ORY OF WAVE TRANSMISSION

GEORGE CONSTANTINESCO

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A TREATISE ON TRANSMISSION OF POWER BY VIBRATIONS

BY

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ENGINEER

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A TREATISE ON WAVE TRANSMISSION OF POWER

CHAPTER I

INTRODUCTORY

One of the fundamental problems of mechanical engineering is that of transmitting energy found in nature, after suitable transformation, to some point at which it can be made available for performing useful work.

The methods of transmitting energy known and practised by engineers are broadly included in two classes: mechanical, including hydraulic, pneumatic and wire rope methods; and electrical methods.

The present volume deals with a new method by which the problem has been solved by the author.

All methods of transmitting power through liquids, known as hydraulic methods, as hitherto applied, depend on the continuous transmission of pressure through a liquid so that pressure generated at one end of the line is utilised at the other end. The liquid in this form of transmission merely acts as an incompressible flexible connecting-rod.

The known pneumatic methods involve a flow in the pipes always in one direction, pressure being generated at one end of the system and utilised at the other end, but in this case the elasticity of the air employed is sometimes taken advantage of in the power utilisers.

In the wire-rope methods, the motive power is, as it were, attached by a string, as near as possible inextensible, to the power utilisers; the system depends on the longitudinal motion of the wire as a whole.

In all these known methods of applying mechanical means to the transmission of power from one point to a distant point elasticity has no direct function and is generally avoided or ignored.

The author's system depends on the elasticity of the medium through which the energy is transmitted. The essential feature of the system is that the particles of the medium employed, whether solid, liquid, or gaseous, are in a state of vibration about a mean position.

According to the new system, energy is transmitted from one point to another, which may be at a considerable distance, by means of impressed periodic variations of pressure or tension producing longitudinal vibrations in solid, liquid, or gaseous columns. The energy is transmitted by periodic changes of pressure and volume in the longitudinal direction, and may be described as wave transmission of power, or mechanical wave transmission.

CHAPTER II

ELEMENTARY PHYSICAL PRINCIPLES

THERE are many instances in nature of transmission of energy by vibrations; wave motion may almost be said to be the natural method of transmitting energy.

Let us consider some known phenomena of vibrations of particles of matter.

The transmission of sound through air is due to a vibratory motion set up by the source in the surrounding air; each particle of air in the neighbourhood of the source is put into a state of vibration about a mean position.

A common method of producing sound is to cause an elastic diaphragm to vibrate, impressing its vibrations on the surrounding air. By isolating the air to which the vibrations are transmitted, as, for instance, by means of a speaking-tube, the sound can be directed and a given quantity of energy of vibration produced can thus be transmitted over great distances.

Consider what is taking place in the tube when the contained air is set in motion by a diaphragm in a plane normal to the axis of the tube and vibrated about a mean position.

The first movement of the diaphragm in the direction of the tube displaces some air along the tube; this displacement is resisted by the still air further along the tube, so that a zone of compressed air is produced in the immediate neighbourhood of the diaphragm. At the same time the moving diaphragm is giving velocity to the particles of air in its immediate neighbourhood, and these particles communicate their velocity to those beyond them, and thus any disturbance once produced by the diaphragm must travel forward along the tube. On the return movement of the diaphragm, the compressed air in its immediate neighbourhood, being elastic, expands, and we have then a zone of low-pressure air in contact with the diaphragm.

The continuing vibrations of the diaphragm produce alternate zones of high and low pressure, and the disturbances so produced travel forward along the tube until the whole of the air particles in the tube are in a state of vibration; it has been found that the zones of high and low pressure travel along the tube with a definite velocity of about 330 metres per second, this velocity varying slightly with the diameter of the tube.

In a similar manner sound energy travels through other elastic media. The velocity through water has been found to be about 1435 metres per second.

As hitherto employed for the transmission of power in hydraulic and telpherage systems, liquid and solid connections have been considered as movable *en bloc*, and for practical purposes incompressible and inextensible. Both liquid and solid columns, however, are elastic, and this property can be made use of to transmit energy by vibrations of the particles of matter of which they are built up. We will first consider the case of liquid columns.

Assume that we have 150 metres of wrought-iron steam-pipe, of 2·5 cm. diameter and 0·5 cm. thickness of metal, closed at one end and filled with water; and suppose a fluid-tight piston is forced into the pipe under a steady pressure of 35 kg. per sq. cm. If the liquid were incompressible the increase in volume of the containing pipe under the pressure would allow the piston to enter about 1·5 cm.

If the pipe were absolutely inexpansible the pressure would compress the water to an extent that would allow the piston to enter about 26 cm.

It is seen, therefore, that the compression of the water in a wrought-iron steam-pipe of the size considered is the chief factor in the changes of volume which take place under pressure, and that the expansion of the containing pipe is almost negligible.

On removing the pressure from the piston, the water will again expand to its original volume. With other liquids similar results will be obtained. Assume now that the pipe, instead of being closed rigidly at one end, is closed by a light floating piston held always in contact with the liquid column, but free to move with the liquid; assume further that the working piston, instead of being slowly pushed into the pipe, is connected to a rapidly rotating crank, so that it moves with a simple harmonic motion, and that in addition to the piston impulses a steady pressure acts on the liquid column at both ends. The only resistance to the movement of the piston is then the inertia of the liquid column, and if the column is short the liquid will move as a solid mass. If, however, the column is of considerable length, the motion of the layers of liquid nearer the working piston is resisted by the inertia of the more remote layers, and on the in-stroke of the piston the liquid in its neighbourhood will be compressed and its volume diminished; it follows that the motion of the layers of liquid remote from the piston will be less than that of layers nearer to it.

At any given speed of rotation of the crank there will be a point in the liquid column at which, on the completion of the in-stroke of the piston, no movement of the liquid has occurred. The liquid between this point and the piston will at this moment be in a state of compression varying from a maximum at the piston to zero.

At the moment of maximum velocity of the piston, the velocity of the layer of liquid in contact with it will necessarily be greater than the velocity of the more remote layers, and the kinetic energy of the layers nearer the piston will, therefore, be transmitted in the forward direction along the column. The energy expended by the piston in its forward stroke at the end of this stroke is present in the liquid column, partly in the form of potential energy due to the decreased volume of the liquid under compression and partly as kinetic energy.

On the return stroke of the piston, the compression of the layer of liquid in contact with it decreases, and expansion of the liquid takes place between the piston and the point in the column at which the pressure is a maximum. As the point of maximum pressure moves away from the piston at the commencement of the return stroke, the velocity of the layer of liquid in contact with the piston is reversed, while the pressure of this layer diminishes until the piston is at the end of its out-stroke. At the end of this out-stroke the layer of liquid in contact with the piston is instantaneously at rest.

As the crank continues rotating, there are thus impressed on the liquid column a series of impulses sending a series of changes of pressure and volume along the column, the particles of liquid each vibrating about a mean position.

The considerations dealt with above as regards vibrations in liquid columns apply also to solids; this may be shown by considering the case of a long helical spring, one end of which is subjected to periodic shocks in the longitudinal direction. At each shock the end of the spring will be compressed and will again expand when the impulse is removed; the effect of the impulse, however, will travel along the spring in the direction of the shock with a definite velocity. The inertia of the coils of the spring remote from the end provides the resistance necessary to compress the first coils, but on the removal of the impulse expansion takes place in both directions, so that the wave of pressure and displacement travels along the spring.

An example of this occurs in practice in the case of the recoil springs of heavy ordnance, in which it has been noticed that pulses in the movements of the gun take place, due to the zones of compression in the recoil springs produced by the sudden shock of firing.

Consider now a very long steel wire connected to a crank so that the end is given a simple harmonic motion in the longitudinal direction, and suppose that the tension given by the crank is superposed on a steady tension in the wire so that no part of the wire is ever in a state of longitudinal compression.

As the crank rotates the end of the wire will be subjected to alternate maximum and minimum tensions occurring periodically; under certain conditions the wire, being elastic, will not move *en bloc*, but the periodic changes of tension will produce periodic displacements of the particles of the wire in the longitudinal direction, each particle vibrating about a mean position as in the case of the fluid columns discussed above.

In the transmission of sound through air we have seen that a series of vibrations is imparted to the air particles, causing them to move about a mean position; and thus a series of waves of alternate compression and rarefication travels forward from the source. If these waves fall on a sensitive receiver, such as the drum of the human ear, the receiver is set in vibration and the sound is heard. This is, in fact, an example of the transmission of energy by mechanical wave motion. Similarly, sound is transmitted through liquids and solids.

In order that a receiver may be able to respond to the vibrations falling on it, certain conditions are essential. The part of the receiver which is to be put in motion must be capable of vibrating at the periodicity of the vibrations which fall on it.

In the case of the human ear very sensitive receivers are found, which are tuned to or capable of adapting themselves readily to vibrations of different periodicity within certain limits of frequency. When, however, we come to the problem of detecting vibrations by mechanical means, and still more so when it is desired to transmit power economically by means of these vibrations, it is necessary that the part moved should be designed so that it can respond to the particular periodicity of vibration by which the power is transmitted. It is further necessary, if the part moved has to perform useful work, that the work should be performed in such a manner that the ability of the receiver to vibrate in unison with the impressed vibrations is not interfered with.

Although in some cases in which energy has been heretofore transmitted by vibrations in matter—as, for instance, the case of a tuning-fork made to respond to sound waves of its own frequency—the question of the period of vibration of the receiver has been considered; in no case, up to the present, has the tuned receiver been adapted to the performance of work. For the transmission of power by mechanical wave motion it is therefore necessary to devise means by which the vibrations in the transmission line may be received and converted to use.

Let us now consider further the case of a rapidly rotating crank causing a piston to reciprocate at the end of a long pipe containing liquid. We have seen above that a series of zones of high pressure and compression of the liquid alternating with zones of low pressure and expansion of the liquid are produced, and that these zones travel forward along the pipe.

In Fig. 1 suppose the crank *a* to be rotating uniformly, causing

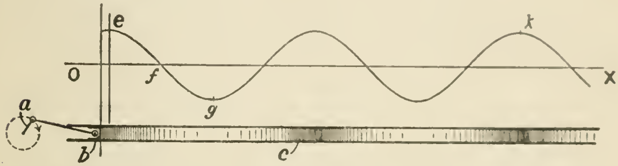

FIG. 1.

the piston *b* to reciprocate in the pipe *c*, which is full of liquid. At each in-stroke of the piston a zone of high pressure is formed, and these zones of high pressure, shown by shading, travel along the pipe away from the piston; between every pair of high pressure zones is a zone of low pressure shown light in the figure. The pressure at any point in the pipe, therefore, will go through a series of values from a maximum to a minimum, and these values will repeat periodically. Let the line *ox* represent the value of the mean pressure, then with the piston in the position illustrated, the instantaneous pressures at different points along the pipe may be represented by the ordinates of the sine curve *efg . . . k*. As the rotation of the crank is uniform, it will be evident that the distances between successive points of maximum pressure will be equal. This uniform distance along the pipe at which the values of the pressure are repeated is the wave length of the vibrating movement of the liquid.

If v is the velocity with which these waves travel along the pipe, and n is the number of revolutions in unit time of the crank a, it will readily be seen that the wave length λ must be $\frac{v}{n}$.

Assume now that the pipe is of finite length and is closed at the point r at a distance from the piston b equal to an exact multiple of the wave length, and suppose that the stroke of the piston is small compared with the wave length as shown in Fig. 2.

The wave of compression will be stopped at *r* and reflected, and the reflected wave will travel back along the pipe.

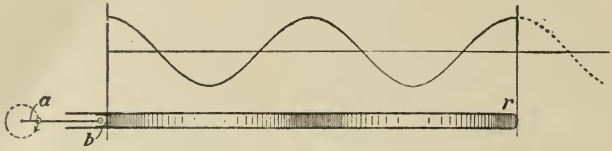

Fig. 2.

If the crank continues its rotation at uniform speed, with the length of pipe and speed of rotation we have taken—*i.e.*, with the distance from the piston *b* to the stop *r* an exact multiple of the wave length—a zone of maximum pressure will be just starting from the piston at the instant the reflected zone of maximum pressure reaches it; so that we shall have a wave of double the original amplitude travelling forward along the pipe. The next revolution of the crank will again add to the amplitude of the wave sent forward; and so on with successive revolutions. The result of this continual pouring in of energy is that the maximum pressure increases without limit till ultimately the pipe bursts.

It should be noticed that, in a wave of greater amplitude, the maximum pressures are increased, and the maximum velocities and distance of travel of the oscillating particles are also increased.

Suppose now that instead of closing the pipe rigidly at *r* we have at *r* a piston *m* connected to a crank *n* similar to *a* as shown in Fig. 3.

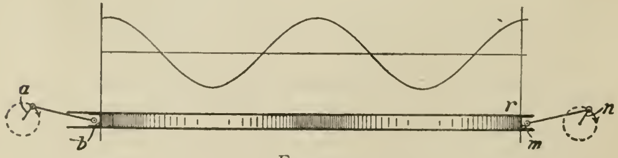

Fig. 3.

Suppose that the crank *n* is rotating at the same angular velocity and in the same phase as the crank *a*. If the liquid column were continued beyond the piston *m*, it is evident that the movement of the piston would produce in this column a series of waves which would be exactly similar to and a continuation of the waves between *b* and *m*.

The piston *m*, therefore, if moving synchronously with *b*, will be able to take up the whole energy of the waves produced by *b* and travelling along the pipe.

It will be seen, further, that the piston will be able to take up and utilise the whole of the energy of the waves travelling to it if placed at any point of the pipe, provided its time period of reciprocation is the same as that of the piston *a*, and provided that the phase of its movement is such as would produce a continuation beyond it of the impinging waves; that is to say, provided the piston movement is in phase with the movement of the layer of liquid in contact with it.

