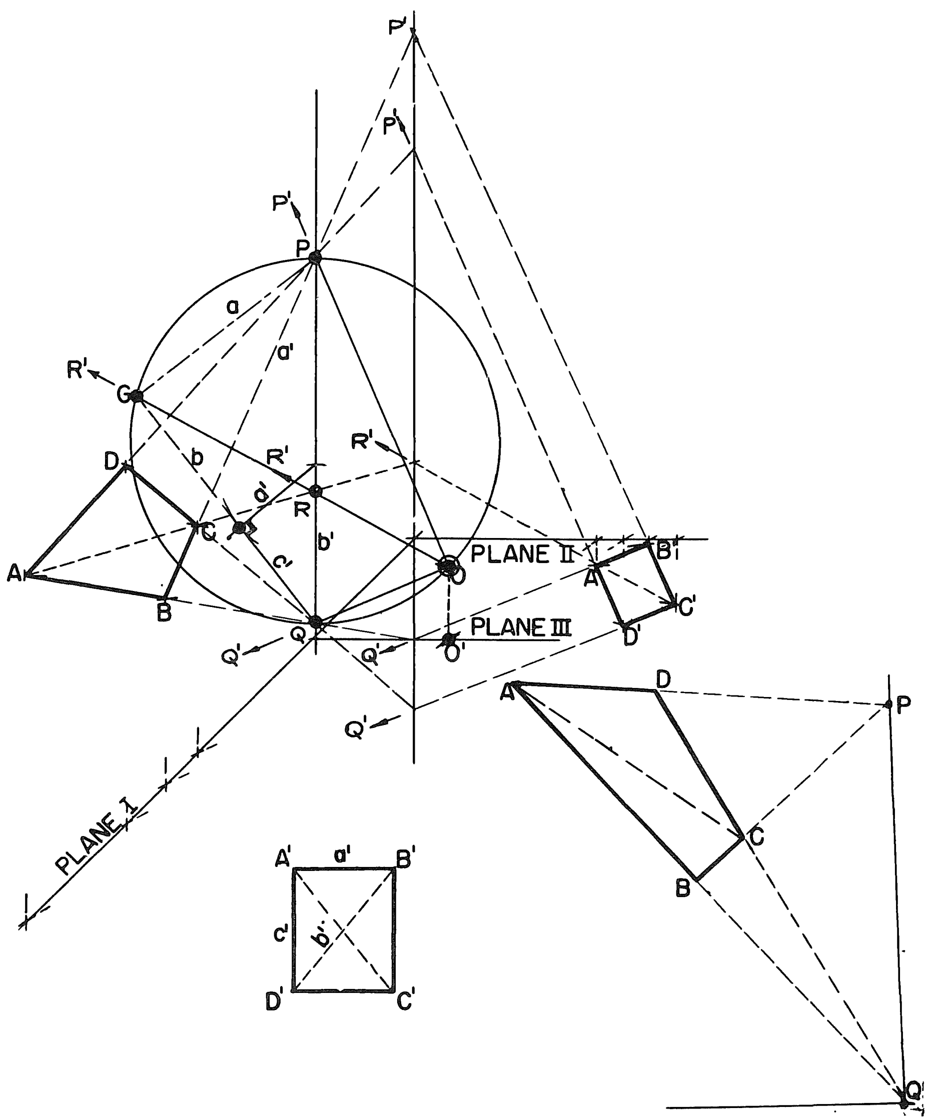


Figure 13–10.



Figures 13–11 & 13–12.



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PLANE I
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C'

Figure 13–13.

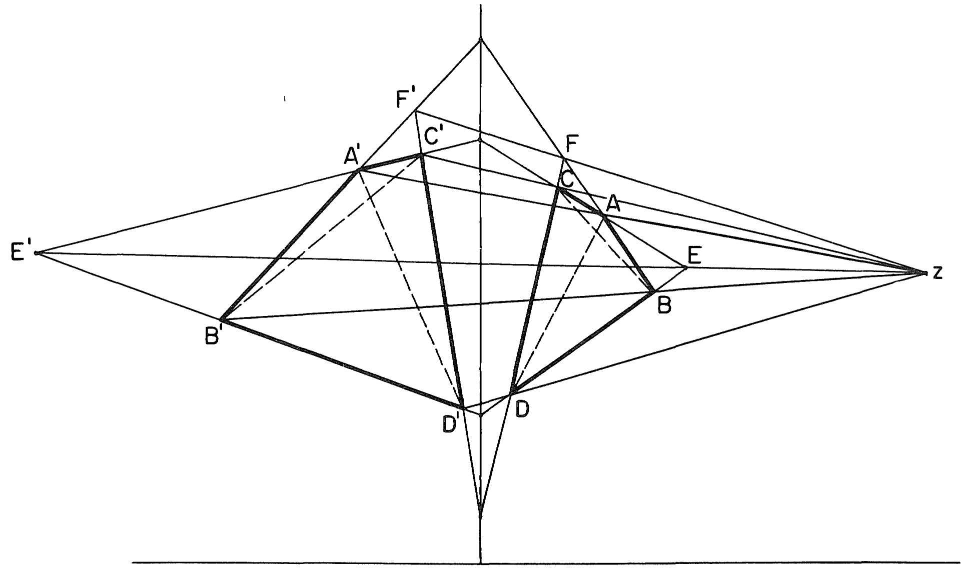


Figure 13–14a.

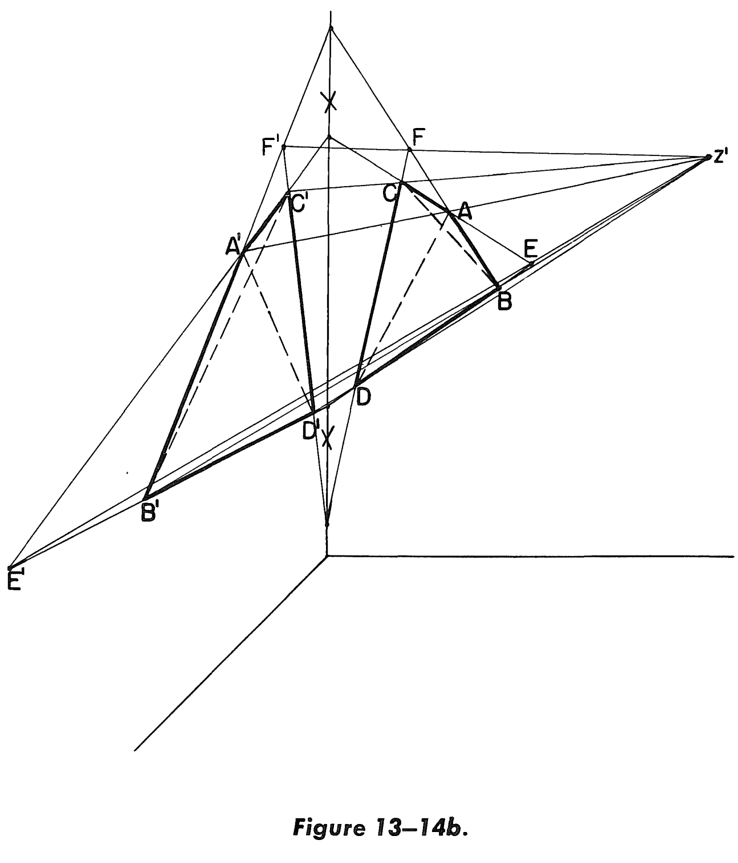


Figure 13–14b.

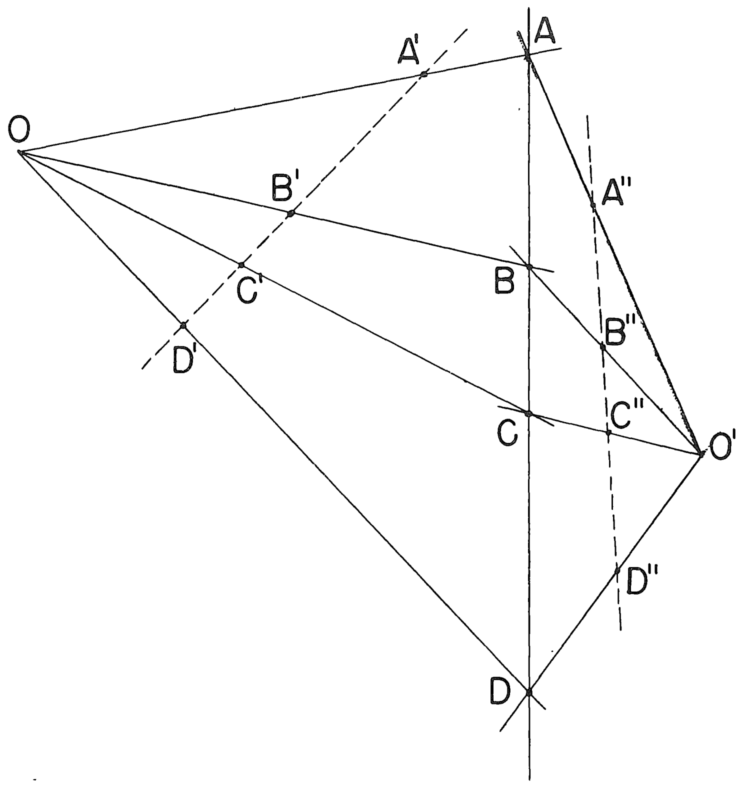


Figure 13–15.

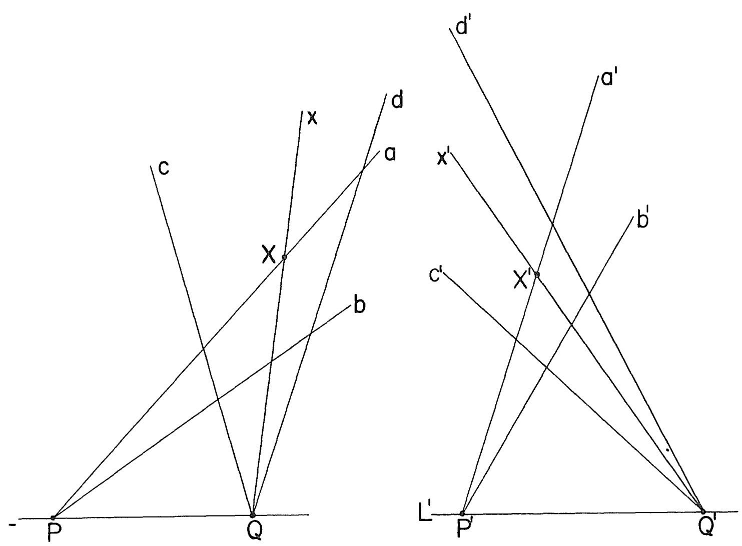


Figure 13–16.