In the transmission of power by wave motion in this example, the maximum pressure in the pipe will at no point exceed the maximum pressure in the neighbourhood of the working piston, however long the transmission line may be; and will be the same whether the line

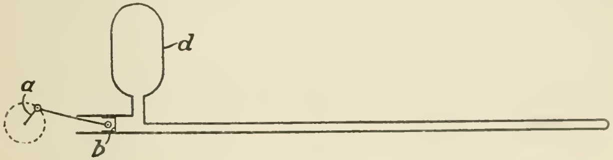

FIG. 4.

is a single wave length or any number of wave lengths. Also the two pistons may be moving in the same or in opposite directions, and their motions may differ in phase by any angle according to the relation between the distance from one to the other and the wave length.

In the example above discussed, the whole of the energy put into the liquid column by the piston *b* can be taken up by the piston *m*. If more energy is put in by *b* than is taken up by the piston *m*, assuming no frictional losses, it is obvious that reflected waves must be formed as the direct waves fall on the piston *m*. The result of this will be that the surplus energy will remain in the liquid and the continuation of the rotation will continually pour in energy, increasing the maximum pressure indefinitely till ultimately, as in the case of the closed pipe, the pipe will burst.

Suppose that, in the case of a closed pipe having a length of several wave lengths, a vessel *d* completely filled with liquid, of considerable volume in proportion to the stroke volume of the piston *b*, and with rigid walls, is placed in communication with the pipe in the neighbourhood of the piston, as shown in Fig. 4. At each in-stroke of the piston a flow will take place through the entrance to the vessel *d*, and the liquid in this vessel will be compressed, and at each out-stroke of the piston the liquid in the vessel will again expand; according to the volume of the vessel more or less liquid will flow into it and out of it at each revolution of the crank. The capacity *d* will thus act as a spring, taking up the energy of the direct and reflected waves when the pressure is high, and giving back this energy when the pressure falls; the mean pressure in the vessel *d* and in the pipe will be the same, so that when the successive reflected waves in the pipe have been produced and have reached a certain amplitude equivalent to this mean pressure, the piston will merely exert energy in compressing the liquid in the vessel *d* on its in-stroke, and the liquid acting as a spring will restore this energy to the piston on its out-stroke. The result of this is that when the reflected waves have been produced, there will be a series of *stationary* waves in the pipe, and no further increase of energy in the liquid

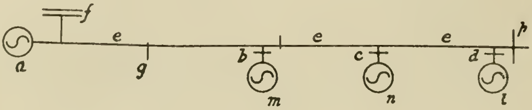

FIG. 5.

will take place and the pressures in the pipe will never exceed the fixed limit.

By using a vessel such as *d*, therefore, the pipe can be completely or partially closed. It is therefore possible to place at the far end or other point of the pipe apparatus for utilising only part of the energy of the wave, and the rotating crank *a* will only require to perform work to the extent of the energy actually utilised.

Consider now a case (Fig. 5) in which waves are transmitted by a reciprocating piston *a* along a line *eee* provided with branches. Assume that the pipe *e* is closed at *p* at a distance of one complete wave length from the wave generator *a*; and that there are branches *bcd* at the half, three-quarter and full wave-length distances respectively. We know from the cases discussed above that if the cock *p* is closed and the cock *d* opened, leading to a motor *l* rotating at the synchronous speed, the motor *l* will be able to take up the whole of the energy put into the liquid by the pump.

We also know that if all the cocks are closed stationary waves will be produced in the pipe *e* having maximum variations of pressure at the end *p* and at the half wave length *b*. At these points the flow will always be zero, while the pressure will alternate between maximum and minimum values determined by the capacity *f*, consisting of a closed vessel filled with liquid. At the quarter and three-quarter wave length *g* and *c* respectively the flow will alternate between maximum and minimum values, but the variation of pressure will remain zero.

In this case the points of maximum pressure and maximum movement do not travel along the pipe, but are fixed in position, and theoretically no energy flows from the generator. At the points of maximum movement no variation of pressure will occur; and at the points of maximum pressure variation there will be no movement of the liquid.

It is evident, therefore, that if the cock *b* leading directly to a motor *m* be opened, the motor *m*, running at the synchronous speed, will be able to take up all the energy given to the line. The stationary half wave between *a* and *b* will therefore disappear, its place being taken by the forward travelling wave, while between *b* and *p* the stationary wave will persist. If the cock *c* leading to the motor *n* at the three-quarter wave length be opened, all other cocks being closed, since at the point *c* the variation of pressure is always zero, no energy can be taken up by the motor, and the stationary wave will persist in the whole length of the pipe.

If the motor be connected at any intermediate point, part of the energy will be taken up by the motor, while the stationary wave will persist but will be of reduced amplitude between the generator *a* and the motor. The state of the liquid between the generator *a* and the motor may be considered as the resultant of two superposed waves: one a stationary wave and the other a travelling wave of flowing energy.

Assume now that the motor *l* is not capable of taking up all the energy which can be transmitted to the line by the generator *a*; then we shall have superposed in the pipe a system of stationary waves and a system of waves travelling along the pipe, so that there will be no point in the pipe at which the variation of pressure will always be zero, consequently a motor connected at any point of the pipe will be able to take up and utilise a portion of the energy which is transmitted to the line.

We see, therefore, that if we have a number of motors all connected to the line, every one of them will be able to take some energy and do some useful work. It is only when no energy is being utilised that points at which the variation of pressure is *permanently* zero can exist.

It is seen from the preceding discussion that periodic variations of pressure and volume can be impressed on columns of gases, liquids or solids; and can be made to travel along such columns, causing the several particles to be set in vibration about their mean positions.

It is further clear that under certain conditions the mechanical energy so transmitted can be made available for the performance of useful work.

The theoretical considerations on which such mechanical wave transmission depends are investigated analytically in the following chapters.

CHAPTER III

DEFINITIONS

Alternating Fluid Currents.—For any flow of fluid in full pipes, if

ω = the sectional area of the pipe in square centimetres,
v = the velocity of the fluid at any instant in centimetres per second,

and

i = the flow of liquid in cubic centimetres per second, we have,

$$i = v\omega.$$

Suppose that the current is produced by a piston moving in a cylinder of section Ω sq. cm. with a simple harmonic motion.

Let

r = the equivalent length of the driving crank in centimetres,
a = the angular velocity of the crank or the *pulsation* in radians per second,
n = the number of revolutions of the crank per second,

Then the flow from the cylinder to the pipe at any instant will be

$$i = I \sin (at + \phi) \quad \ldots \quad (1)$$

where

$I = ra\,\Omega$ = the maximum alternating flow in cm.³ sec. or the *amplitude* of the flow.
t = the time in seconds,
ϕ = the angle of phase,

and if

T = the period of one complete alternation, equal to the time of one complete revolution of the crank,

we have

$$a = 2\pi n,$$

$$n = \frac{1}{T}$$

Let us define the *effective current* $I_{\text{eff.}}$ by the equation

$$I^2_{\text{eff.}} = \frac{1}{T}\int_0^T i^2 dt = \frac{I^2}{2} \qquad (2)$$

and the *effective velocity*

$$v_{\text{eff.}} = \frac{I_{\text{eff.}}}{\omega}$$

The stroke volume δ will be given by the relation—

$$\delta = 2r\Omega = 2\frac{I}{a}$$

This corresponds to the volume displaced by the piston in the cylinder during a single stroke and is measured in cubic centimetres. Thus the stroke f of the liquid in the pipe in the immediate neighbourhood of the piston will be—

$$f = \frac{\delta}{\omega} = \frac{2I}{a\omega}$$

Alternating Pressures.—The consideration of alternating pressures is similar to that of alternating currents. In a pipe in which the current is flowing the pressure p will be of similar form; and we have

$$p = H \sin(at + \psi) + p_{\text{m.}} \qquad (3)$$

where

H = the maximum alternating pressure in kilogrammes per square centimetre,

ψ = the angle of phase,

and

p_{m} = the mean pressure in the pipe.

The minimum pressure in the pipe will then be

$$p_{\text{min.}} = p_{\text{m.}} - H$$

and the maximum pressure will be

$$p_{\text{max.}} = p_{\text{m.}} + H$$

If p_1 is the pressure at any point in the pipe and p_2 the pressure at another point,

the difference

$$h = p_1 - p_2 = H \sin(at + \psi) \qquad (4)$$

will be defined as the *instantaneous hydromotive force* between the two points, and H is its amplitude.

The effective hydromotive force will be

$$H_{\text{eff.}} = \frac{H}{\sqrt{2}}$$

Friction. In an alternating current of liquid flowing in a pipe, certain friction will occur at the surface of the pipe and in the body of the liquid itself; we shall assume that the pressure difference or hydromotive force required to produce the flow is substantially proportional to the current.*

The relation between the hydromotive force and current may therefore be written

$$h = Ri \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where

$$R = \text{coefficient of friction in } \frac{\text{kg. sec.}}{\text{cm.}^5}$$

From the best experiments at present available the value of R may be calculated from the formula—

$$R = \epsilon \frac{\gamma l v_{\text{eff.}}}{2g\omega d} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where

γ = the density of the liquid in kg. cm.^{3},
l = the length of pipe in centimetres,
g = the acceleration due to gravity in cm. sec.^{2},
ω = section of the pipe in square centimetres,
$v_{\text{eff.}}$ = the effective velocity in cm. sec.,
d = the internal diameter of the pipe in centimetres,

and for water—

$$\epsilon = 0{\cdot}02 + \frac{0{\cdot}18}{\sqrt{v_{\text{eff.}} d}}$$

Exact experiments are lacking on this subject, so this formula must only be considered as an approximation.

From (6), substituting for ϵ, we have

$$R = \frac{\gamma l}{g\omega}\left(\cdot 01 \frac{v}{d} + \frac{\cdot 09}{d}\sqrt{\frac{v_{\text{eff.}}}{d}}\right)$$

or, writing

$$100k = \frac{v_{\text{eff.}}}{d} + \frac{9}{d}\sqrt{\frac{v_{\text{eff.}}}{d}} = \frac{v_{\text{eff.}}}{d}\left(1 + \frac{9}{v_{\text{eff.}}}\sqrt{\frac{v_{\text{eff.}}}{d}}\right)$$

$$R = k\frac{\gamma l}{g\omega} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

* For further developments see page 86.

It should be noted that comparatively greater effective velocities may be employed in pipes of greater diameters for the same value of k.

The loss of power W due to friction will be

$$W = \frac{1}{T}\int_0^T hidt \qquad (8)$$

putting

$$h = Ri$$

we get

$$W = \frac{1}{T}\int_0^T Ri^2 dt = \frac{R}{T}\int_0^T i^2 dt = \frac{RI^2}{2} \qquad (9)$$

We have, therefore—

$$W = \frac{RI^2}{2} = \frac{HI}{2} = H_{eff.} \times I_{eff.}$$

Example

It is required to transmit power between two points. Find the relative weights of metal when the transmission is made through one, or through m pipes; the conditions being that the same power is transmitted, the same hydromotive force employed and the loss of power equal in the two cases.

Assume that the diameters d are large enough to allow of the simplification

$$100k \sim \frac{v_e}{d}$$

Let v and ω be the effective velocity and section respectively of the single pipe, and v_1 and ω_1 the same quantities for one of the m pipes.

As the same power is to be transmitted we must have

$$I = v\omega = mv_1\omega_1 = mI_1$$

In order that the loss of power may be the same we must have

$$RI^2 = mR_1I_1^2$$

hence

$$mR = R_1$$

But from (7), as the length of each pipe is the same—

$$\frac{R_1}{R} = \frac{k_1}{\omega_1} \cdot \frac{\omega}{k}$$

it follows that

$$\frac{k_1\omega}{k\omega_1} = m$$

Also we have

$$\frac{k_1}{k} = \frac{v_1}{d_1} \times \frac{d}{v} \cdot \frac{1 + \frac{9}{v_1}\sqrt{\frac{v_1}{d_1}}}{1 + \frac{9}{v}\sqrt{\frac{v}{d}}} \sim \frac{v_1 d}{d_1 v}$$

therefore

$$\frac{v_1}{v} \cdot \frac{d\omega}{d_1\omega_1} = m$$

but since

$$\frac{v_1}{v} = \frac{\omega}{m\omega_1}$$

we have

$$\left(\frac{\omega}{\omega_1}\right)^2 \frac{d}{d_1} = m^2$$

So that

$$\frac{d}{d_1} = m^{\frac{2}{5}}$$

The weight G and G_1 of the two systems of pipes will be approximately in proportion to their sections, so that

$$\frac{G_1}{G} = \frac{md_1^2}{d^2} = m^{\frac{1}{5}}$$

The single pipe, therefore, is the more economical solution.

Thus if two pipes are used $m = 2$
and

$$\frac{G_1}{G} = 2^{\frac{1}{5}} = 1{\cdot}15$$

So that it is more economical to employ a single pipe.

Numerical Example

Find the power lost in a pipe 10 km. long having an internal diameter 10 cm. and in which there is flowing an alternating current of water with effective velocity 100 cm. per second.

We have

$$W = RI_{\text{eff.}}^2$$

$$I_{\text{eff.}} = v_{\text{eff.}} \cdot \omega$$

and

$$R = \frac{\gamma l}{g\omega} k$$

where

$$k = 0{\cdot}01 \frac{v_{\text{eff.}}}{d} + \frac{0{\cdot}09}{d} \sqrt{\frac{v_{\text{eff.}}}{d}}$$

From which

$$W = 13{\cdot}1 \text{ h.p.}$$

Capacities and Condensers.—A *hydraulic capacity* is a reservoir to receive the flow of an alternating current, such as a large vessel filled with liquid, an elastic bottle, a diaphragm, or a light piston working in a cylinder and held by a spring device placed in communication with a pipe in which an alternating current of liquid is flowing. Its value depends on elasticity.