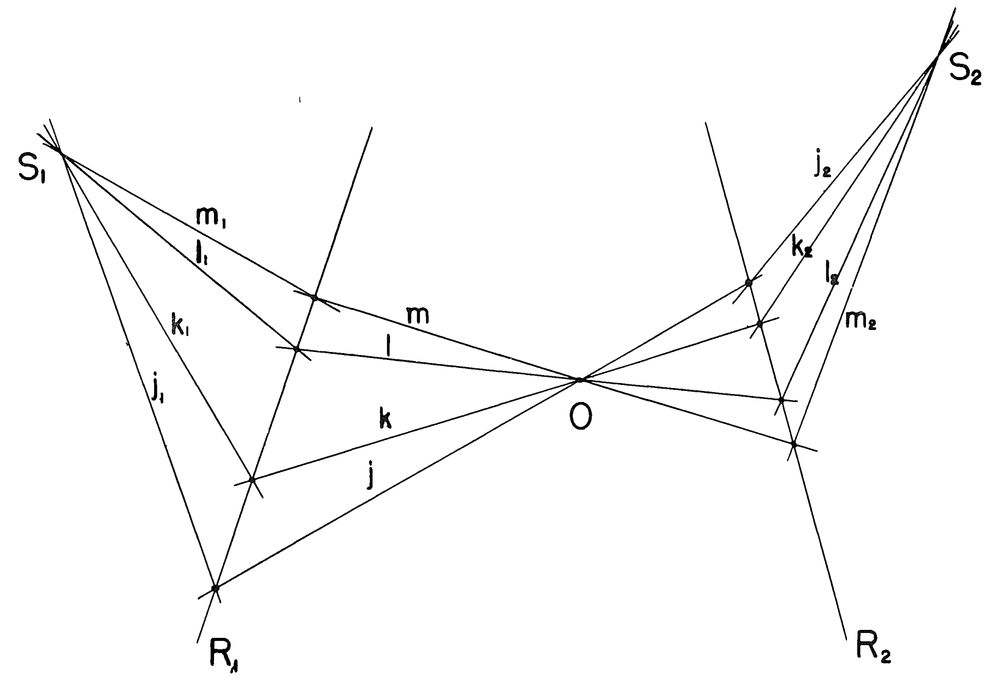


Figure 13–17.

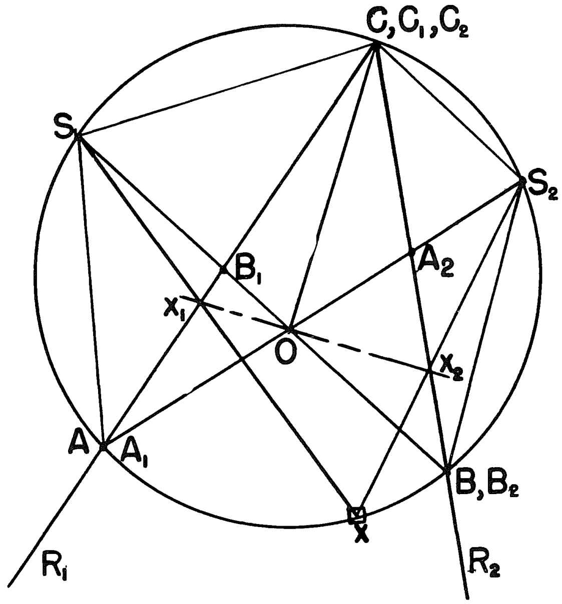


Figure 13–18.

CHAPTER 14 (APPENDIX D)

NOMOGRAPHIC-ELECTRONIC COMPUTATION

It is possible to bring together graphical work and modern electronic techniques so as to present time-honored graphical procedures within a modern technical framework. We describe a graphical computing technique which is fast, efficient and not subject to the conventional limitations on accuracy of the old-fashioned graphical method. To do this, we use a definition of "graphics" which states that wherever numerical *value* is correlated with space *position*, a graphical process has occurred. For this computing technique, it has been figured that before long for many relationships 10,000 solutions per second could be obtained with three figure accuracy – a goal far below a theoretical limit of 50,000 *solutions* per second. Such rates would be useful, for instance, in solving systems of partial differential equations. For oppositely moving satellites travelling about 100 miles above the earth's surface, a ten thousandth of a second can represent a relative displacement of a few feet so as to be useful in proximity problems. Present investigating rates are at about 500 *solutions* per second. Both graphical and nomographic techniques are used in conjunction with electronics.

The notion at the heart of the NOEL (NOmographic-ELectronic) computer is a simple one, first, that an equation or relationship hard to solve in its original variables (let us call them the *blue* variables) can be changed into one (in what we may call the *red* variables) which is easy to solve. When the red answer is known, the blue one can be found by an inverse correlation. A second and vital part of the technique is that there are *many* equations in blue variables which can *all* become changed under this technique into one and the same equation in red variables (namely, a linear equation).

It is the simplifying change from the blue variables to the red variables and how to set up this correlation graphically that first concerns us. Imagine a scale in U, a curved line on the page graduated and calibrated in values of U, so that for each value of U it is easy to read off the X and Y coordinates U_x and U_y for each value and conversely easy to ascertain the value of U that corresponds to every U_x, U_y defining a point on the scale. This is clearly a graphical device within the definition we set up earlier. Now the potentialities of the alignment diagram suddenly become clear. An equation or relationship F (U,V,W) = 0 in the blue values U, V, and W is placed in such a form, Figure 14—1a, b, c, as to define a pair of coordinates (red values) U_x, U_y for U (similarly for V and W) so that the original blue equation F (U,V,W) = 0 is replaced by a red *linear* equation in U_x,U_y, V_x,V_y, W_x,W_y. If U and V are specified in blue, the corresponding red pairs U_x,U_y, V_x,V_y are now known. We could now run through a table of red W_x,W_y pairs and find one such that it satisfied with the former a linear relationship.

$$\frac{U_y - V_y}{U_x - V_x} = \frac{W_y - V_y}{W_x - V_x} \qquad 1)$$

The blue W-value corresponding inversely to that red W-*pair* just found (Figure 14—1d) is the W-value corresponding to the given U, V values in the equation F (U,V,W) = 0. For our pains, we have a system where the red equation has been easy to solve and it has led back to the blue answer value.

In the old-fashioned nomogram, the correlation between a blue U *value* and a red U-*position pair* U_x,U_y is shown by drawing a blue *scale* in U defined by red coordinates, Figure 14—1d. We now show this same correlation correspondence between blue value and red position pair – but in a different way.

We set up the same correlation, Figure 14—1e, of the value of U with U_x,U_y, V with V_x,V_y, W with W_xW_y by representing each of the nine quantities by a column of *countable bits* – a vertical column of bits for each of the quantities we have mentioned. We physically place the vertical columns U, U_x, U_y, V, V_x, V_y together with a clock track on a memory film so that an optical image of the columns of bits is passed across six photoelectric cells as the memory is moved. At any instant the *cumulative count* of the bits in a given column is taken to be the value of that variable in the respective column at that instant. If we know the value of the independent variable U and the independent variable V, we can cause counting to stop when the prearranged known value is this cumulative count. At the same time we then know U_x and U_y values through their simultaneous cumulative counts. If we now, in a "second pass", treat three W columns W, W_y and W_x (plus a clock track) to the same counting process, trying out cumulative counts of W_x and W_y for a linear relationship 1) with the known U_x, U_y, V_x, V_y counts, there will come an instant when they will satisfy this linear relation. If we arrange to shut off the W counting at that instant, we will then have trapped in the cumulative W-count the value of W satisfying $F(U,V,W) = 0$ for the specified values of U and V introduced at the outset.

The functional behavior $F(U,V,W) = 0$ is thus brought about solely by the use of funny little black marks called countable bits together with the relative placement of these in their respective columns. This is the purest form of graphical process, fortunately amenable to speedy electronic counting devices and achieved today at the rate of 1 million bits per second using comparatively inexpensive equipment.

EXAMPLE:

A simple case will bring together most of these ideas, which later can be extended. In Fig. 14–2, the equation $F(U,V,W) = 0$ appears at the lower left. It is assumed that an alignment diagram of the form shown there can be drawn to represent this equation. The alignment diagram is shown embedded in an XY axis system and the equation of the straight line of colineation appears below the diagram. The blue variables are U, V, W and the red variables are X_1, Y_1 $(X_1 = 0)$; X_2, Y_2 $(X_2 = G)$; and X_3, Y_3. In the lower center of the diagram there is a typical example of a piece of film with "first pass" columns V: $V_x = G$, $V_y = Y_2$ and V; U: $U_x = 0$, $U_y = Y_1$ and U; and "second pass" columns W: $W_x = X_3$ and $W_y = Y_3$ and W. The constancy of V_x and U_x ends the need of their being present in this representation.

We now carry out the above mentioned scheme in the following way. Values of V come from the tape (upper left), are cumulatively stored in the binary pulse counter 1 until their negative sum equals the present input positive sum in V, leaving a zero count in this counter and causing two things to happen: (1) The S_2 gate opens, interrupting the flow of the Y_2 count to the binary pulse counter 2 in the upper right. 2 The switch 0 is activated causing the assimilated Y_2 count in the pulse counter 1 to be deposited in the Accumulator 5 . Corresponding events in the lower left counter 3 have happened to input U and Y_1, causing a $-Y_1$ count to be terminated by switch S_1 when the U cumulative count reached 0. This count was taken from the reversible counter 4 and placed in the Accumulator 5

As the film moves upward, by the time the reading gets to the dotted line separating pass 1 from 2, the final result in the Accumulator 5 reads $Y_2 - Y_1$ and the reversible counter 4 reads $-Y_1$. Pass 2 is now ready to begin. On the first X_3 pulse (and not again) the upper right binary pulse counter 2 is cleared and the Accumulator 5 is placed in that counter leaving there the count $Y_2 - Y_1$. On every subsequent X_3 pulse acting through gate 0 the value of the Binary Pulse Counter $Y_2 - Y_1$ is dumped into the Accumulator 5 creating the product $X_3(Y_2 - Y_1)$. Each Y_3 pulse enters the reversible counter 4 after a short delay establishing there the sum $Y_3 - Y_1$. At the same time the lower right binary pulse counter 7 is collecting the cumulative pulse value of the W count. Meanwhile there is a steady comparison going on through the Coincidence Circuit 6 of the values existing (1) in the Accumulator 5 and (2) in the Reversible Counter. 4 Assuming for simplicity that $G = 1$, the red equation shown in Fig. 14–2 will be satisfied when the Coincidence Circuit 6 detects identity in the magnitude of the Accumulator 5 and the Reversible Counter 4 . When this occurs, the Accumulator 4 opens the switch S_3 and the answer value, the cumulative blue W-value, appears trapped in the lower right binary pulse counter 7

It will be seen right away that many problems arise in implementing the above scheme. We have referred glibly to the transformation of film memory of countable bits into a pulse pattern receivable by the arithmetic element. This requires a form of scanning or *reading*, introducing a wide variety of mechanical and electronic problems.

For this pattern to be "read", the bit pattern has to be transformed into a pattern of pulses as described above on which the arithmetic element can work. For this purpose various types of reading devices can easily be imagined and some have been developed. In Figs. 14–3-6 it will be easy to see that in some cases the memory is fixed and a system of mechanical-optical scanning is employed. In others, optical elements remain fixed and the memory moves so as to bring raster after raster of bits into contact with the optical elements which, in turn, bring their images to the photoelectric cells. It is even possible to conceive of reading devices in which there are no mechanical optical moving parts, for instance, that a cathode-ray-tube moving electron beam could read this memory pattern and register it on photoelectric tubes to yield the desired pulse pattern used by the arithmetic element.

Still other helpful simplifications can be broadly envisaged. Let us imagine the bit pattern replaced by an identical pattern of tiny charged ferrite cells, one for every bit. Imagine that each column of cells can be unloaded in sequence to a "delay" column, which changes each electric pulse to a mechanical disturbance passing down the column, then reverting in turn to an electric pulse at the end of the column. The latter is conveyed to the top of the column by a closed circuit, so that the pattern runs repeatedly through the cycle – a device in recent extensive use. If we conceive that all the columns can be made to do this at a uniform rate, we have the picture of the memory pattern moving repeatedly down the delay lines, instead of around on a film. Counting and arithmetic can be done as before, and here we have achieved the desired end of "no moving parts" for reading the bit memory.

Limiting ourselves to photographic memory, before such pulses can be read they must have been *written* into the memory, that is, they must have been inscribed by the thousands and with complete accuracy. The solution developed at the Engineering Projects Laboratory at M.I.T. required that bits should never appear in the memory except upon certain evenly spaced rulings called rasters. In a given column of bits it then becomes the question whether or not, on the cumulative counting of these bits, one should or should not be inserted in order that the cumulative count of that variable should be in sufficiently close step with the cumulative counts of the other variables in their respective columns. There can also be no escape from the conclusion that the value or intrinsic worth of a bit may well have to change from time to time as progress occurs up a column of bits representing a non-uniform function. The signalling required to do this is fairly extensive but does not pose any problems beyond those of basic electronic circuitry.

A law, or algorithm, for determining whether or not a bit should be written in for a given raster for proper representation of the function is now assumed to have been worked out, leaving the question of how a bit is "written" once the command is given to draw it. One way is to use the oscilloscope output of the IBM 7094 which computed the yes-or-no decision for the bit in the first place. A subroutine can be prepared and called into play for each bit to be written. This subroutine scintillates a dot pattern, Figure 14–7, over the desired bit area.

A second type of writing technique employs an enlarged raster pattern of bits, (that is, a line of bits perpendicular to the column direction). Each bit "window" is the end of a chamber containing a well diffused strobotac flash tube of one microsecond duration. These are then "fired" from the same IBM 7094 tape output that would have been used to "write" the bits by oscilloscope scintillation. The large raster is then photographed into a small raster in a radial position on a steadily rotating disc. Figure 6 shows an enlarged pattern of these at 400 bits per inch on a three inch radius.

Changing the memory is all that is required to adapt the computer to any other equation for which a memory has been prepared – a matter of microseconds. The same arithmetic element will be used in all cases, expressing each time the linear relationship in terms of the "red" variables as previously described. The hoped-for solution-times previously referred to indicate that fast or slow cheap control can be had by this device. One imagines that all of the processes in a given plant could perhaps be controlled by a single central unit of this nature.

Use in connection with differential equations has been mentioned and can perhaps best be illustrated by showing first how nomography can be used in the normal way to help solve such a problem:

We employ a technique presented by Professors Morita and Simokawa, Kanazawa University, Japan which utilizes a well-known Runge-Kutta series development of fourth order accuracy in such a way that nomographic techniques are effective. Briefly, a solution is developed for an ordinary differential equation

$$g(x,y,y') = 0 \qquad (14\text{–}1)$$

from point (x_n, y_n) to point (x_{n+1}, y_{n+1}), in which

$$x_{n+1} = x_n + h \qquad (14\text{–}2)$$

$$y_{n+1} = y_n + (1/6)\,(k_1 + 2k_2 + 2k_3 + k_4) \qquad (14\text{–}3)$$

where

$$k_1 = h \cdot f(x_n, y_n)$$

$$k_2 = h \cdot f(x_n + (1/2)h, y_n + (1/2)\,k_1)$$

$$k_3 = h \cdot f(x_n + (1/2)h, y_n + (1/2)\,k_2)$$

$$k_4 = h \cdot f(x_n + h, y_n + k_3) \qquad (14\text{–}4)$$

and where the original relationship $g(x,y,y')$ is also soluble in the form

$$y' = f(x,y).$$

The latter restriction upon form will be found not to be binding in the nomographic case we use. Equation (14–1) must be able to be put into canonical nomographic form or else a series of such forms. As a matter of practice, differential equations tend to have a *relatively* simple aspect, which, while they may be very hard to wring formal solutions from, nevertheless frequently indeed can be put into the required nomographic form.

EXAMPLE: Figure 14–8

(14–5)

A nomographic form of this equation is:

$$g(x,y,y') = \begin{vmatrix} x/50 & y/50 & e^{-y'}/10 \\ 0 & G & G\cdot e^{y'}/10 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

(14–6)

where G is the width of the chart and yielding parametric equations

$X = x/50$:	$Y = 0$	for the x-scale
$X = y/50$;	$Y = G$	for the y-scale
$X = e^{-y'}/10$;	$Y = G\cdot e^{y'}/10$	for the y′-scale

(14–7)

Figure 14–8 shows the old-fashioned use of the chart for the following values, with answers, as required by the formulas (14–4), in extending a solution from $x = 1.5$, $y = 2.0$ by steps of $h = 0.5$.

$x = 1.5$;	$y = 2.0$; $y' = f = 2k_1 = 1.11$;	$k_1 = 0.55$
$x = 1.5 + 0.25 = 1.75$;	$y = 2. + 0.2775 = 2.278$	$2k_2 = 0.98$; $k_2 = 0.49$
$x = 1.75$;	$y = 2.245$;	$k_3 = 0.490$
$x = 2.0$;	$y = 2.490$;	$k_4 = 0.430$

$$y_{n+1} = y_n + 1/16\ (0.55 + 0.98 + 0.98 + 0.430) = 2.49$$

The nomogram of Figure 14–8 is, of course, never reproduced in the form shown but *is presented only in permanent memory form as columns of countable bits.*

EXAMPLE 2.

A second ordinary differential equation $y' + y\cdot e^x - e^x = 0$ is shown in Figure 14–9, together with its canonical form, and the nomogram expressing this equation. A solution also appears, practically identical with a classical solution, though separated in the drawing.

EXAMPLE 3.

Figures 14–10a and 14–10b. A non-linear differential equation $y' + xy^2 = x$ with its nomogram and solution are shown. The nomographic, Runge-Kutta solution cannot be distinguished from the classical solution even though an increment as large as $x = 0.1$ was used. Under an Euler method, modified Euler method, or Runge-Kutta techniques, the nomographic solution differs from the classical by about 10^{-3}.

It is worth mentioning that many problems of the required circuitry can be simplified by using the projective geometry or central projection referred to earlier in this text, since this gives a nomogram great flexibility in shape and hence flexibility in the nature and needs of the various columns of countable bits.

The examples have shown non-linear ordinary differential equations of the first degree and order, but those of higher degree or order can be represented by systems of equations whose members are all of first order.

The case of partial differential equations, as well as systems of them, is much more difficult but shows promise. Here is the point where the teaming of such a fast, special-purpose computer with a fast general-purpose computer of relatively small memory could be especially helpful. Many forms of attack on this problem along nomographic lines are currently under investigation because of its importance in modern technology.

In resume, we view the nomographic technique as an organization of a computation conducive to a correlation of the old variables with new ones. The new variables satisfy a *linear* relation. This can be done for a large class of equations, making it worthwhile to put the *correlation* in a countable bit form and have the linear relation worked out by circuitry. Computation and writing of the thousands of bits in memory can be done *very* quickly by IBM 7094 with oscilloscope output, and other methods. Reading can be done in a variety of optical-mechanical and other ways. Speed and cheapness are the reward.

Figure 14—1a.

The equation has been placed in nomographic determinant form.

$$\frac{U^2 + V^2}{U + V} = W; \text{ can be written}$$

$$F(U, V, W) = W - \frac{U^2 + V^2}{U + V} = \begin{vmatrix} \frac{1}{U} & U & 1 \\ \frac{1}{V} & V & 1 \\ 0 & W & 1 \end{vmatrix} = 0$$

Blue Variables

Figure 14—1b.