Hydraulic condensers are appliances for making alterations in the value of currents, pressures or phases of alternating current. The apparatus usually consists of a movable solid body dividing the liquid column and supported elastically in a mean position so that it follows the movements of the liquid column. An example of a condenser is shown at Fig. 6.

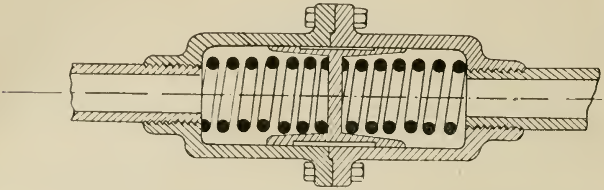

FIG. 6.

The principal function of hydraulic condensers is to counteract inertia effects due to moving masses.

The capacity C of a condenser consisting of a piston of section ω on which the liquid pressure is acting, held in a mean position by means of springs, is given by the equation

$$\triangle V = \omega \triangle f = C \triangle p \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

where

$\triangle V$ = the variation of volume of the space for liquid,

$\triangle f$ = the variation of longitudinal position of the piston,

and

$\triangle p$ = the variation of pressure in the liquid.

If the piston is held by a spring we have at any instant

$$f = AF$$

where

A = a constant depending on the spring

and

F = the force acting on the spring.

In the condenser we have

$$\triangle F = \omega \triangle p$$

consequently

$$\triangle f = A \, \omega \triangle p \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad ; \qquad (11)$$

From equations (10) and (11) we get

$$C = A\omega^2$$

and

$$F = \frac{f}{A} = \frac{f\omega^2}{C} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (12)$$

For springs of wire of circular section wound on a cylinder we have

$$B = \frac{2G}{\sigma^2} Ff$$

where

B = volume of metal in the spring in cubic centimetres,

σ = the allowable stress of the metal in kilogrammes per square centimetre,

and

G = the coefficient of transverse elasticity of the metal.

This may be written simply—

$$B = mFf \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

m being a constant depending only on G and σ.

If d is the diameter of the wire of the spring and D the mean diameter of the spiral, we have

$$F = 0{\cdot}4 \frac{d^3}{D} \sigma$$

so that

$$d = \sqrt[3]{\frac{FD}{0{\cdot}4\sigma}}$$

or simply

$$d = n\sqrt[3]{FD}$$

n being a constant depending only on σ.

The values of m and n for steel springs for different values of σ are given in the following table—

σ . .	1500	1750	2000	2250	2500	2750	3000	3250	3500
m . .	0·67	0·49	0·38	0·30	0·24	0·20	0·17	0·142	0·123
n . .	0·118	0·113	0·107	0·103	0·100	0·096	0·094	0·091	0·089

These equations, therefore, enable us to calculate the springs required for a condenser of a given capacity required to work at a given maximum stress.*

For springs made of wire of square section, taking the same values for C, D, f, and G, if B' be the volume of the spring and d' the side of the square of the section,

we have

$$B' = 1{\cdot}4B$$

and

$$d' = 0{\cdot}88d$$

B and d being the volume and diameter of wire of the equivalent spring with wire of circular section.

* A table of springs will be found in Appendix.

The square-sectioned wire will thus require a much heavier spring which would be about 25 per cent. longer when fully compressed than the spring made with wire of circular section. Other forms will also be less economical than the circular section.

Consideration of Several Springs acting together. — If several springs act directly on the piston of a condenser, the springs exert pressures given by

$$F_1 = \frac{f\omega^2}{C_1} \quad F_2 = \frac{f\omega^2}{C_2} \quad . \; . \; . \quad F = \frac{f\omega^2}{C_n}$$

The resultant pressure

$$F = \Sigma F_n$$

$$= f\omega^2 \Sigma \frac{1}{C_n}$$

If we denote the resultant capacity by C we have

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \; . \; . \; . + \frac{1}{C_n}$$

This equation is general whether the springs are on the same side of the piston or some on one side and some on the other; and whether they are under initial pressure or not, provided they are in contact with the piston throughout its stroke. Suppose the piston is controlled by a number of springs under different degrees of compression at the instant at which the change of pressure occurs we still have

$$\Sigma F_n = \omega^2 \Sigma \frac{f_n}{C_n}$$

If we vary the pressure

$$\Delta F = \Delta \Sigma F_n = \omega^2 \Delta \Sigma \frac{f_n}{C_n}$$

but

$$f_1 f_2 \; . \; . \; . \; f_n \text{ all vary with } \Delta f$$

so that

$$\Delta \Sigma \frac{f_n}{C_n} = \Delta f \Sigma \frac{1}{C_n}$$

and

$$\Delta F = \omega^2 \Delta f \Sigma \frac{1}{C_n} = \omega^2 \Delta f \frac{1}{C}$$

and

$$\frac{1}{C} = \Sigma \frac{1}{C_n}$$

Thus if we have a condenser in which the piston is held by two equal springs acting on opposite sides and C_1 be the capacity of one spring; the capacity of the condenser will be given by the relation

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_1}$$

and

$$C = \frac{C_1}{2}$$

Capacity of Liquid Columns in Pipes.—Capacity C has above been defined by the relation

$$C = \frac{\Delta V}{\Delta p}$$

where ΔV is the change in the volume of the liquid under a change of pressure Δp.

If E is the coefficient of elasticity of the liquid we have

$$\Delta p = E \frac{\Delta V}{V}$$

so that

$$C = \frac{V}{E}$$

V being the volume of liquid in cubic centimetres.

For water the value of E is about 2×10^4.

For steel E is about 2×10^6.

So that the value for water is about a hundred times smaller than for steel.

For lubricating oil $E = 1{\cdot}4 \times 10^4$ approximately.

Thus the capacity of a volume of liquid V for water will be

$$C = \frac{V}{20000}$$

and for lubricating oil

$$C = \frac{V}{14000}$$

It should be noted that the coefficient of elasticity is not absolutely independent of the pressure, but increases as the pressure increases. A table of coefficients of elasticity of different liquids under various pressures is given in Appendix.

Capacity due to Elasticity of the Walls of the Pipes.—The change of volume of a pipe of length l and internal diameter $2r$ will be

$$\Delta V = 2\pi r l \Delta r + \pi r^2 \Delta l$$

The capacity will therefore be

$$C = \frac{\Delta V}{\Delta p} = 2\pi rl \frac{\Delta r}{\Delta p} + \pi r^2 \frac{\Delta l}{\Delta p}$$

If E_1 is the coefficient of elasticity of the metal of the pipe and σ and τ the tension per unit section in the walls circumferentially and longitudinally respectively

$$\sigma = E_1 \frac{\Delta r}{r}$$

and

$$\tau = E_1 \frac{\Delta l}{l}$$

but if e is the thickness of the walls

$$\sigma e = r \Delta p = 2\tau e$$

so that

$$\Delta r = \frac{\sigma r}{E_1} = \frac{r^2 \Delta p}{E_1 e}$$

and

$$\Delta l = \frac{l\tau}{E_1} = \frac{rl \Delta p}{2E_1 e}$$

and

$$C = 2{\cdot}5\pi rl \frac{r^2}{E_1 e} = \frac{1{\cdot}25\omega ld}{E_1 e}$$

where d is the diameter and ω the section of the pipe.

The total capacity of a pipe full of liquid, taking into account the compressibility of the liquid and the elasticity of the metal, will therefore be

$$C = l\omega\left[\frac{1}{E} + \frac{1{\cdot}25}{E_1}\left(\frac{d}{e}\right)\right]$$

For iron pipes and water

$$E_1 = 100\,E \text{ approximately}$$

so that

$$C = \frac{l\omega}{E}\left(1 + \frac{d}{80e}\right)$$

As the term $\frac{d}{80e}$ is in practice small in comparison with unity, we may treat the pipe as rigid, and we have

$$C = \frac{l\omega}{E_1}$$

and take or E_1 the value

$$E_1 = E\left(1 - \frac{d}{80e}\right)$$

Inertia.—We have seen that capacity is a characteristic depending on elasticity. *Inertia* is a property which depends on the mass in motion. Let us consider a body of mass M kept in motion by a pressure acting in one direction over a surface of ω square centimetres normal to the direction of movement.

If $h = \Delta p$ is the hydromotive force at a given instant and v the velocity of the mass M we shall have the equation of motion

$$h\omega = M\frac{dv}{dt}$$

from which, since the current $i = v\omega$,

$$h = \frac{Mdi}{\omega^2 dt}$$

We will define as the coefficient of inertia L the quantity

$$L = \frac{M}{\omega^2} = \frac{P}{g\omega^2}$$

where

$P = Mg$; g being the acceleration due to gravity.

We thus get

$$h = L\frac{di}{dt}$$

This equation gives the relation between the hydromotive force acting upon a mass and the current i created by the movement, the mass being supposed to be rigidly connected to a piston moving in a cylinder of section ω. Such a mass and its piston will be referred to as an inertia.

The convention of a piston attached to the mass is only for the purpose of giving a positive meaning to the quantities ω and i on which the hydromotive force is supposed to act. Thus if a force is acting directly on a mass we have to assume that the force is acting through a piston of size chosen arbitrarily in order to give a meaning to the current i.

A column of liquid of length l will have a coefficient of inertia

$$L = \frac{\gamma l}{g\omega}$$

where γ is the density of the liquid in kilogrammes per cubic centimetre, l the length of the column in centimetres and ω the internal sectional area of the pipe in square centimetres.

Leakage.—Under the term *leakage* we shall include all loss of liquid through joints or small apertures or even due to the porosity of the material of which the pipe is made.

The loss of a fluid under pressure through small apertures is proportional to the difference of pressure.

Denoting by S a coefficient which we shall term the *coefficient of porosity* we have the general law

$$i = Sh$$

With the units we have selected S will be measured in $\frac{\text{cm.}^5}{\text{kg. sec.}}$, and it will be seen that the dimensions are the inverse of the dimensions of friction.

In dealing with a long pipe whose joints are not perfect, we may regard the total loss of liquid as uniformly distributed along the pipe; and the coefficient of porosity due to leakage through joints may be defined per unit length of the pipe.

Example I

Capacity due to Circular Plate. — Consider a condenser formed by a circular elastic plate restrained at its edges as shown in Fig. 7. A pressure difference $\Delta p = p_1 - p_2$ will correspond with a displacement of liquid ΔV represented by the volume displaced by the elastic plate during its deformation to the position indicated by the dotted line.

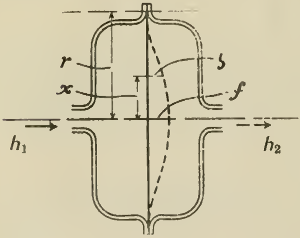

Fig. 7.

Let Ω be the area of surface of the plate, then the total force acting on this plate is $F = \Omega \Delta p$).

If f is the deflection of the plate at its centre, we have

$$f = AF = A\Omega\Delta p \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where A is a constant depending only on the dimensions and nature of the plate and the method by which it is held at its periphery. For the volume displaced we have

$$\Delta V = 2\pi\int_0^r yxdx = \pi\int_0^r yd(x^2)$$

Let

$$z = \left(\frac{x}{r}\right)^2$$

then this expression is simplified to

$$\Delta V = \pi r^2 \int_0^1 y dz = \Omega \int_0^1 y dz$$

The form of the deflected surface may be generally defined by a relation of the form

$$y = f\phi(z)$$

$\phi(z)$ being a function of z
so that

$$\Delta V = \Omega f \int_0^1 \phi(z) dz \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

By definition $C = \frac{\Delta V}{\Delta p}$; if, therefore, we replace ΔV and Δp by their values we get

$$C = A\Omega^2 \int_0^1 \phi(z) dz$$

If the pressure is only acting on a part of the surface of the plate near the centre, as, for instance, where the pressures are applied through a piston of sectional area ω the equation (1) will become

$$f = AF = A\omega \Delta p_1$$

and the change of volume ΔV is the volume swept by the piston on the small area ω.

This volume is

$$\Delta V_1 = f\omega$$

and we get

$$C = \frac{\Delta V_1}{\Delta p_1} = A\omega^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In this case, therefore, the diaphragm acts like a spring and the formula (3) may be compared with that found for the capacity of a spring (*see* p. 18).

From the theory of elasticity of circular plates we have generally

$$A = u \frac{\Omega}{Es^3}$$

where

u = a constant depending on the method of holding the plate and the distribution of pressure,

E = the coefficient of elasticity of the plate,

and

s = the thickness of the plate,

Let Ω_1 be the area of a piston which would produce the same displacement ΔV as the circular plate for the same stroke f; by definition we have

$$\Delta V = \Omega_1 f = \Omega f \int_0^1 \phi(z) dz$$

so that

$$w = \frac{\Omega_1}{\Omega} = \int_0^1 \phi(z) dz$$

and we get

$$C = A \Omega \Omega_1$$

and in the case in which the pressures are applied to the plate by a piston over a small surface ω around the centre of the plate, we have

$$C = A \omega^2$$

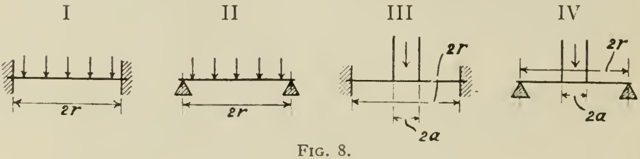

FIG. 8.