The nomographic interpretation of the determinant form on the left is the diagram on the right.

$$\begin{vmatrix} U_x & U_y & 1 \\ V_x & V_y & 1 \\ W_x & W_y & 1 \end{vmatrix} = 0$$

Red Variables

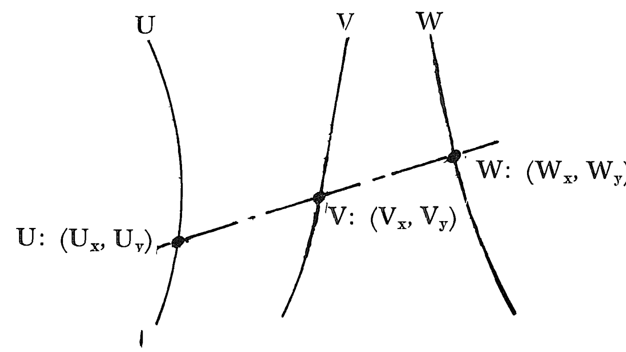


Figure 14—1c.

Here we correlate the two determinant forms.

1. $U_x = \frac{1}{U}; \quad U_y = U$

$V_x = -\frac{1}{V}; \quad V_y = V;$

$W_x = 0; \quad W_y = W$

Red Blue; Red Blue

obeying the relations

2. $$\frac{U_y - V_y}{U_x - V_x} = \frac{W_y - V_y}{W_x - V_x}$$

Red Variables

or

$$\frac{U - V}{\frac{1}{U} - (\frac{-1}{V})} = \frac{W - V}{0 - (\frac{1}{V})}$$

or

$$\frac{U^2 + V^2}{U + V} = W$$

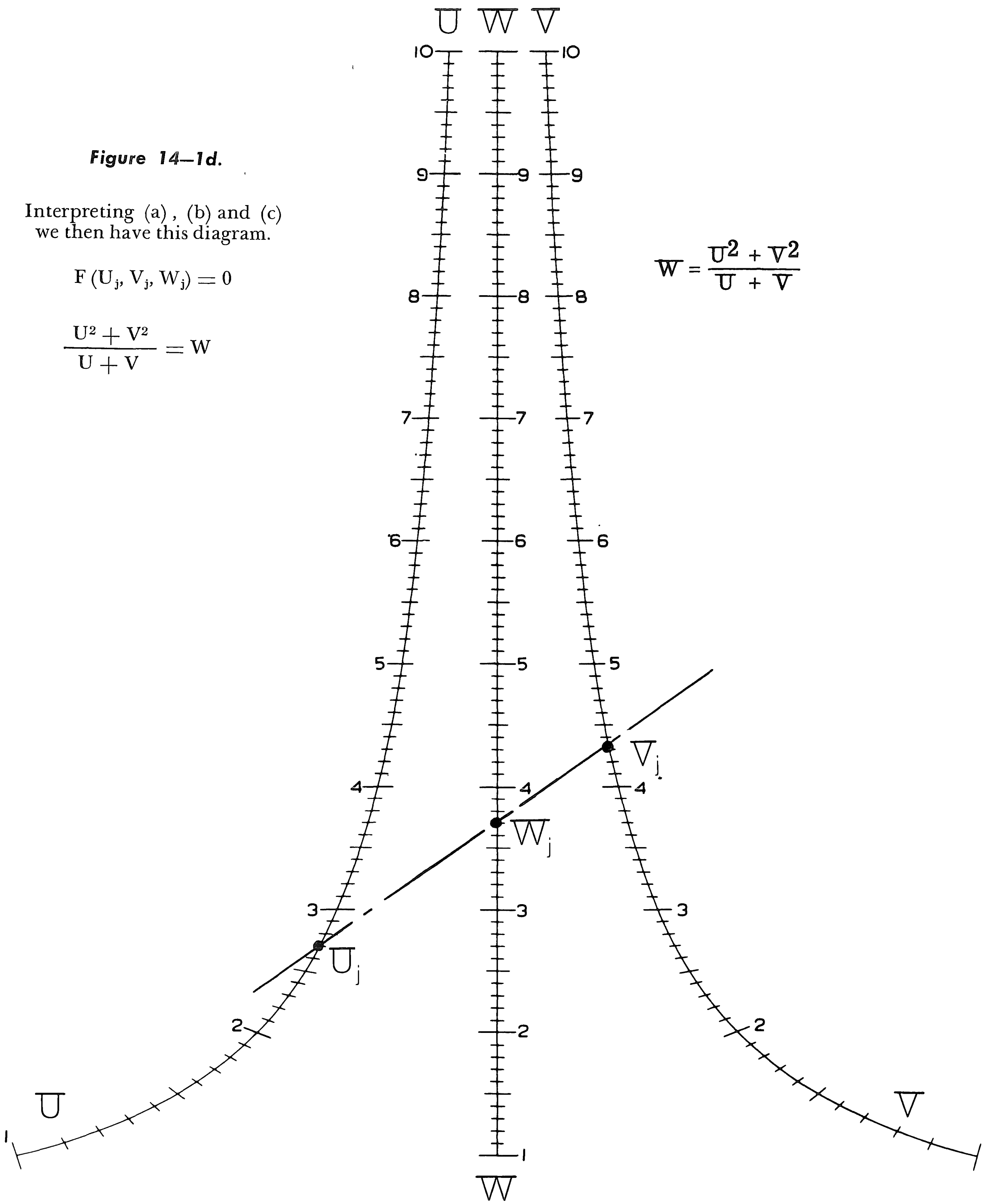
Blue Variables

Figure 14–1d.

Interpreting (a), (b) and (c) we then have this diagram.

$F(U_j, V_j, W_j) = 0$

$$\frac{U^2 + V^2}{U + V} = W$$



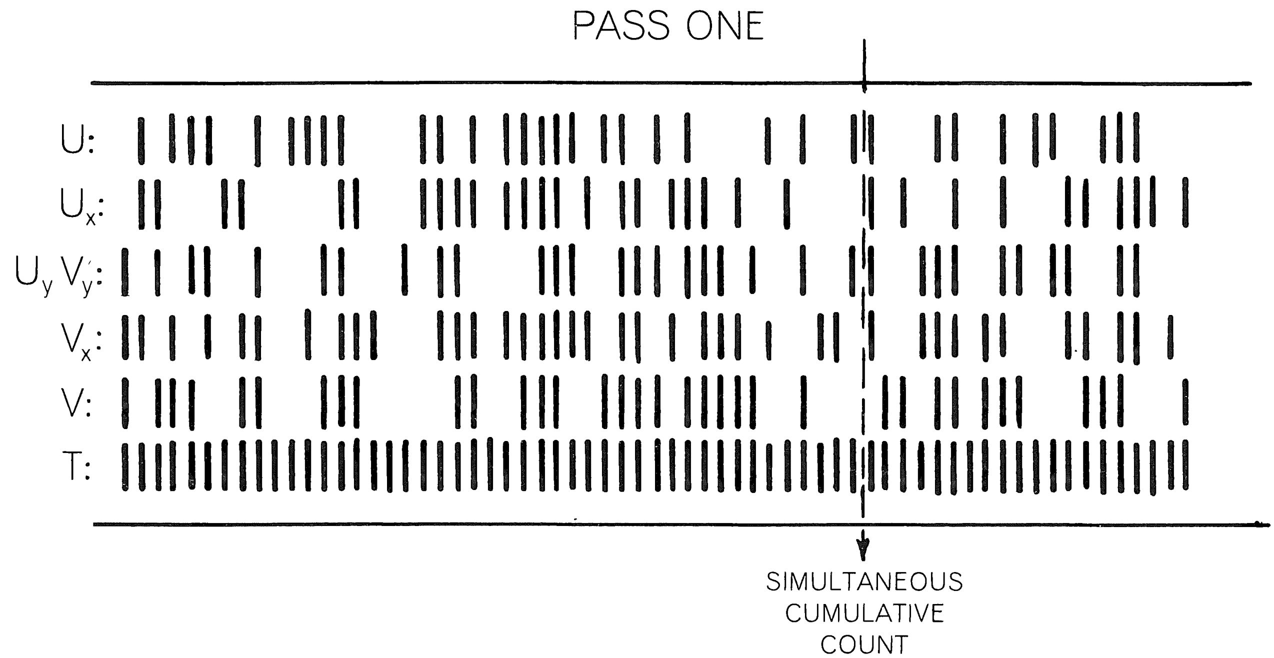


Figure 14—1e.

Here simultaneous cumulative counts of bits in the columns express the relations (c) *1.* and an electronic circuit effectuates (c) *2.*

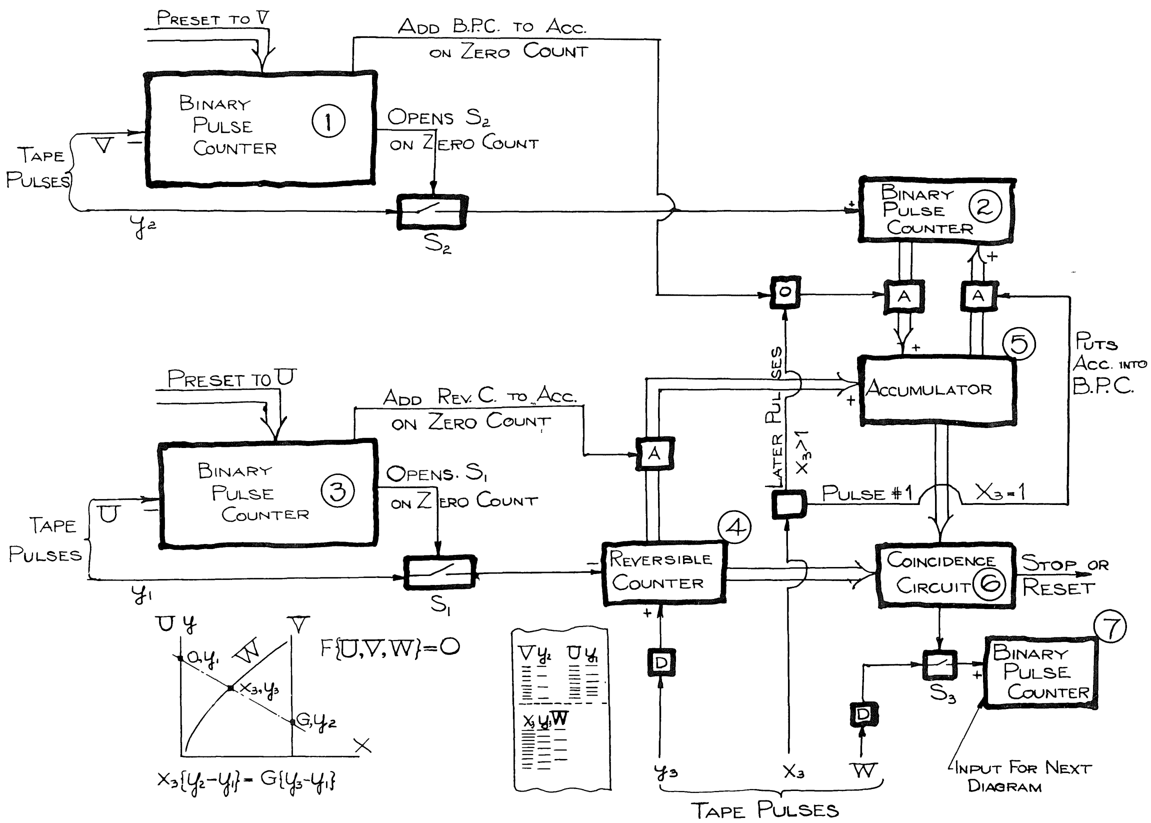


Figure 14–2a.

A simple circuit to carry thru computation by countable bit.

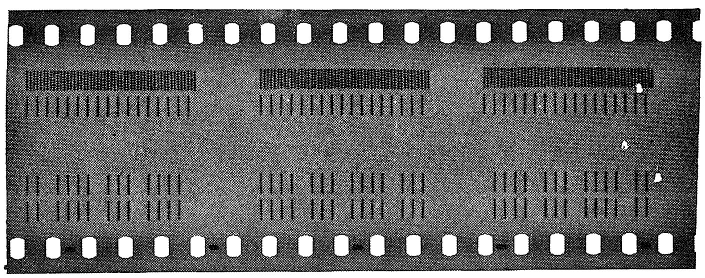


Figure 14—2b.

A portion of a nomogram prepared for nomographic-electronic (NOEL) countable-bit computation.

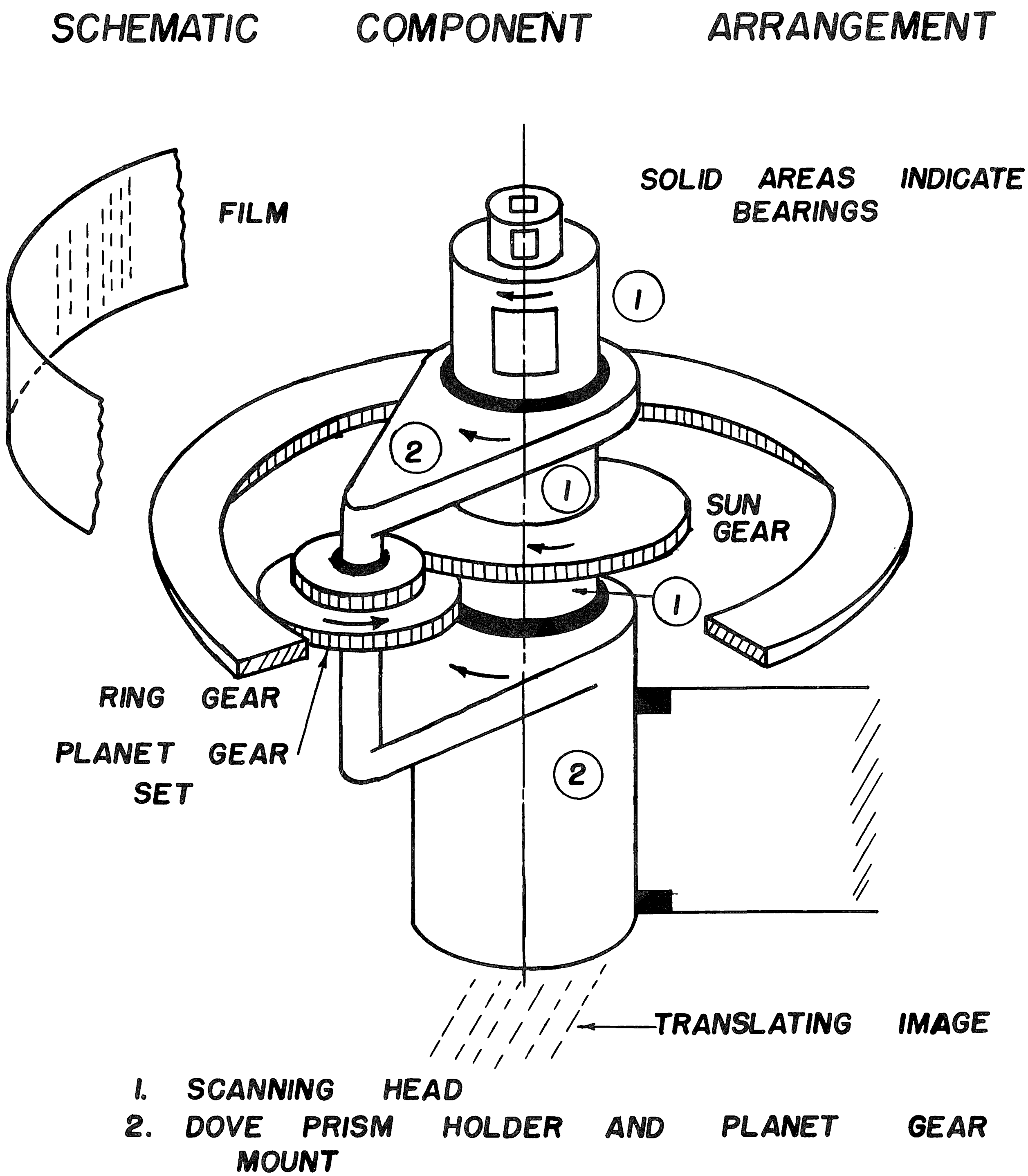
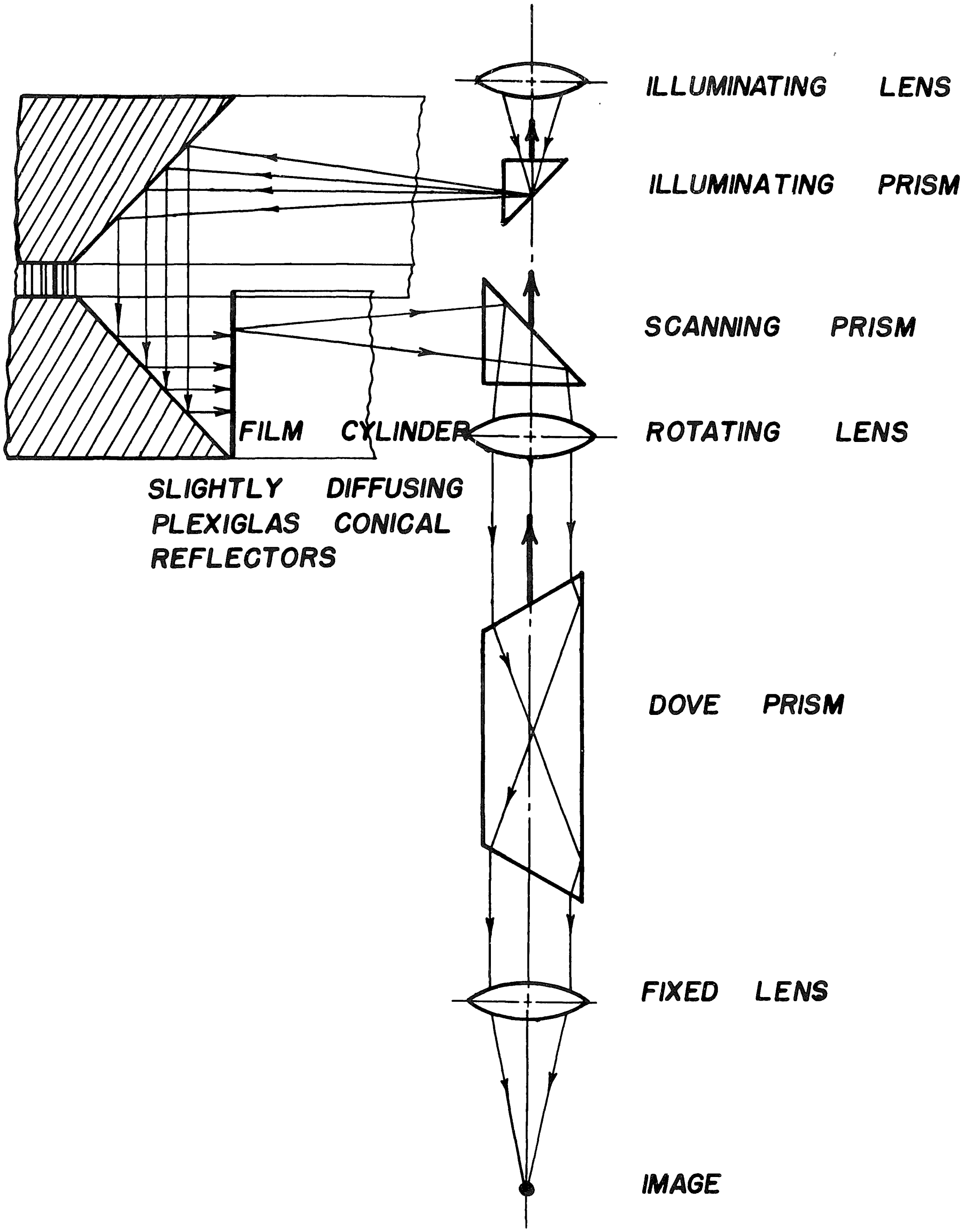


Figure 14—3a.

This shows a fixed-memory, moving-optics type of bit-memory reader. Details of the light path appear in Figure 14—3b.



SCHEMATIC OF OPTICS

Figure 14—3b.

Rough schematics of optical path of light in the fixed-memory, moving-optics reader shown in Figure 14—3a.