In the following table the values of the constants u and $w = \frac{\Omega_1}{\Omega}$ are given for the four cases illustrated in Fig. 8.

	I	II	III	IV
u	0·017	0·07	0·07	0·17
w	0·33	0·46	0·25	0·40

The four cases are—

I. Plate rigidly clamped and pressure uniformly distributed.
II. Plate freely supported and pressure uniformly distributed.
III. Plate rigidly clamped and pressure concentrated at the centre over a circle of radius a.
IV. Plate freely supported and pressure concentrated at the centre over a circle of radius a.

The maximum stress in the metal in the four cases would be

$$\sigma_1 = 0{\cdot}22 \frac{F}{s^2}$$

$$\sigma_2 = 0{\cdot}28 \frac{F}{s^2}$$

$$\sigma_3 = 0\cdot43\frac{F}{s^2}\log\left(\frac{r}{a}\right)$$

$$\sigma_4 = \left(1 - \frac{2a}{3r}\right)\frac{F}{s^2}$$

F in each case being the total pressure on the plate.

The function $\phi(z)$ is given by the theory of elasticity of circular plates, and we have in the four cases

$$\phi_1(z) = (z - 1)^2$$
$$\phi_2(z) = 1 - 1\cdot24z + 0\cdot245z^2$$
$$\phi_3(z) = 1 - z + z\log z$$
$$\phi_4(z) = 1 - z + 0\cdot4\log z$$

Example II

Inertia of a Circular Plate.—Let dM be the mass of an element of the plate at a distance x from the centre.

The force due to the inertia of this mass would be

$$dF = \frac{d^2y}{dt^2}dM$$

but

$$dM = \frac{\gamma}{g}s2\pi x dx = \frac{\gamma s\pi r^2}{g}dz = \frac{P}{g}dz$$

where

γ = the density of the plate in kg. cm.^{3}

g = the acceleration due to gravity and

P = the weight of the plate of radius r.

But we have found

$y = f\phi(z)$, so that as z is independent of the time

$$\frac{d^2y}{dx^2} = \phi(z)\frac{d^2f}{dt^2}$$

Thus we get

$$dF = \frac{P}{g}\cdot\frac{d^2f}{dt^2}\phi(z)dz$$

and

$$F = \frac{P}{g}\cdot\frac{d^2f}{dt^2}\int_0^1\phi(z)dz = u\frac{Pd^2f}{gdt^2}$$

If we denote by P_1 the equivalent weight assumed concentrated at the centre of the plate, and which would produce the same inertia force F on the stroke f, we have

$$P_1 = wP$$

Comparing this with the relation found between Ω_1 and Ω we have

$$w = \frac{P_1}{P} = \frac{\Omega_1}{\Omega}$$

We therefore come to the conclusion that a vibrating plate may be considered as a piston of the same thickness and density as the plate having a surface Ω_1 given by the formula $\frac{\Omega_1}{\Omega} = w$; this piston being held by a virtual spring device giving a capacity $C = A\Omega\Omega_1$.

In the analysis given above it is assumed that the plate is vibrating in the direction normal to its surface, points situated on circles of the same radius having equal movement; and it is assumed that there are no vibrations in the radial direction. For frequencies usually employed, this assumption is accurate; but for very high frequency other phenomena occur, and other vibrations take place. For practical purposes the analysis given above is sufficiently accurate and for calculation of condensers in which circular plates are used instead of springs, the formulæ arrived at are substantially correct.

Exercises

1. Find the capacity due to the expansion of a sphere of diameter d and thickness e, the coefficient of elasticity of the metal from which the sphere is made being E_1

Answer $$C = \frac{\pi d^4}{8E_1 e}$$

2. Find the capacity due to the expansion of a cylinder of length l and diameter d, the ends being closed by two hemispheres.

Answer $$C = 0{\cdot}98\frac{ld^3}{E_1 e}\left(1 + 0{\cdot}4\frac{d}{l}\right)$$

CHAPTER IV

EFFECTS OF CAPACITY, INERTIA, FRICTION AND LEAKAGE ON ALTERNATING CURRENTS

Relation between Current, Capacity, and Hydromotive Force.—If we consider a current flowing in a line containing a capacity, say a condenser, the variation of volume due to the current will be

$$\triangleleft V = \int i dt$$

but

$$h = \Delta p = \frac{\Delta V}{C}$$

so that

$$h = \frac{1}{C}\int i dt$$

If

$$i = I \sin (at + \phi)$$

we have

$$h = -\frac{I}{aC} \cos (at + \phi)$$

$$= \frac{I}{aC} \sin \left(at + \phi - \frac{\pi}{2}\right)$$

Thus the maximum value of the hydromotive force will be

$$H = \frac{I}{aC}$$

and at any instant we have

$$h = H \sin \left(at + \phi - \frac{\pi}{2}\right) = H \sin (at + \psi)$$

where

$$\psi = \phi - \frac{\pi}{2}$$

We see, therefore, that the hydromotive force varies according to a sine law and is of the same period as the current in the condenser, but the phase angle ψ is less than that of the current by $\frac{\pi}{2}$.

If, then, in Fig. 9 we take a vector $OA = I$ to represent the current, the hydromotive force will be represented by the vector $OB = H$ turned back through an angle $\frac{\pi}{2}$. The two vectors turn as though rigidly connected to each other and their projections on the axis OX represent at any instant the values of i and h respectively.

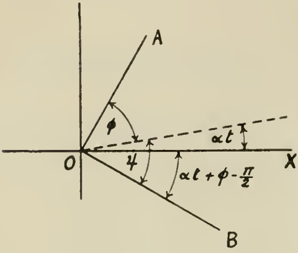

FIG. 9.

Since numerically

$$H = \frac{I}{aC}$$

we may write symbolically

$$(H) = -j\frac{I}{aC}$$

$-j$ being a symbol indicating that the vector H is $\frac{\pi}{2}$ or 90° behind the vector I.

Multiplying both sides of the symbolic equation by jaC we have

$$-j^2I = jaCH$$

which represents that the quantity $-j^2I$ is $\frac{\pi}{2}$ in advance of the hydromotive force H.

But we have found that the hydromotive force H is $\frac{\pi}{2}$ behind the current I, *i.e.*, the current I is $\frac{\pi}{2}$ in advance of the hydromotive force H. It follows, therefore, that $-j^2 = 1$ and $j^2 = -1$.

Thus $-j \times -j = j^2 = -1$ will signify a retardation of phase of $-\frac{\pi}{2} - \frac{\pi}{2} = -\pi$, and $j \times j = j^2 = -1$, an advance of phase of $\frac{\pi}{2} + \frac{\pi}{2} = \pi$.

The equation found may be written

$$(I) = jaCH \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

We shall use the equation connecting the current capacity and hydromotive force in this form.

Relation between Current, Inertia, and Hydromotive Force.—Let L be the coefficient of inertia, then if the current is of the form of

$$i = I \sin (at + \phi)$$

we have

$$h = L\frac{di}{dt} = LIa \cos (at + \phi)$$

$$= LIa \sin \left(at + \phi + \frac{\pi}{2}\right)$$

If we put $H = LIa$ we have

$$h = H \sin\left(at + \phi + \frac{\pi}{2}\right)$$

It is seen, therefore, that the hydromotive force is of a sine form and is $\frac{\pi}{2}$ in advance of the current; we may, therefore, write, using the symbolic notation

$$(H) = jLaI \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

which indicates that the hydromotive force is represented by the projection on OX of the vector H equal numerically to LaI and in advance of the vector representing I by the angle $\frac{\pi}{2}$.

Leakage.—If the current is of sine form, since we have $i = Sh$ where S is the *coefficient of porosity*, h is, therefore, of sine form and in phase with the current, and we may write (*see* p. 24)

$$I = SH \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Friction.—For friction also we have

$$H = RI \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

in phase with the current (*see* p. 18).

Combined Effect of Friction, Capacity, Inertia and Leakage in a Pipe.—If we have in series in a pipe, friction, capacity, inertia and leakage, and we write

$$H_1 = RI, \quad H_2 = -j\frac{I}{aC}, \quad H_3 = jLaI, \quad H_4 = \frac{I}{S}$$

the hydromotive force between the ends of the pipe will be

$$(H) = H_1 + H_2 + H_3 + H_4$$

$$= \left(R + \frac{1}{S}\right) I + j\left(La - \frac{1}{Ca}\right) I \quad . \quad . \quad . \quad . \quad (5)$$

It will be seen that the vector H is the resultant of one vector $\left(R + \frac{1}{S}\right) I$, in phase with I; and another vector $\left(La - \frac{1}{Ca}\right) I$, $\frac{\pi}{2}$ in advance of I.

Drawing the vector diagram it is seen that the numerical value of H will be

$$H = I\sqrt{\left(R + \frac{1}{S}\right)^2 + \left(La - \frac{1}{Ca}\right)^2}$$

and that the vector H will be in advance of the vector I by the angle ψ given by the equation

$$\tan\psi = \frac{La - \frac{1}{Ca}}{R + \frac{1}{S}}$$

We see that $\psi = 0$ when $LCa^2 = 1$.

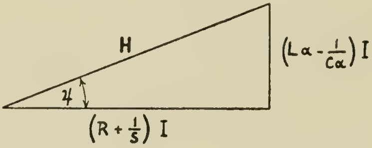

FIG. 10.

This condition corresponds with a state of *resonance* between the capacity and the inertia.

The quantity $La - \frac{1}{Ca}$ we shall term *reactance*; it is of the same dimensions as the coefficient of friction, but differs from friction in that the hydromotive force due to a frictional resistance is in phase with the current, while the hydromotive force due to a reactance differs in phase by 90° from the current.

It should be noted that the reactance may be considered to be produced only by inertia putting $L_1 = \frac{1}{Ca^2}$ and considering capacity as a negative inertia; thus the reactance would be simply

$$(L - L_1)a$$

An inertia may be considered as a negative capacity given by

$$C_1 = \frac{1}{La^2}$$

In the same way we may consider a resistance as a symbolical inertia $(R) = jaL_2$; and a leakage as a symbolical capacity $(S) = jaC$; for example, the value of the equivalent inertia for a resistance R would be

$$(L_2) = -j\frac{R}{a}$$

The effect of putting resistances, inertias, capacities and leakages in parallel may be similarly found.

In this case, if we denote by R, S, L, C the several values in parallel we have —

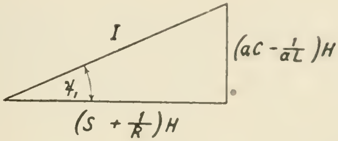

FIG. 11.

$$(I) = \left(S + \frac{1}{R}\right)H + j\left(aC - \frac{1}{aL}\right)H$$

and

$$\tan\psi_1 = \frac{aC_1 - \dfrac{1}{aL_1}}{S + \dfrac{1}{R}}$$

If we have several resistances, inertias, capacities and leakages in series, we have generally

$$(H) = \left(\Sigma R + \Sigma\frac{1}{S}\right)I + j\left(a\Sigma L - \frac{1}{a}\Sigma\frac{1}{C}\right)I$$

and the condition of resonance will be

$$a^2\Sigma L = \Sigma\frac{1}{C}$$

In the same way for several currents at a point we have

$$I = I_1 + I_2 + I_3 + \ldots$$

I being the geometrical sum of the vectors representing

$$I_1, I_2, I_3, \text{etc.}$$

Similarly we get the formula for I

$$(I) = \left(\Sigma S + \Sigma\frac{1}{R}\right)H + j\left(a\Sigma C - \frac{1}{a}\Sigma\frac{1}{L}\right)H$$

If the current or sonomotive pressure are general functions, we have the corresponding relations

$$h_1 = Ri; \quad h_2 = \frac{1}{C}\int idt; \quad h_3 = L\frac{di}{dt}; \quad h_4 = \frac{i}{S}$$

and the sonomotive pressure between the ends of a pipe having friction, capacity, inertia and leakage *in series* will be

$$h = h_1 + h_2 + h_3 + h_4$$

$$= Ri + \frac{1}{C}\int idt + L\frac{di}{dt} + \frac{i}{S}$$

and similarly the current required in a pipe to which friction, capacity, inertia and leakage are connected *in parallel* or to a common point is given by

$$i = Sh + C\frac{dh}{dt} + \frac{1}{L}\int hdt + \frac{h}{R}$$

Mechanical Work.—The mechanical work done by an alternating current may be calculated as follows:

Let dx be the movement of a cross section of the liquid in a pipe of area ω during a time dt. The work done by the displacement of the liquid between two points under a hydromotive force h would be

$$dW = h\omega dx = h\omega\frac{dx}{dt}dt$$

but since $\frac{dx}{dt}$ represents the instantaneous velocity in the pipe

$$\omega\frac{dx}{dt} = i$$

so that we have

$$dW = hidt$$

and

$$W = \int_0^t hidt$$

If h and i are simple sine functions of the form

$$h = H \sin (at + \psi_1)$$

$$i = I \sin (at + \psi_2)$$

we have

$$W = HI\int_0^t \sin (at + \psi_1) \sin (at + \psi_2)dt$$

taking into consideration that

$$\sin (at + \psi_1) \sin (at + \psi_2) = \tfrac{1}{2}\Big[\cos (\psi_1 - \psi_2) - \cos (2at + \psi_1 + \psi_2)\Big]$$

we get

$$W = \frac{HI}{2}\left[(t - t_0)\cos(\psi_1 - \psi_2) - \frac{1}{2a}\{\sin(2at + \psi_1 + \psi_2) - \sin(2at_0 + \psi_1 + \psi_2)\}\right]$$

t and t_0 being the limits of time between which we are considering the work done.