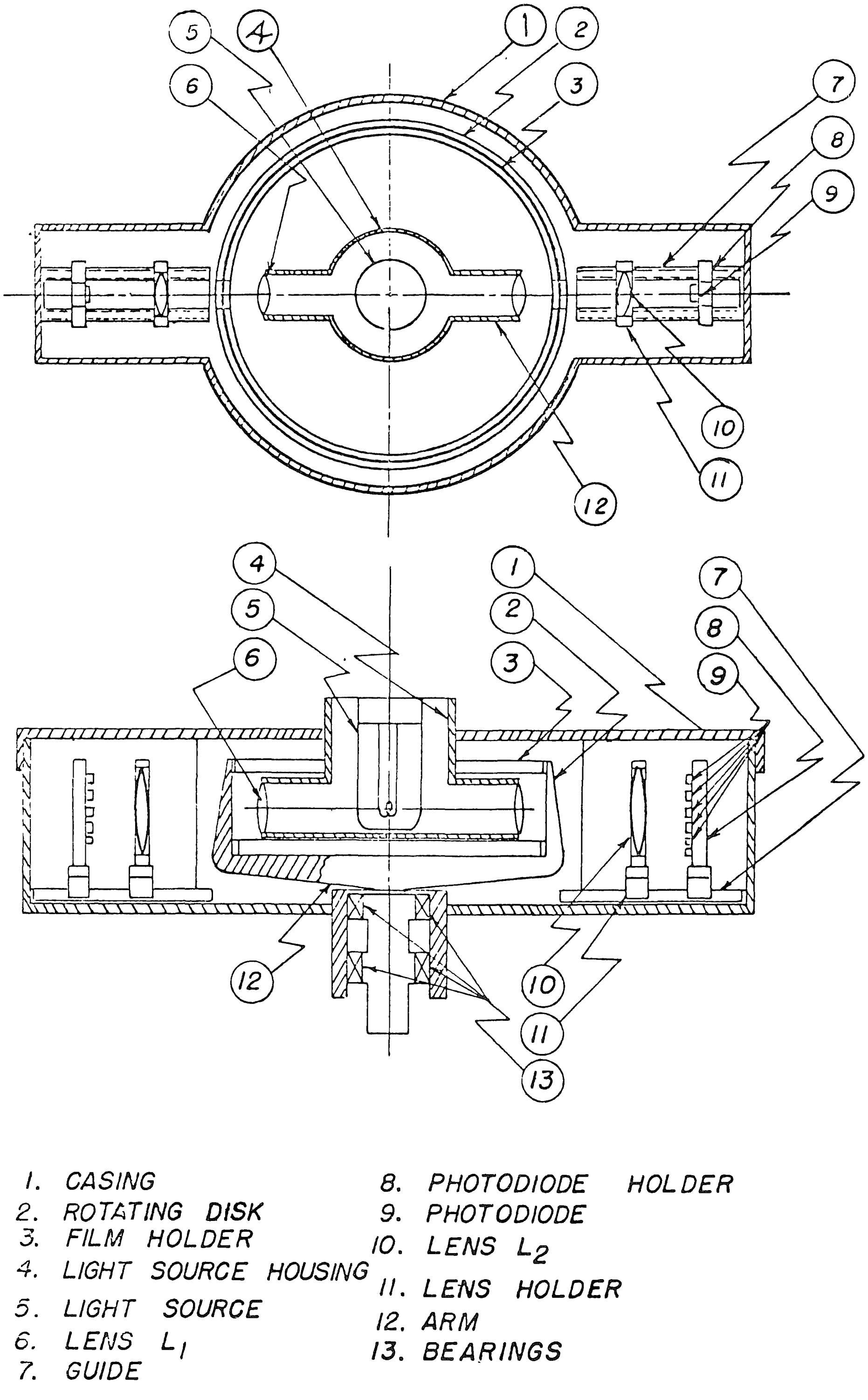


Figure 14—4.

Film is held in film-holder *3* in this fixed-optics, moving-memory type of reader.

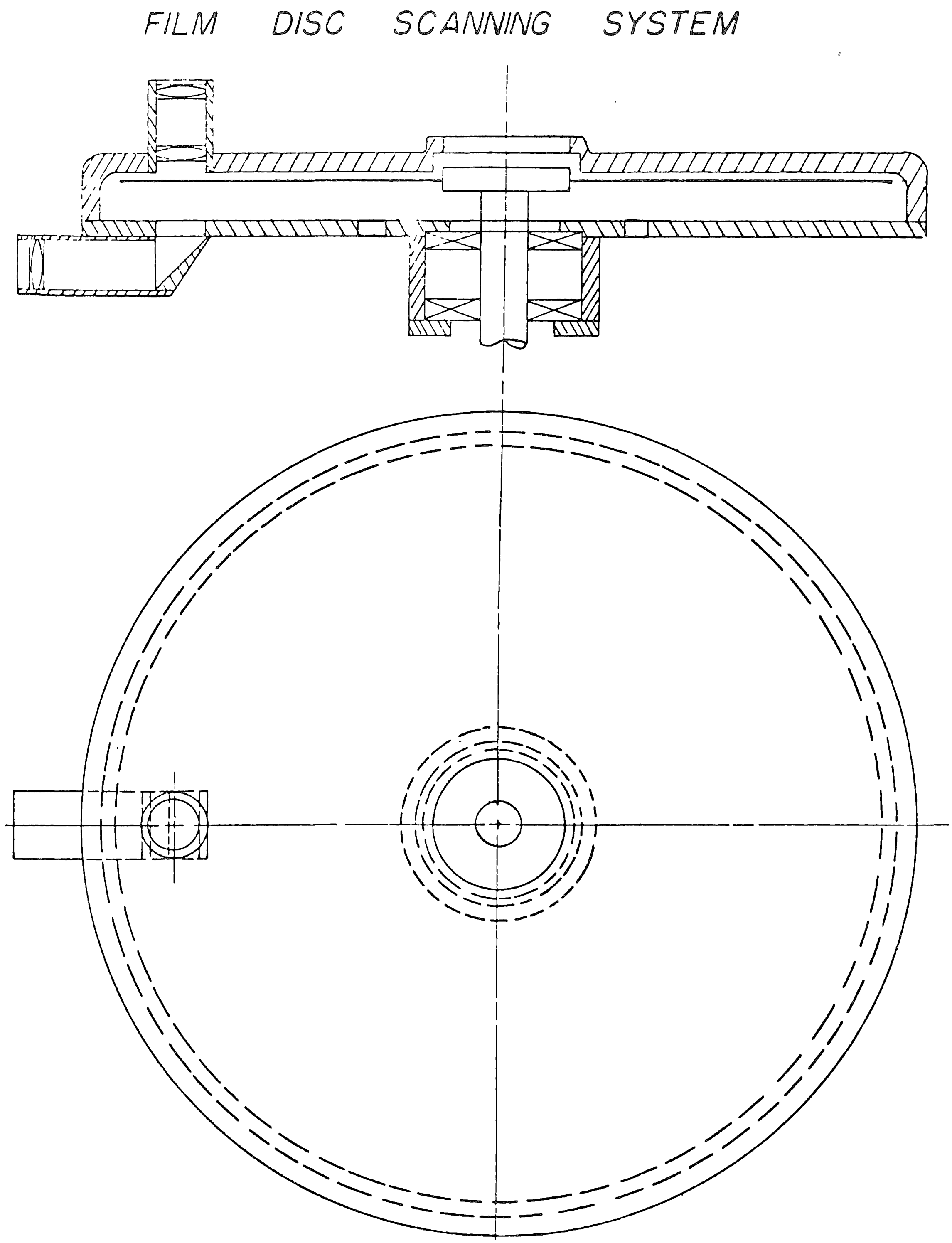


Figure 14—5.

Schematic of the elements of a disc-memory reading system. The optics are fixed and the disc turns, carrying the bit-pattern of the memory.

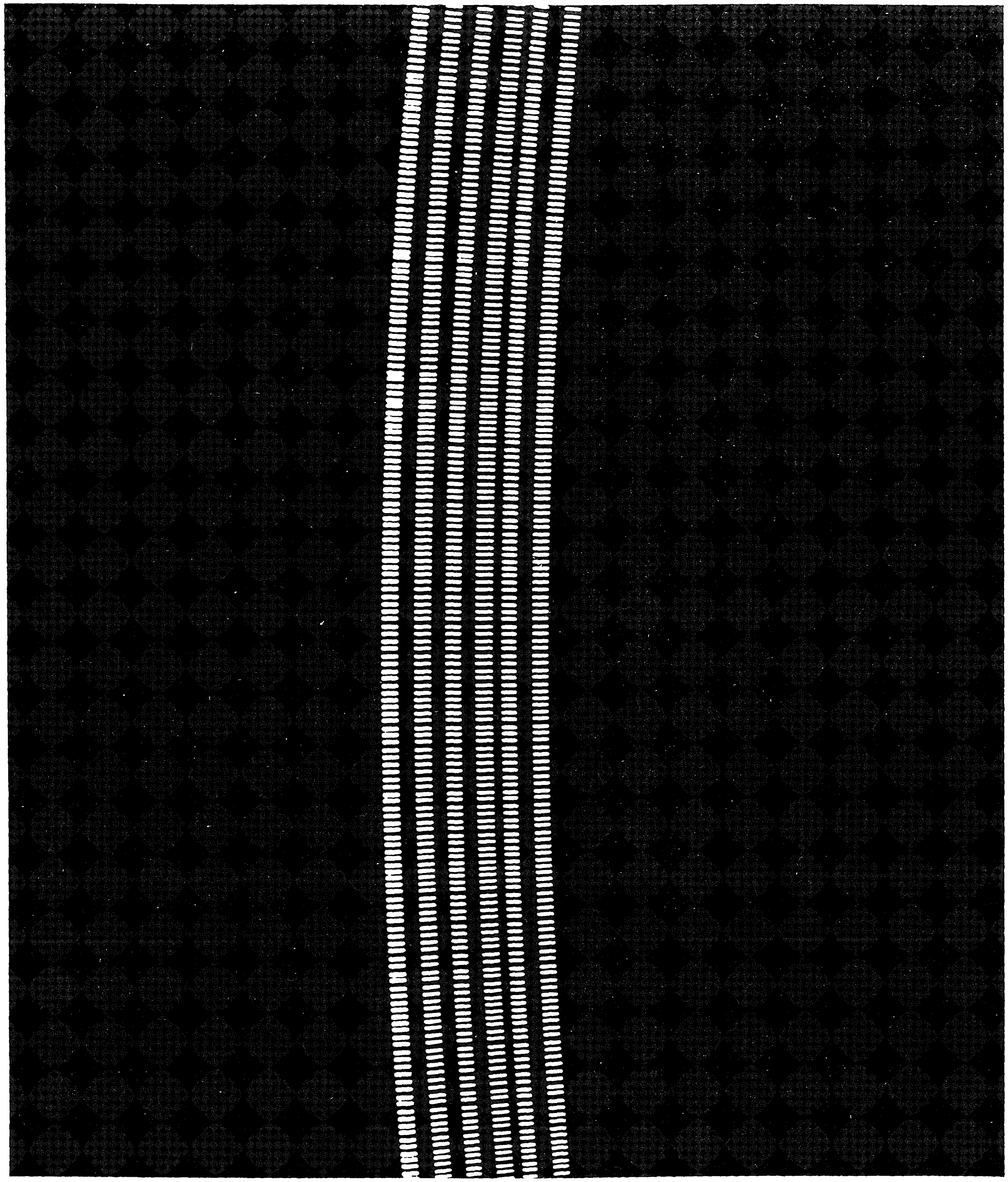


Figure 14–6.