We see from this formula that the mean work per second during one period is

$$W = \frac{HI}{2}\cos(\psi_1 - \psi_2)$$

The angle $\psi_1 - \psi_2 = \phi$ is the angle between the vectors representing H and I in Fig. 12.

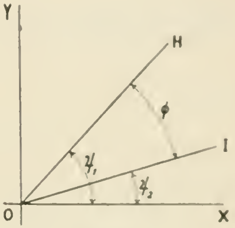

FIG. 12.

Since

$$\frac{HI}{2} = \frac{H}{\sqrt{2}} \times \frac{I}{\sqrt{2}} = H_{\text{eff.}} \times I_{\text{eff.}}$$

we can write

$$W = H_{\text{eff.}} \times I_{\text{eff.}} \times \cos\phi$$

Thus the work per second is represented by the product of the vectors representing the effective hydromotive force and the effective current multiplied by the cosine of the angle between these vectors. This cosine may be termed the *power factor.*

If, as in Fig. 13, we represent the current by I and the hydromotive force by

$$(H) = H_1 + jH_2$$

we see that

$$W = \tfrac{1}{2}HI\cos\phi = \tfrac{1}{2}H_1 I$$

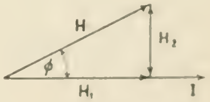

FIG. 13.

Thus only the component H_1 produces work, the other component of H, namely H_2, is a *workless* component. Generally we may say that work is produced only by vectors H and I which are parallel or in phase.

If we have

$$(H) = H_1 + jH_2$$

$$(I) = I_1 + jI_2$$

the work would be the sum of the work corresponding to the two pairs of vectors which are parallel to each other, namely, H_1 and I_1, and H_2 and I_2; thus

$$W = \tfrac{1}{2}H_1I_1 + \tfrac{1}{2}H_2I_2$$

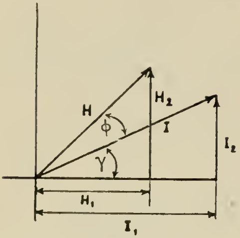

FIG. 14.

This result can be arrived at in another way; noting that if γ (Fig. 14) is an arbitrary angle and we have—

$$H_1 = H \cos(\phi + \gamma)$$
$$H_2 = H \sin(\phi + \gamma)$$
$$I_1 = I \cos\gamma$$
$$I_2 = I \sin\gamma$$

we have

$$\cos\phi = \cos\gamma\cos(\phi + \gamma) + \sin\gamma\sin(\phi + \gamma).$$

Multiplying this expression by $\tfrac{1}{2}HI$ we find the formula given above

$$W = \tfrac{1}{2}H_1I_1 + \tfrac{1}{2}H_2I_2$$

Care must be taken not to multiply the symbolical values of H and I in order to get the work in symbolic form, because the work cannot be expressed in symbolic form. The mean work per second is not a periodic function of the time, but is simply an arithmetical value; a symbolic expression for the work, therefore, is meaningless.

In order to obtain the expression for the mechanical power or the energy per second from the symbolic expressions of H and I, we must take the mean of the products of the parallel components. As a general rule components of H and I which are parallel are performing work, and components which are in *quadrature* (*i. e.* which differ in phase by 90°) do not perform work.

Thus the pressures which result from a current passing through inertias or capacities do not produce work or do not require any work to maintain the motion, because these pressures differ in phase by 90° from the current.

Potential Energy of a Capacity.—The work stored in a capacity C can be calculated from the formula

$$W = \int hidt$$

but we have found that the relation between the current and hydromotive force is

$$h = \frac{1}{C}\int idt$$

or

$$idt = Cdh$$

Substituting, we get

$$W = C\int h dh = \frac{Ch^2}{2} + A$$

A being a constant of integration.

If we assume that the work is zero when $h = 0$, we get simply

$$W = \frac{Ch^2}{2}$$

If h is a periodic function the maximum value of W would be

$$W = \frac{CH^2}{2}$$

Kinetic Energy of an Inertia.—In the same way we get the work accumulated in an inertia by the formula

$$W = \int hidt$$

but we have found that

$$h = L\frac{di}{dt}$$

So that by a method similar to that followed above we get for the maximum value of the kinetic energy stored in an inertia

$$W = \frac{LI^2}{2}$$

The following are examples of the application of the theory to certain cases of alternating currents in circuits containing separate capacities and inertias.

Example I

Find the period of resonance of a vibrating circular plate (*see* page 24).

The coefficient of inertia of the equivalent piston would be

$$L = \frac{P_1}{g\Omega_1^2}$$

The capacity of the plate

$$C = A\Omega\Omega_1$$

The condition of resonance of such a plate is $LCa^2 = 1$, a being equal to $2\pi n$ where n is the number of vibrations per second. Thus we have

$$1 = LCa^2 = \frac{P_1 A\Omega\Omega_1 a^2}{g\Omega_1^2} = \frac{P_1 A\Omega}{g\Omega_1}a^2$$

but

$$P_1 = P\frac{\Omega_1}{\Omega}$$

$$\therefore \frac{APa^2}{g} = 1$$

and

$$a = \sqrt{\frac{g}{AP}} = \frac{s}{\Omega}\sqrt{\frac{Eg}{\gamma u}}$$

where γ is the specific gravity of the plate.

Example II

Assume we have a generator *a* producing an alternating flow of liquid in pipes *bc* which are connected between the points *b*, *c* by a pipe containing a condenser *d* whose capacity is *C*.

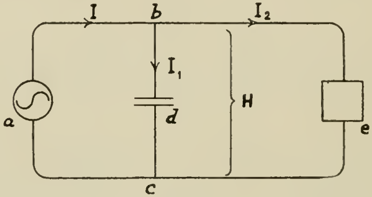

Fig. 15.

Suppose that the circuit is closed through a reciprocating implement *e*, whose coefficient of inertia as above defined is *L*. We can find the different currents and the hydromotive force.

Let

I = current in *ab*
I_1 = current in *bdc*
I_2 = current in *bec*
H = hydromotive force between *b* and *c*.

Then, considering the circuits *bdc* and *bec* separately

$$(H) = jLaI_2$$

and

$$(I_1) = jCaH$$

from which

$$I_1 = -LCa^2I_2$$

but

$$I = I_1 + I_2$$

therefore

$$I = I_2(1 - LCa^2)$$

hence

$$I_2 = \frac{I}{1 - LCa^2}$$

$$I_1 = -\frac{LCa^2}{1 - LCa^2}$$

and

$$(H) = j\frac{La}{1 - LCa^2}I$$

It will be seen that I_2 may be greater than I and that H and the currents I_1 I_2 become infinite if the condition of resonance $LCa^2 = 1$ is fulfilled.

Assume that there is friction R in the circuit *bec*.

Then we have

$$(H) = (R + jLa)I_2$$

$$(I_1) = jCaH$$

So that

$$(I_1) = (-LCa^2 + jRCa)I_2$$

and

$$(I) = (1 - LCa^2 + jRCa)I_2$$

If

$LCa^2 = 1$, the condition for resonance,

we have

$$(I) = jRCaI_2$$

i.e.

$$(I_2) = -j\frac{I}{RCa}$$

and

$$(H) = \left(-j\frac{1}{Ca} + \frac{L}{RC}\right)I$$

So that arithmetically

$$H = \frac{I}{Ca}\sqrt{1 + \left(\frac{La}{R}\right)^2}$$

With R sufficiently small and the frequency sufficiently high to neglect the term 1 in comparison with

$$\left(\frac{La}{R}\right)^2$$

we get

$$H = \frac{LI}{CR} = \frac{L^2a^2I}{R}$$

Example III

Let a (Fig. 16) be a generator producing alternating currents in pipes which are connected by two pipes containing capacities, inertias and resistances $C_1L_1R_1$ and $C_2L_2R_2$ respectively, the two connecting pipes being in parallel. This corresponds in practice with the case of two working implements placed in parallel across the transmission lines.

Find the currents I_1 and I_2 which will give a hydromotive force H.

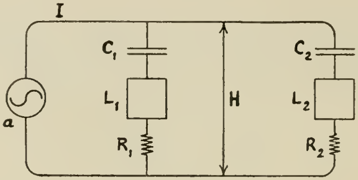

Fig. 16.

We have the equations

$$(H) = I_1(R_1 + jaL') = I_2(R_2 + jaL'')$$

where

$$L' = L_1 - \frac{1}{a^2C_1} \text{ and } L'' = L_2 - \frac{1}{a^2C_2}$$

$$\therefore \frac{I_1}{I_2} = \frac{R_2 + jaL''}{R_1 + jaL'}$$

but

$$(I) = (I_1) + (I_2)$$

so that

$$(I_1) = I\frac{R_2 + jaL''}{R + jaL}$$

$$(I_2) = I\frac{R_1 + jaL'}{R + jaL}$$

where

$$R = R_1 + R_2 \text{ and } L = L' + L''$$

Thus we find

$$(H) = I\frac{(R_1 + jaL')(R_2 + jaL'')}{R + jaL}$$

and to find the numerical values of the vectors I_1 and I_2

we have

$$I_1 = I\sqrt{\frac{R_1^2 + (aL'')^2}{R^2 + a^2L^2}}$$

$$I_2 = I\sqrt{\frac{R_1^2 + (aL')^2}{R^2 + a^2L^2}}$$

while for the numerical value of H we have

$$H = I\sqrt{\frac{[R_1^2 + (aL')^2][R_2^2 + (aL'')^2]}{R^2 + a^2L^2}}$$

Example IV

Suppose, as in Figure 17, we have a generator a having a single piston generating an alternating current in a single line. An iron vessel containing a volume of liquid of capacity C is placed in parallel on the line near the generator. Such a capacity may be considered as an ordinary condenser one of whose sides is maintained at a constant pressure equal to the mean pressure in the main pipe. In such a case the difference of pressure on the two sides of the condenser is equal

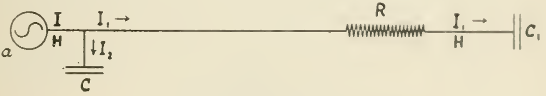

Fig. 17.

to the variation of pressure in the main pipe above and below the mean pressure. Let R be a friction device absorbing energy, for example a tube of small diameter adapted to be heated by the alternating current, and let C_1 be a second capacity formed by a second iron vessel. Assume that we know the current I and the angular velocity a of the equivalent crank. Find the hydromotive force H and the work done by the generator and absorbed in heating the friction device.

We have the equations

$$(I_2) = jaCH$$

$$(I_1) = jaC_1H_1$$

$$H - H_1 = RI_1$$

$$(I) = (I_1) + (I_2)$$

from which

$$(I_1) = \frac{jaC_1H}{1 + jaC_1R}$$

and

$$(I) = \frac{jaC_1H}{1 + jaC_1R} + jaCH$$

$$= \frac{ja(C + C_1) - a^2CC_1R}{1 + jaC_1R} H$$

Arithmetically

$$I = Ha\sqrt{\frac{(C + C_1)^2 + (aCC_1R)^2}{1 + (aC_1R)^2}}$$

and

$$H = \frac{I}{a}\sqrt{\frac{1 + (aC_1R)^2}{(C + C_1)^2 + (aCC_1R)^2}}$$

In symbolic notation we may write

$$(I) = \frac{[j(C + C_1) - aCC_1R][1 - jaC_1R]}{1 + (aC_1R)^2}Ha$$

$$= \frac{Ha}{1 + (aC_1R)^2}\left[RaC_1^2 + j(C + C_1 + RCa^2C_1^2)\right]$$

The current I thus has a component in phase with H given by

$$I' = \frac{HRa^2C_1^2}{1 + (aC_1R)^2}$$

so that the mechanical power absorbed will be

$$W = \frac{HI'}{2} = \frac{H^2a^2C_1^2R}{2[1 + (aC_1R)^2]}$$

Or, replacing H by its arithmetical value in terms of I,

$$W = \frac{I^2C_1^2R}{2[(C + C_1)^2 + (aCC_1R)^2]}$$

The value of the friction for which the mechanical power W is a maximum is that given by

$$\frac{dW}{dR} = 0$$

$$R = \frac{C + C_1}{aCC_1}$$

If R has this value, we have

$$W_{max.} = \frac{I^2 C_1}{4aC(C + C_1)}$$

and

$$H = \frac{I}{a(C + C_1)} \sqrt{1 + \frac{C_1}{C} + \frac{1}{2}\left(\frac{C_1}{C}\right)^2}$$

Numerical Example

Suppose we have a generator consisting of an oscillating piston whose stroke is 6 cm., section of piston 5 sq. cm. and number of revolutions per minute 955. Assume that the capacity C is a steel bottle of volume 5000 cubic cm. and the capacity C_1 a steel bottle of volume 2000 cubic cm., the liquid employed being water having a coefficient of elasticity 20000 kg. per sq. cm., neglecting the effect of the liquid in the pipes,
we have

$$a = 2\pi n = 2\pi \frac{955}{60} = 100$$

$$I = ra\omega = 3 \times 100 \times 5 = 1500 \text{ cm.}^3 \text{ sec.}$$

$$C = \frac{V}{E} = \frac{5000}{20000} = 0{\cdot}25$$

$$C_1 = \frac{2000}{20000} = 0{\cdot}1$$

The maximum mechanical power at the generator will correspond to a resistance

$$R = \frac{C + C_1}{aCC_1} = \frac{0{\cdot}25 + 0{\cdot}1}{100 \times 0{\cdot}25 \times 0{\cdot}1} = 0{\cdot}14$$

and the power will be

$$W = \frac{1500^2 \times 0{\cdot}1}{4 \times 100 \times 0{\cdot}25(0{\cdot}25 + 0{\cdot}1)}$$

$$= 6440 \text{ kg. cm.} = 0{\cdot}85 \text{ h.p.}$$

The hydromotive force H will be

$$H = \frac{1500}{100(0{\cdot}25 + 0{\cdot}1)} \sqrt{1 + \frac{0{\cdot}1}{0{\cdot}25} + \frac{1}{2}\left(\frac{0{\cdot}1}{0{\cdot}25}\right)^2}$$

$$= 52 \text{ kg. cm.}^2$$

It is interesting to note that the product

$$\frac{HI}{2} = \frac{52 \times 1500}{2} = 39000 \text{ kg. cm.}$$

gives an apparent horse-power of 5·1, so that the power factor at the generator will be

$$\cos\phi = \frac{0{\cdot}85}{5{\cdot}1} = 0{\cdot}167$$

The numerical value of I_1 and I_2 can also be found.
We have

$$I_2 = aCH = 100 \times 0{\cdot}25 \times 52$$

$$= 1300 \text{ cm.}^3 \text{ sec.}$$

$$I_1 = \frac{aC_1H}{\sqrt{1 + (aC_1R)^2}} = \frac{100 \times 0{\cdot}1 \times 52}{\sqrt{1 + (100 \times 0{\cdot}1 \times 0{\cdot}14)^2}}$$

$$= 302 \text{ cm.}^3 \text{ sec.}$$

Assume that the friction is in the form of a tube of 0·6 cm. internal diameter; the section ω will be

$$\omega = \frac{\pi}{4} \cdot 0{\cdot}6^2 = 0{\cdot}282$$

The maximum velocity will be

$$v = \frac{I}{\omega} = \frac{302}{0{\cdot}282} = 1070 \text{ cm. sec.}$$

and the effective velocity will be

$$v_{\text{eff.}} = \frac{1070}{\sqrt{2}} = 756 \text{ cm. sec.}$$

The coefficient of friction from Chapter III will be

$$k = \frac{v_{\text{eff.}}}{100d} + \frac{9}{100d}\sqrt{\frac{v_{\text{eff.}}}{d}}$$

$$= \frac{756}{100 \times 0{\cdot}6} + \frac{9}{100 \times 0{\cdot}6}\sqrt{\frac{756}{0{\cdot}6}}$$

$$= 17{\cdot}9$$

So that the coefficient of friction

$$R = k\frac{l}{\omega_1 \times 10^6} = \frac{17{\cdot}9l}{10^6 \times 0{\cdot}282} = 0{\cdot}14$$

which gives us

$$l = \frac{0{\cdot}14 \times 10^6 \times 0{\cdot}282}{17{\cdot}9} = 2280 \text{ cm.}$$

We see that such a tube is too long to use in practice, so that it will be necessary to employ a tube of smaller diameter.

If we take a tube of diameter 0·32 cm. we get

$$\omega = \frac{\pi}{4}(0{\cdot}32)^2 = 0{\cdot}08 \text{ cm.}^2$$

The velocity

$$v = \frac{300}{0{\cdot}08} = 3750 \text{ cm. sec.}$$

$$v_{\text{eff.}} = 2650 \text{ cm. sec.}$$

$$k = 105$$

and

$$l = \frac{0{\cdot}14 \times 10^6 \times 0{\cdot}08}{105} = 107 \text{ cm.}$$

It should be noted what a great difference in resistance is produced by a small change in the diameter of the pipe which forms the friction device.

CHAPTER V

WAVES IN LONG PIPES

Alternating Flow in Long Pipes.—In the preceding chapter we have discussed the theoretical conditions governing flow of alternating currents in circuits containing capacity, inertia and resistance, without consideration of the distributed capacity and inertia of the liquid in the pipes. In long-distance transmission it is necessary to consider the effect of the capacity and inertia of the liquid itself.

Let the resistance, inertia, leakage and capacity of a long pipe be per unit length R, L, S and C respectively; and consider a portion dx of the pipe; and let (H), (I) be the symbolic or vectorial values of the hydromotive force and current respectively.

If $d(H)$ be the hydromotive force causing movement of the liquid mass in the portion dx, and $d(I)$ the difference of current between the ends of the portion dx, we have (see pages 31, 33)

$$d(H) = (I)Rdx + ja(I)Ldx$$

$$d(I) = (H)Sdx + ja(H)Cdx$$

so that

$$\frac{d(H)}{dx} = R(I) + jaL(I)$$

and

$$\frac{d(I)}{dx} = S(H) + jaC(H)$$

If we put

$$(L) = L - j\frac{R}{a}$$

and

$$(C) = C - j\frac{S}{a}$$

these equations become

$$\frac{d(H)}{dx} = ja(L)(I)$$

$$\frac{d(I)}{dx} = ja(C)(H)$$

Differentiating these equations, we get

$$\frac{d^2(H)}{dx^2} = -a^2(C)(L)(H)$$

$$\frac{d^2(I)}{dx^2} = -a^2(C)(L)(I)$$

If we now put

$$\mu = a\sqrt{(L)(C)}$$

we get

$$\frac{d^2(H)}{dx^2} + \mu^2(H) = 0$$

and

$$\frac{d^2(I)}{dx^2} + \mu^2(I) = 0$$

The general solution of these equations is

$$(H) = A \sin \mu x + B \cos \mu x \quad \ldots \ldots \quad (1)$$

$$(I) = A_1 \sin \mu x + B_1 \cos \mu x \quad \ldots \ldots \quad (2)$$

To determine the constants, let us consider the end of the pipe remote from the generator, and let the hydromotive force and current at this end be (H) and (I) respectively.

Then we have for $x = 0$

$$B = (H)$$

and

$$B_1 = (I)$$

From equations (1) and (2) we get by differentiation

$$\frac{d(H)}{dx} = ja(L)(I) = \mu(A \cos \mu x - B \sin \mu x)$$

$$\frac{d(I)}{dx} = ja(C)(H) = \mu(A_1 \cos \mu x - B_1 \sin \mu x)$$

so that for $x = 0$

$$A = \frac{ja(L)(I)}{\mu}$$

and

$$A_1 = j\frac{a(C)(H)}{\mu}$$

If (H_0) and (I_0) be the values of (H) and (I) at the generator end, the length of the pipe being l, we have

$$(H_0) = (H) \cos \mu l + j(I) \sqrt{\frac{(L)}{(C)}} \sin \mu l \quad \ldots \ldots \quad (3)$$

$$(I_0) = (I) \cos \mu l + j(H) \sqrt{\frac{(C)}{(L)}} \sin \mu l \quad \ldots \ldots \quad (4)$$

These equations are general, and give the complete solution of the problem. The quantities (L) and (C) are symbolic, so that $\mu = a\sqrt{(L)(C)}$ will also be a symbolic quantity.

If R and S are very small, or when a is very large, the values of (L) and (C) are practically equal to L and C, the complex terms $j\frac{R}{a}$ and $j\frac{S}{a}$ becoming negligible. Under these conditions the general formulæ (3) (4) become

$$(H_0) = (H)\cos\mu l + j(I)\sqrt{\frac{L}{C}}\sin\mu l$$

$$(I_0) = (I)\cos\mu l + j(H)\sqrt{\frac{C}{L}}\sin\mu l$$

and we see that the movement follows a sine law with reference to the variable l as well as with reference to the time. It will also be seen that for values of l given by

$$\mu l = 2k\pi$$

$$(H_0) = (H)$$

and

$$(I_0) = (I)$$

i.e., (H_0) and (I_0) are harmonic functions of l.

The length λ given by

$$\mu\lambda = 2\pi$$

is the *wave length.*

These conclusions, however, only hold in so far as $\frac{R}{a}$ and $\frac{S}{a}$ are negligible in comparison with L and C.

The general formulæ (3) (4) may also be written

$$(H_0)\sqrt{(C)} = (H)\sqrt{(C)}\cos\mu l + j(I)\sqrt{(L)}\sin\mu l$$

$$(I_0)\sqrt{(L)} = (I)\sqrt{(L)}\cos\mu l + j(H)\sqrt{(C)}\sin\mu l$$

putting

$$h = (H)\sqrt{(C)} \qquad i = (I)\sqrt{(L)}$$

$$h_0 = (H_0)\sqrt{(C)} \qquad i_0 = (I_0)\sqrt{(L)}$$

and

$$\mu l = \alpha$$

The equations become

$$h_0 = h\cos\alpha + ji\sin\alpha \quad . \quad . \quad . \quad (5)$$

$$i_0 = i\cos\alpha + jh\sin\alpha \quad . \quad . \quad . \quad (6)$$

also

$$h = h_0\cos\alpha - ji_0\sin\alpha \quad . \quad . \quad . \quad (7)$$

$$i = i_0\cos\alpha - jh_0\sin\alpha \quad . \quad . \quad . \quad (8)$$

We shall call the quantities h and i *proportional pressure* and *proportional current.*

To change formulæ in terms of h and i to formulæ in terms of (H) and (I) we have only to replace h by $(H)\sqrt{(C)}$ and i by $(I)\sqrt{(L)}$.

It will be seen that, neglecting R and S, the proportional pressure

$$h = H\sqrt{C}$$

and the proportional current

$$i = I\sqrt{L}$$

The maximum value of the potential energy in unit length of pipe is

$$\frac{H^2C}{2}$$

and the maximum value of the kinetic energy is

$$\frac{I^2L}{2}$$

We see, therefore, that in this case the proportional pressure is equal to the square root of twice the maximum potential energy, while the proportional current is equal to the square root of twice the maximum kinetic energy.

Since we have

$$h_0 = h \cos \alpha + ji \sin \alpha$$

and

$$i_0 = i \cos \alpha + jh \sin \alpha$$

we see that arithmetically

$$h_0^2 + i_0^2 = h^2 \cos^2 \alpha + i^2 \sin^2 \alpha + i^2 \cos^2 \alpha + h^2 \sin^2 \alpha = h^2 + i^2$$

So that the sum of the maximum kinetic and potential energy per unit length is constant along the pipe.

It will be convenient to apply these ideas of proportional currents and proportional pressures to cases in which appliances such as condensers and inertias are connected to the pipes. Thus the equations giving the relation between current and hydromotive force found in Chapter IV, namely

$$(I) = jaCH$$

and

$$(H) = jaLI$$

may be written

$$(I)\sqrt{(L)} = jaCH\sqrt{(C)}\sqrt{\frac{(L)}{(C)}}$$

$$(H)\sqrt{(C)} = jaLI\sqrt{(L)}\sqrt{\frac{(C)}{(L)}}$$

If we put

$$\psi = \sqrt{\frac{(C)}{(L)}}$$

we have

$$\psi i = jaCh \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and

$$h = jaL\psi i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

It will be seen that these are identical with the ordinary formulæ, I being replaced by ψi and H by h.

Inversely, to transform formulæ in terms of i and h into the ordinary formulæ it is only necessary to replace h by ψH and i by I.

If R and S are negligible, or if the frequency is very high, we have

$$\psi = \sqrt{\frac{C}{L}}$$

For unit length of pipe we have found

$$L = \frac{\gamma}{g\omega}$$

and

$$C = \frac{V}{E} = \frac{\omega}{E}$$

so that

$$\psi = \omega\sqrt{\frac{g}{\gamma E}}$$

For water

$$\psi = 7\omega,$$

For mineral oil

$$\psi = 9\omega$$

approximately.

Some Useful Formulæ.—If R and S are negligible, we have found above

$$\mu\lambda = 2\pi,$$

but

$$a = \mu l$$

and

$$\mu = a\sqrt{LC}$$

Also we have found

$$\sqrt{LC} = \sqrt{\frac{\gamma}{gE}} = \text{constant}.$$

The *wave velocity* v equals the wave length divided by the period of one pulsation.

so that

$$v = \frac{\lambda}{\frac{2\pi}{a}} = \frac{a}{\mu}$$

$$= \frac{1}{\sqrt{LC}} = \sqrt{\frac{gE}{\gamma}}$$

$$\mu = \frac{a}{v} = \frac{2\pi}{\lambda} = \frac{2\pi n}{v} = \frac{\alpha}{l}$$

For water v is 146000 cm. per second, and the wave velocity is equal to the speed of sound in water. The wave velocity varies to a certain extent with the pressure and the temperature; a table giving the values of v and ψ for different pressures and temperatures is given in an Appendix.

For mineral oil v is about 125000 cm. per second.

Application of the Theory of Proportional Pressures and Currents

Example I

Suppose we have a long pipe (Fig. 18) with a condenser at each end. Let h_1, h_2 be the proportional pressures at the ends of the line and i_1, i_2 the proportional currents at these points.

Fig. 18.

Let $h_0 - h_1$ be the fall of proportional pressure in the first condenser and $h_2 - h$ that in the second.