Bit pattern generated on a rotating disc, 400 bit/inch density, on a 3″ radius, enlarged 16 times.

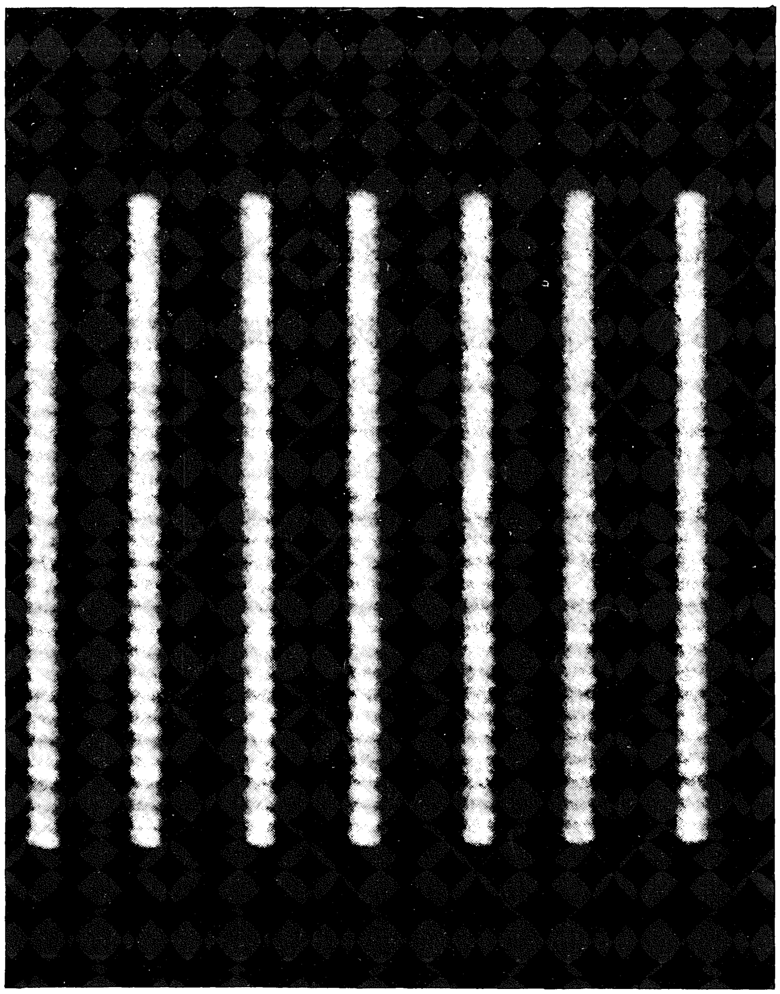


Figure 14—7.

Bit-patterns generated by scintillation techniques as a CRT output of the 7094. A subroutine of the scintillations establishes the bit whenever the bit-or-no-bit 7094 program calls for one. X-20.

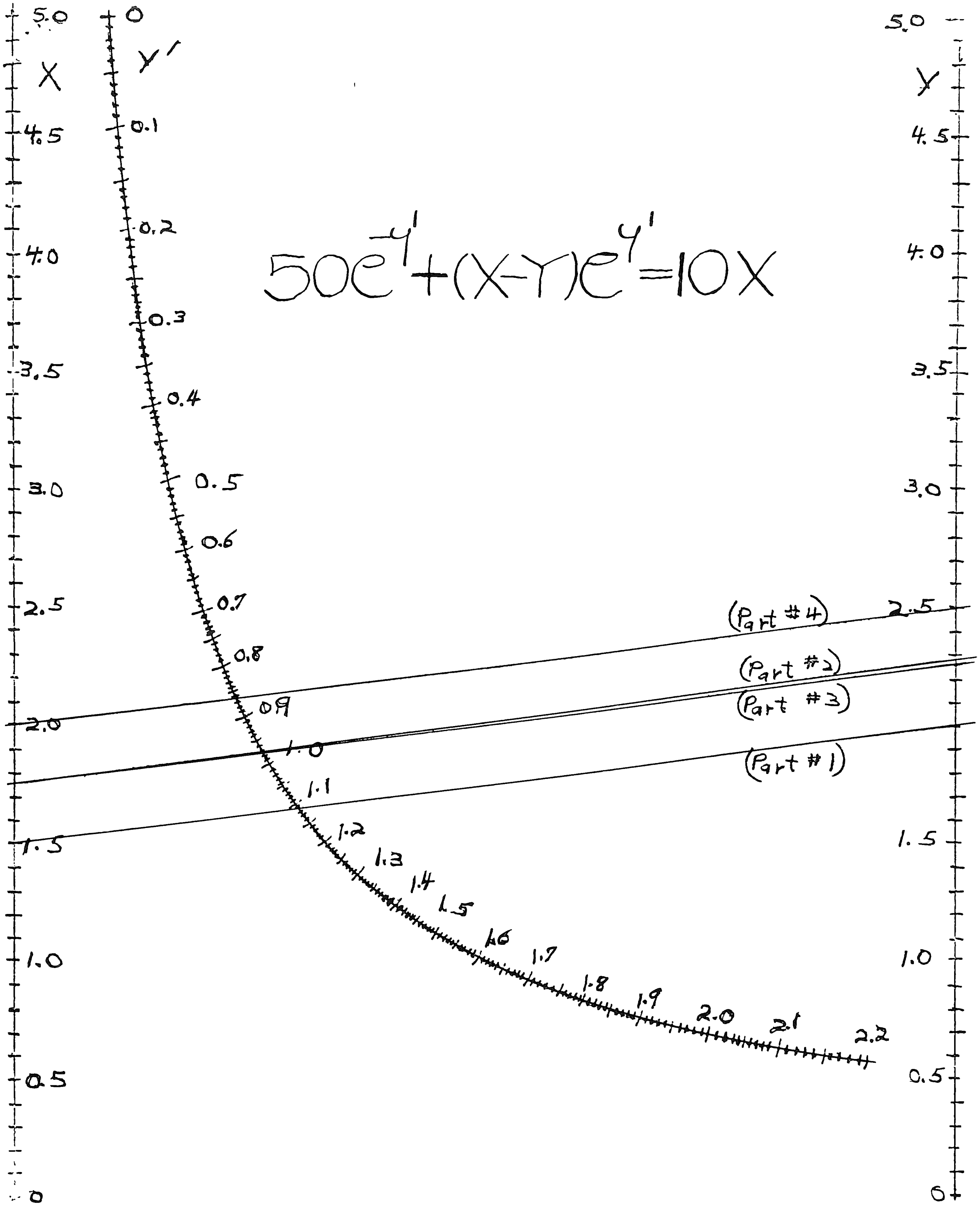


Figure 14—8.

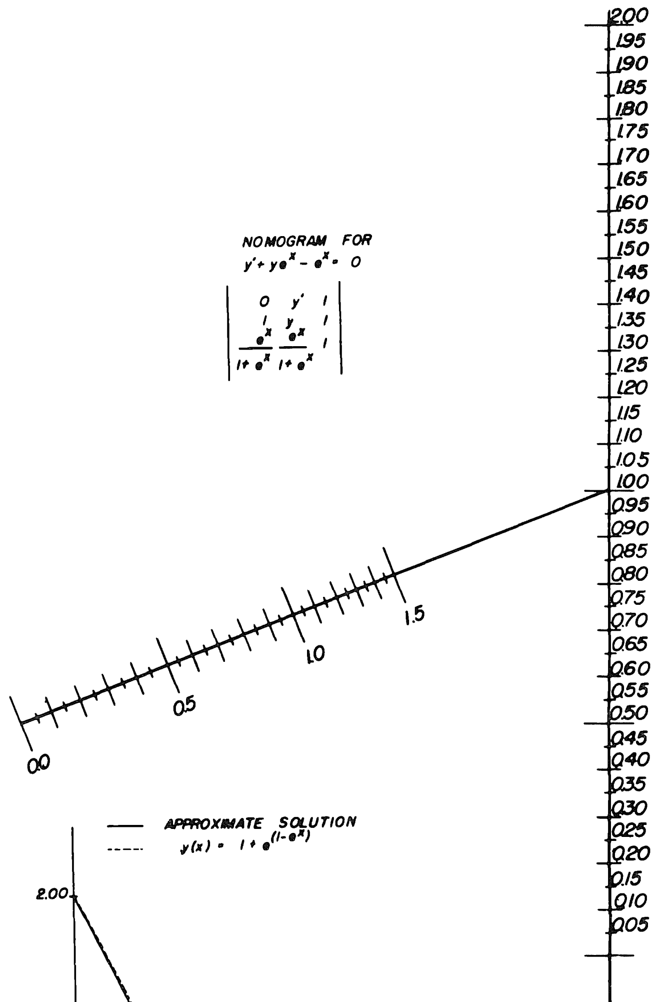
Home-made nomogram for the solution of the differential equation shown, using Runge-Kutta developments.