Let $\mu l = \alpha$ as above. Then we have

$$h_1 = h_2 \cos \alpha + j i_2 \sin \alpha$$

$$i_1 = i_2 \cos \alpha + j h_2 \sin \alpha$$

but

$$i_0 = i_1$$

and

$$i_2 = i$$

so that

$$h_1 = h_2 \cos \alpha + j i \sin \alpha$$

$$i_0 = i \cos \alpha + j h_2 \sin \alpha$$

If the capacities of the condensers be C_1 and C_2

$$\psi i_0 = jaC_1(h_0 - h_1)$$

$$\psi i = jaC_2(h_2 - h)$$

$$\therefore \quad h_0 = h_1 - j\frac{\psi}{aC_1}i_0$$

$$h_2 = h - j\frac{\psi}{aC_2}i$$

or

$$h_0 = h_2 \cos a + ji \sin a - j\frac{\psi}{aC_1}i_0$$

$$h_0 = h\left(\cos a + \frac{\psi}{aC_1}\sin a\right) + ji\left[\left(1 - \frac{\psi^2}{a^2C_1C_2}\right)\sin a - \left(\frac{1}{C_1} + \frac{1}{C_2}\right)\frac{\psi}{a}\cos a\right]$$

and

$$i_0 = i \cos a + j\left(h - j\frac{\psi}{aC_2}i\right)\sin a$$

$$= i\left(\cos a + \frac{\psi}{aC_2}\sin a\right) + jh \sin a$$

If the capacities of the condensers be chosen equal, so that $C = C_1 = C_2$, and if, further,

$$\frac{\psi}{aC} = \frac{1 - \cos a}{\sin a} = \tan\frac{a}{2}$$

we have

$$i_0 = i + jh \sin a$$

and

$$h_0 = h$$

The condition assumed gives

$$C = \frac{\psi}{a}\cot\frac{a}{2} = \frac{\psi}{a}\tan\frac{\pi - a}{2} \quad \ldots \ldots \quad (1)$$

If, instead of (1), we have the condition

$$\frac{\psi}{aC} = \frac{\cos a - 1}{\sin a} = -\cot\frac{a}{2} = \cot\left(\pi - \frac{a}{2}\right)$$

$$C = \frac{\psi}{a}\tan\left(\pi - \frac{a}{2}\right) \quad \ldots \ldots \quad (2)$$

so that

$$i_0 = -i + jh \sin a$$

and

$$h_0 = -h$$

If the condition (1) is observed, we see that the value of the capacity C diminishes to zero as a approaches π, while under the condition (2) C approaches infinity as a approaches π.

It should be remembered that the condition $a = \pi$ is fulfilled when $l = \frac{\lambda}{2}$, *i.e.*, when the length of the pipe is one-half the wave length. It will be seen, therefore, that when a approaches π it is desirable, in order that small condensers may be used, to so design the condensers that there is *resonance in opposition*, i.e., so that $h_0 = -h$ and $i_0 = -i$.

The smallest capacity necessary in practice when there is resonance in opposition will be that corresponding to $a = \frac{3\pi}{2}$, when we have $C = \frac{\psi}{a}$.

This is also the smallest capacity necessary when there is *direct resonance* according to formula (1) with $a = \frac{\pi}{2}$.

It is to be noticed that the smaller the capacity the greater is the volume of the springs in a condenser.

A pipe having two equal condensers, one at each end, will be defined as a *balanced pipe* when we have

$$h_0 = h; \text{ or } h_0 = -h$$

irrespective of the values of the current i, which conditions will be fulfilled when

$$C = \frac{\psi}{a \tan \frac{a}{2}} \qquad \text{if } a < \frac{\pi}{2}$$

and

$$C = \frac{\psi}{a} \tan\left(\pi - \frac{a}{2}\right) \qquad \text{if } a > \frac{\pi}{2}$$

In the general formula found above

$$h_0 = h\left(\cos a + \frac{\psi}{aC_1} \sin a\right) + ji\left[\left(1 - \frac{\psi^2}{a^2 C_1 C_2}\right) \sin a - \left(\frac{1}{C_1} + \frac{1}{C_2}\right)\frac{\psi}{a} \cos a\right]$$

If we eliminate the condenser C_2, so that $C_2 = \infty$, we have

$$h_0 = h\left(\cos a + \frac{\psi}{aC_1} \sin a\right) + ji\left(\sin a - \frac{\psi}{aC_1} \cos a\right)$$

putting

$$\frac{\psi}{aC_1} = \tan \beta$$

we have

$$h_0 \cos \beta = h \cos (a - \beta) + ji \sin (a - \beta)$$

With this condition it is not possible to have $h_0 = h$, so that it is not possible to obtain complete balance by using a condenser at one end of the pipe only.

If we make

$$a = \beta$$

we get

$$h_0 = \frac{h}{\cos a}$$

The pressure would always be greater at the generator end than at the receiver end, but still independent of the current i.

Example II

Suppose we have a long pipe with an inertia at one point. First assume that the inertia is placed near the right-hand end, as in Fig. 19.

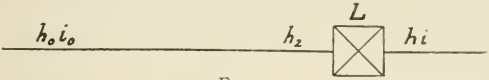

FIG. 19.

Let the proportional pressure and current at the left-hand end be h_0 and i_0 respectively, and the proportional pressure and current at the right-hand end h and i.

Let L be the coefficient of inertia and h_2 the proportional pressure on the left of the inertia.

Then we have

$$h_0 = h_2 \cos a + ji \sin a$$
$$i_0 = i \cos a + jh_2 \sin a$$

and

$$h_2 - h = j\psi aLi$$

so that

$$h_0 = h \cos a + ji(\psi aL + \sin a)$$
$$i_0 = i \cos a + j(h + j\psi aLi) \sin a$$
$$= i(\cos a - \psi aLi \sin a) + jh \sin a$$

If the inertia is at the left, as in Fig. 20,

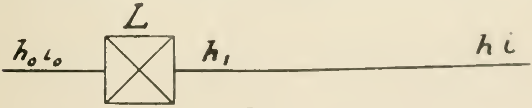

FIG. 20.

$$h_1 = h \cos a + ji \sin a$$
$$i_0 = i \cos a + jh \sin a$$
$$h_0 - h_1 = j\psi aLi_0$$
$$h_0 = h \cos a + ji \sin a + j\psi aL(i \cos a + jh \sin a)$$
$$= h(\cos a - \psi aL \sin a) + ji(\sin a + \psi aL \cos a)$$

putting

$$\psi a L = \tan \phi$$

we get

$$h_0 \cos \phi = h \cos (a + \phi) + ji \sin (a + \phi)$$

From the analogy with the previous example, we come to the conclusion that a pipe may be balanced by inertias instead of condensers.

It will also be readily seen that a balanced pipe is a length of pipe containing such inertia and capacity that it is equivalent, as regards the hydromotive force and current at its ends, to a uniform pipe of length equal to a multiple of half wave lengths of the alternating flow in the uniform pipe at the given periodicity.

Consider now the case in which two inertias are placed at the ends of the pipe, as in Fig. 21.

FIG. 21.

Since an inertia corresponds to a negative capacity (*see* page 32) it is only necessary to substitute in the equations obtained for the case of two condensers, putting $-La$ for $\frac{1}{Ca}$; we then get from the general equation, Example I,

$$h_0 = h(\cos a - \psi a L \sin a) + ji[(1 - \psi^2 a^2 L_1 L_2) \sin a + (L_1 + L_2) a \psi \cos a]$$

$$i_0 = i(\cos a - \psi a L_2 \sin a) + jh \sin a$$

The condition for balance will be

$$L_1 = L_2 = L$$

and

$$\cos a - \psi a L \sin a = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or

$$\cos a - \psi a L \sin a = -1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In the first case

$$h_0 = h$$

and

$$\psi a L = \frac{\cos a - 1}{\sin a} = -\tan \frac{a}{2} = \tan \left(\pi - \frac{a}{2}\right)$$

for values of

$$\left(\pi - \frac{a}{2}\right) < \frac{\pi}{2} \text{ or } > \pi \text{ and } < \frac{3\pi}{2}$$

In the second case

$$h_0 = -h$$

and

$$\psi a L_1 = \cot \frac{\alpha}{2}$$

for values of $\alpha < \pi$

EXAMPLE III

Assume we have a capacity C in parallel with a generator A on a closed pipe, as in Fig. 22.

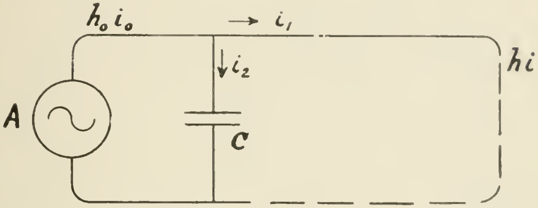

FIG. 22.

We have

$$h_0 = h \cos \alpha + ji \sin \alpha$$

$$i_1 = i \cos \alpha + jh \sin \alpha$$

$$\psi i_2 = ja C h_0$$

so that

$$i_0 = i_1 + i_2 = i \cos \alpha + jh \sin \alpha + j\frac{aC}{\psi} h_0$$

$$= i \cos \alpha + jh \sin \alpha + j\frac{aC}{\psi} h \cos \alpha - \frac{aC}{\psi} i \sin \alpha$$

$$= i\left(\cos \alpha - \frac{aC}{\psi} \sin \alpha\right) + jh\left(\sin \alpha + \frac{aC}{\psi} \cos \alpha\right)$$

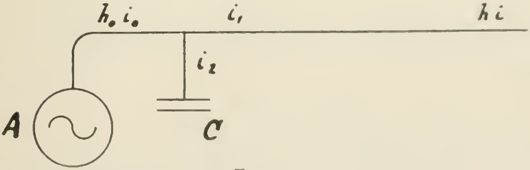

FIG. 23.

This equation is also applicable to the case shown in Fig. 23, in which the condenser C is replaced by a vessel of capacity C, and in which the pipe is closed by a plug or cock at the end remote from the generator.

In this case

$$i = 0$$

and we get

$$h_0 = h \cos \alpha$$

$$i_0 = jh\left(\sin \alpha + \frac{aC}{\psi} \cos \alpha\right) = jh_0\left(\tan \alpha + \frac{aC}{\psi}\right)$$

We therefore have

$$(I_0) = j\psi H_0\left(\tan \alpha + \frac{aC}{\psi}\right)$$

$$= jH_0(\psi \tan \alpha + aC)$$

and arithmetically

$$H_0 = \frac{I}{aC + \psi \tan \alpha}$$

H_0 is therefore infinite if

$$\tan \alpha = \frac{-aC}{\psi}$$

and is zero if

$$\alpha = \frac{\pi}{2}$$

Example IV

Assume that we have a generator at A, Fig. 24, giving a current I_0 at the generator end and I at a working apparatus at B, the value of the hydromotive force at the generator being H_0

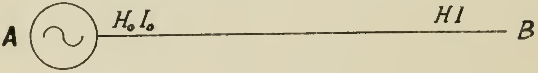

Fig. 24.

We can find the value of H at B and the available power.

We have

$$h_0 = h \cos \alpha + ji \sin \alpha$$

$$i_0 = i \cos \alpha + jh \sin \alpha$$

In the special case in which the working apparatus at B has no inertia or capacity, H and I will always be in phase, and we get arithmetically

$$I_0^2 = (I \cos \alpha)^2 + \psi^2(H \sin \alpha)^2$$

If H' be the value of H on shutting off the working apparatus

$$I = 0$$

and

$$I_0 = \psi H' \sin \alpha$$

$$H' = \frac{I_0}{\psi \sin \alpha}$$

It is evident that H' is always greater than H; but if I_0 is considerably greater than $I \cos \alpha$ there will be little change of sonomotive pressure at the generator on shutting off the working apparatus.

Numerical Example

Let

$$I = 500 \text{ cm.}^3; \ \cos \alpha = 0{\cdot}44; \ \sin \alpha = 0{\cdot}9, \text{ i. e. } \alpha = 63° \ 50'$$

$$\psi = 35; \ H = 100 \text{ kg./cm.}^2$$

Then we have

$$I_0 = \sqrt{(500 \times 0{\cdot}44)^2 + (90 \times 35)^2}$$

$$= 3157 \text{ cm.}^3$$

and

$$H^1 = \frac{3157}{35 \times 0{\cdot}9} = 100 \text{ kg./cm.}^2$$

The hydromotive force at the generator will be

$$H_0 = {\cdot}44 \times 100 = 44 \text{ kg./cm.}$$

when the working apparatus is shut off;
and

$$H_0 = \sqrt{(H \cos \alpha)^2 + \left(\frac{I \sin \alpha}{\psi}\right)^2}$$

$$= 45{\cdot}6 \text{ kg./cm.}^2$$

with the working apparatus in operation. It should be observed that the hydromotive force at the receiver is more than twice that at the generator.

Graphic Method of Calculating Currents in Pipes

By employing the graphic method given below we can arrive at the values of the different quantities which have to be determined in problems concerning alternating fluid currents in pipes without arithmetical calculation; the method, however, is applicable only if friction and leakage can be neglected.

Let h_0, i_0 be the proportional pressure and current at the generator and hi the values of these quantities at the receiver. We will take the general case in which h and i are out of phase.

We may then write

$$(h) = h_1 + jh_2$$

h_1 being the component of (h), which is in phase with i and h_2, the component which is in quadrature. The general equations connecting these quantities are

$$(h)_0 = (h) \cos \alpha + ji \sin \alpha$$

$$(i)_0 = i \cos \alpha + j(h) \sin \alpha$$

Substituting for (h), we get

$$\left.\begin{aligned}(h)_0 &= h_1 \cos \alpha + j(i \sin \alpha + h_2 \cos \alpha)\\ (i)_0 &= i \cos \alpha - h_2 \sin \alpha + jh_1 \sin \alpha\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (1)$$

In these equations α represents the quantity

$$\frac{2\pi l}{\lambda},$$

l being the length of the pipe and λ the wave length. Let us put

$$\frac{h_2}{i} = \tan \beta = \psi\left(La - \frac{1}{Ca}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

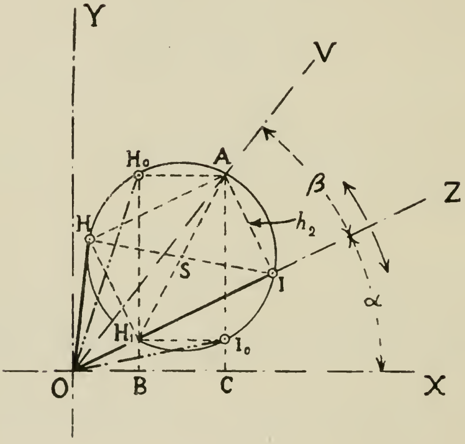

FIG. 25.