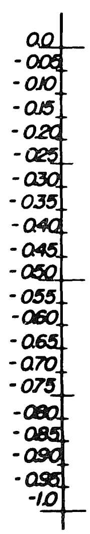
DIFFERENTIAL EQUATION: $y' + ye^x - e^x = 0$

INITIAL CONDITIONS: $x = 0$, $y = 2$

ANALYTICAL SOLUTION: $y = 1 + e^{(1-e^x)}$

X	Y (NOMO.)	Y (ANALYT.)	ERROR
0.00	2.0000	2.0000	0.0000
0.10	1.9001	1.9001	0.0000
0.20	1.8009	1.8013	0.0004
0.30	1.7037	1.7047	0.0010
0.40	1.6109	1.6115	0.0006
0.50	1.5221	1.5227	0.0006
0.60	1.4376	1.4395	0.0019
0.70	1.3607	1.3628	0.0021
0.80	1.2915	1.2936	0.0021
0.90	1.2302	1.2323	0.0021
1.00	1.1773	1.1794	0.0021
1.10	1.1327	1.1348	0.0021
1.20	1.0960	1.0982	0.0022
1.30	1.0666	1.0693	0.0027
1.40	1.0450	1.0471	0.0021
1.50	1.0296	1.0308	0.0012





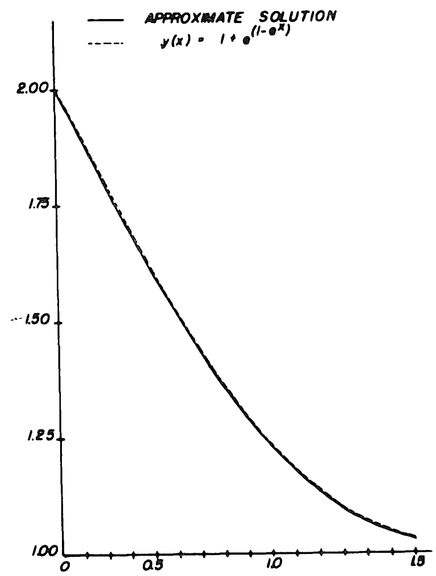


Figure 14—9.

An ordinary differential equation and its solution by nomographic techniques based upon Runge-Kutta developments.

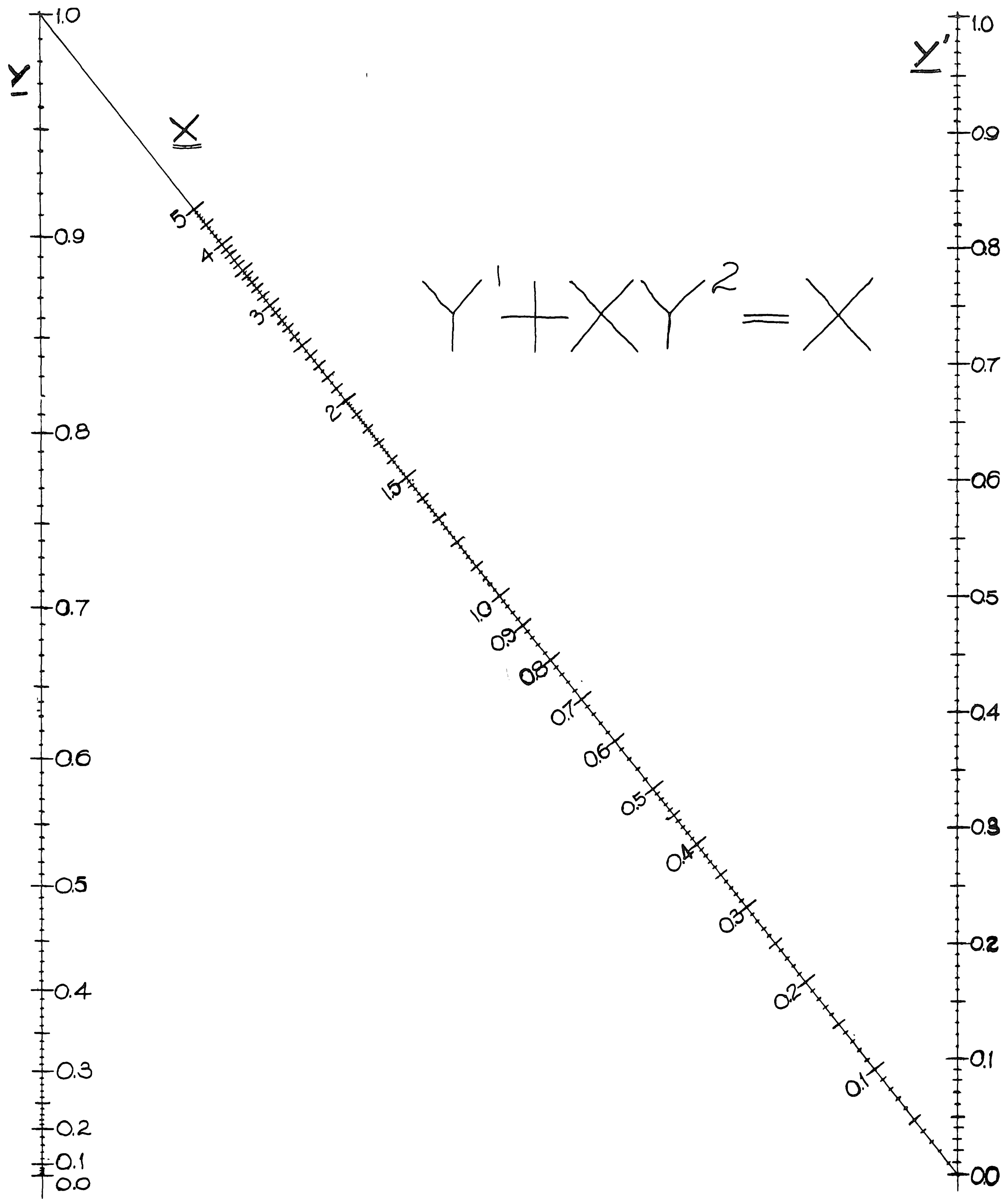


Figure 14—10a.

A home-made nomogram for the non-linear ordinary differential equation shown.

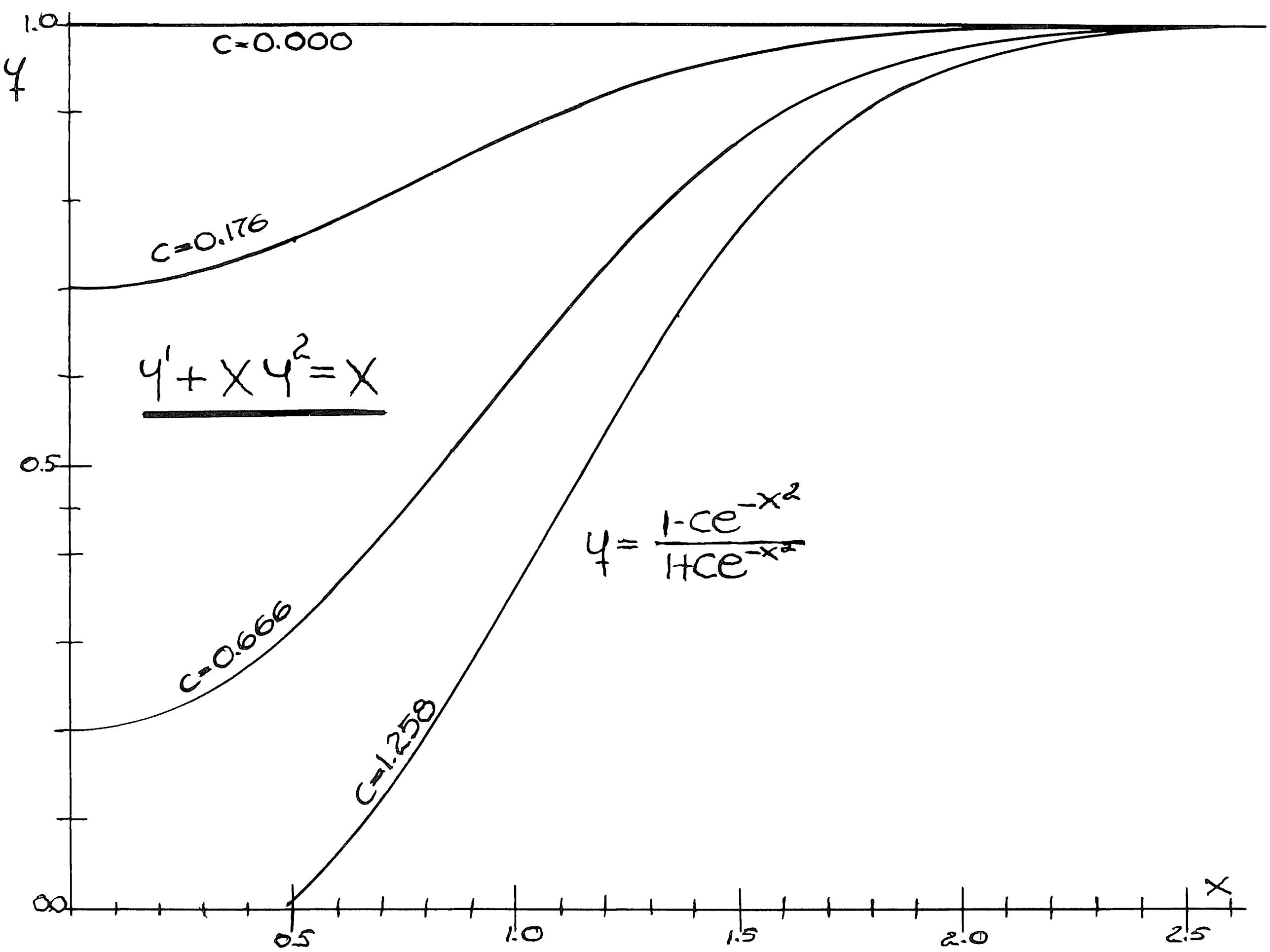


Figure 14—10b.

Solutions to the non-linear ordinary differential equation in Figure 14—10 (a). The classical and nomographic solution curves cannot be distinguished in the region shown.

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