L and C being the inertia and capacity which is producing the difference of phase in the receiver. The constant ψ is a known quantity, its value for water being approximately given by

$$\psi = 7\omega$$

ω being the sectional area of the pipe.

Let us now construct the diagram Fig. 25, as follows—

Draw the straight line $OI = i$ making an angle α with the horizontal OX.

On the line OI cut off a part $OH_1 = h_1$.

Draw a straight line $H_1H = h_2$ at right angles to OI.

Join HI and on the line HI as diameter draw a circle; since the angle HH_1I is a right angle this circle will pass through the point H_1.

Through H_1 draw the horizontal line H_1I_0 and the ordinate H_1H_0, cutting the circle at the points I_0 and H_0 respectively.

We shall then have the vectors (i_0), and (h_0) given by

$$(i_0) = OI_0$$

and

$$(h_0) = OH_0$$

In order to prove this let us consider the points B, C at which the ordinates through H_1 and I_0 cut the line OX, and let A be the point of intersection of the ordinate through C with the circle.

The straight line H_0A is parallel to OX, so that the figure H_1I_0AH is a rectangle.

Similarly, the figure H_1IAH is a rectangle.

We have, therefore,

$$AI = HH_1 = h_2$$

On the other hand, we have symbolically

$$(OH_0) = OB + jBH_0$$

but

$$OB = OH_1 \cos a = h_1 \cos a$$

and

$$BH_0 = OI \sin a + IA \cos a$$

$$= i \sin a + h_2 \cos a$$

Substituting these values, we get

$$(OH_0) = h_1 \cos a + j(i \sin a + h_2 \cos a)$$

which, from the first of the formulae (1) set out above, gives

$$h_0 = OH_0$$

Similarly, we have

$$(OI_0) = OC + jCI_0$$

but

$$CI_0 = BH_1 = OH_1 \sin a = h_1 \sin a$$

and

$$OC = OI \cos a - IA \sin a$$

$$= i \cos a - h_2 \sin a$$

so that we have

$$OI_0 = i \cos a - h_2 \sin a + jh_1 \sin a$$

which, from the second of the formulæ (1), gives

$$i_0 = OI_0$$

Join the points O and A.

The angle AOI is of importance, for we have

$$\tan AOI = \frac{AI}{OI} = \frac{h_2}{i}$$

Comparing this with the relation (2) given above, we see that

$$\tan AOI = \frac{h_2}{i} = \psi\left(La - \frac{1}{Ca}\right) = \tan \beta$$

so that AOI is the angle β of formula (2). We see, therefore, that in the graphical construction the direction of the lines OX, OI and OA depends only on the constants of the pipe and the reactance of the receiver, and in no way depend on the values of h, i, h_0 and i_0.

By the help of this diagram several problems can be solved in a very simple manner; some of these we will now deal with.

Problem I

Suppose we have given the current i at the receiver, and that the work to be done gives the component of the hydromotive force which is in phase with h_1. Assume that we also know the reactance of the receiver and the angle a. We have to find the hydromotive force at the receiver, the angle of phase this makes with the current, and the hydromotive force, current and phase angle at the generator. We first draw the lines OX, OZ and OV, since the angles a and β are known.

Then on the line OZ (Fig. 26) set off the segments

$$OI = i$$

and

$$OH_1 = h_1$$

The centre of the circle of the diagram will lie on perpendicular to H_1I through its middle point.

The centre will also be the middle point of H_1A, but as we do not know the position of the point A on the line OV we trace the locus of the middle point of H_1A for different positions of A. It is evident that this locus is a straight line parallel to OV and bisecting the perpendicular from H_1 on OV.

The intersection of this line with the perpendicular to H_1I will give the centre S of the circle.

Drawing the circle with centre S and radius SI, the points H_0, H and I_0 are at once found, and the diagram is complete. To determine the values of h, h_0 and i_0 it is only necessary to scale the different lengths OH, OH_0 and OI_0.

The angle HOI is the angle of phase at the receiver, and the angle H_0OI_0 that at the generator.

PROBLEM II

Suppose that we have given the quantities i_0, h_1, α and β.

To construct the diagram, with centre O and radius OI_0 describe a circle, draw the lines OX, OZ, OV as before.

On OZ cut off $OH_1 = h_1$.

Through H_1 draw a line parallel to OX, cutting the circle at the point I_0.

Through I_0 draw a line perpendicular to OX; this will intersect the line OV at the point A.

The circle required for the diagram is then drawn on H_1A as diameter.

It will be evident that when we have found this circle all the other points are at once readily obtained.

Numerical Example

It is required to drive a pump which takes 2·5 hp. = 19000 kg. cm. sec.

The distance of transmission is 35 metres and frequency of the alternating current 400 per minute, *i. e.* 6·7 per second, so that

$$a = 2\pi \times 6{\cdot}7 = 42$$

The hydromotive force at the pump must not exceed 50 kg./cm.² The sectional area of the transmission pipe is $\omega = 2{\cdot}85$ cm.², so that

$$\psi = 7 \times 2{\cdot}85 = 20$$

The wave length will be

$$\lambda = \frac{1430^m}{6{\cdot}7} = 214^m = 21400 \text{ cm.}$$

Since the length of the pipe $l = 3500$ cm., we have

$$\alpha = \frac{2\pi l}{\lambda} = \frac{360^\circ \times 3500}{21400} = 59^\circ$$

If h_1 is the component of (h) in phase with the current at the pump (which is the receiver) and i is the proportional current, we have for the mechanical power

$$W = \frac{H_1 I}{2} = 19000 \text{ kg./cm.}$$

Let us assume an angle of phase at the receiver of 45°

We then have

$$H_1 = \frac{H}{\sqrt{2}} = \frac{50}{\sqrt{2}} = 35 \text{ kg./cm.}^2$$

from which we get

$$I = \frac{2 \times 19000}{35} = 1090 \text{ cm.}^3 \text{ sec.}$$

Since the general formulæ for h, i, h_0 and i_0 are homogeneous as regards these variables, we may replace h, h_0 throughout by their corresponding values H and H, and i, i_0 by $\frac{I}{\psi}$ and $\frac{I_0}{\psi}$.

Thus we can take

$$h_1 = 35, \quad h = 50, \quad i = \frac{1090}{20} = 54 \cdot 5$$

The reactance in the pump motor is provided by a spring whose capacity is C.

We then have

$$\tan \beta = \frac{h_2}{i} = -\frac{\psi}{Ca}$$

Since the angle of phase is 45°, we have

$$(h)_2 = -jh_1$$

or

$$h_2 = h_1$$

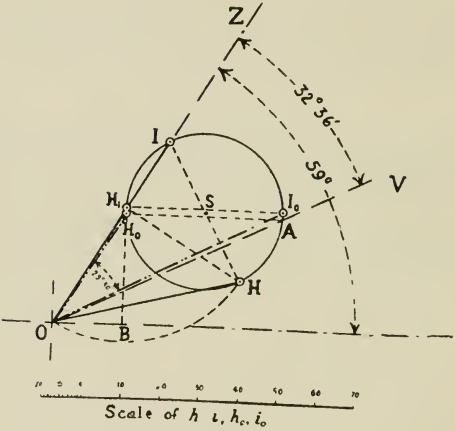

FIG. 26.

and, as β is negative, we have

$$\tan \beta = -\frac{35}{54 \cdot 5} = -0 \cdot 64$$

so that

$$\beta = -32° \, 36'$$

The necessary capacity will be

$$C = \frac{a \tan \beta}{\psi} = \frac{42 \times 0 \cdot 64}{20} = 1 \cdot 35$$

Constructing the diagram (Fig. 26) in the manner indicated in Problem I, we find

$$h_0 = OH_0 = 33$$

and

$$i_0 = OI_0 = 65$$

with an angle of phase

$$\phi = +29° \, 30'$$

From this we get

$$H_0 = 33 \text{ kg./cm.}^2$$

$$I_0 = \psi i_0 = 20 \times 65 = 1300 \text{ cm.}^3\text{/sec.}$$

As a means of verifying this we know that the product

$$W = \frac{H_0 I_0}{2} \cos \phi$$

gives the mechanical power. So that we have

$$W = \frac{33 \times 1300}{2} \cos 29^\circ\ 30' = 18600$$

The error is

$$19000 - 18600 = 400$$

that is, a relative error of

$$\frac{400}{19000} = 0{\cdot}021$$

or about 2 per cent., which is quite satisfactory.

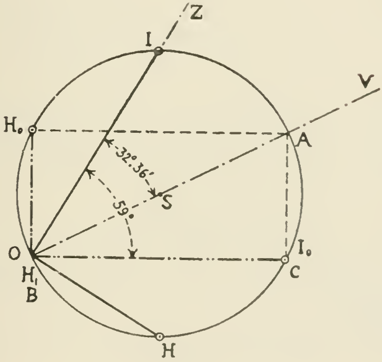

FIG. 27.

We will now find the values of the hydromotive force at the generator and receiver, and the current at the receiver when pump is running without doing work. The generator produces the same current I_0. In the case we are considering

$$h_1 = 0$$

and the circle of the diagram, Fig. 27, will pass through the point O.

Also the points H_1, B coincide with O.

It follows that if we draw a horizontal through O and on this set off $OI_0 = i_0$, we obtain the point I_0 on the circle.

Then by drawing the ordinate through I_0, meeting OV at A, we obtain the point A.

The circle can now be drawn on OA as diameter, so that the points I, H_0 and H can be at once obtained.

Thus we find

$$i = OI = 66{\cdot}5$$
$$h_0 = OH_0 = 39$$
$$h = OH = 36{\cdot}5$$

The angle of phase at the receiver and also at the generator is 90°; we have further

$$H_0 = 39 \text{ kg./cm.}^2$$
$$H = 36{\cdot}5 \text{ kg./cm.}^2$$
$$I = 66{\cdot}5 \times 20 = 1330$$

It is of interest to determine the limit of the mechanical power available at the receiver. We have to find the particular value of H_1 at which the value of I ceases to be a real quantity. Looking at the diagram, Fig. 28, we see that this is the case when the points H_1 and I on the circle coincide. Under these

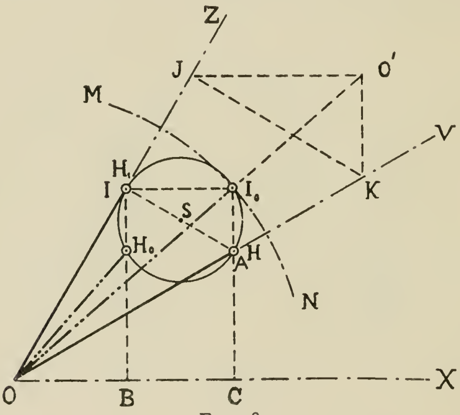

FIG. 28.

conditions the points A and H must also be coincident. The circle, therefore, will be such that the line OZ is a tangent, also the horizontal through the common point H_1, I should cut the circle at the point I_0. The construction to find the circle is, therefore, as follows—

With centre O and radius $OI_0 = i_0$ describe the arc MI_0N; draw the lines OZ, OV, OX as in the previous examples.

Then from any point J on the line OZ draw JK perpendicular to OZ, cutting OV at K. Through K draw the vertical line KO', cutting the horizontal through J at O'. Join OO' and let this cut the arc $MI'N$ at I_0.

Draw I_0I horizontally through I_0; then the point of intersection of this horizontal with OZ gives the point I, H_0.

Draw I_0C vertically through I_0; the point of intersection of this vertical with OV gives the point A, H.

Draw the circle through the points II_0H.

Drawing the ordinate through I, the point of intersection of this with the circle gives the point H_0.

This completely solves the problem; and we have

$$h_1 = i \div OI = 50$$
$$h_0 = OH_0 = 38$$
$$h = OH = 57.5$$

or

$$H_1 = 50 \text{ kg./cm.}^2$$
$$H_0 = 38 \text{ kg./cm.}^2$$
$$H = 57.5 \text{ kg./cm.}^2$$
$$I = 50 \times 20 = 1000 \text{ cm.}^3$$

The limiting value of the work is, therefore,

$$W = \frac{H_1 I}{2} = \frac{50 \times 1000}{2} = 25000 \text{ kg. cm. sec.} = 3.3 \text{ hp.}$$

We see from the figure that H_0 and I_0 are almost in phase, so that $\cos\phi \sim 1$. This is confirmed by the known relation

$$W = \frac{H_0 I_0}{2} = \frac{38 \times 1300}{2} = 24600 \text{ kg. cm. sec.}$$

which only differs by 1·6 per cent. from the value found above.

This example shows how considerably the calculation is simplified by using the graphic method. If it is desired to construct the characteristic curves for different powers absorbed at the receiver the method becomes almost indispensable.

Uniformly Loaded Pipes

The differential equations for an ordinary pipe, taking into account inertia and capacity of the liquid, have been found (*see* p. 45) to be

$$\frac{d(H)}{dx} = ja(L)(I)$$

and

$$\frac{d(I)}{dx} = ja(C)(H)$$

Consider now a pipe having a number of branches uniformly distributed along its length, these branches supplying equal currents to a number of receivers of current. Then the variation of the current I is not only due to the compressibility of the liquid, but also to an output per unit length. Let this output be q.

Then we have

$$d(I) = ja(C)(H)dx + qdx$$

and

$$\frac{d(I)}{dx} = ja(C)(H) + q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and for the hydromotive force

$$\frac{d(H)}{dx} = ja(L)(I) \quad \ldots \ldots \quad (12)$$

Differentiating (12) and substituting from (11), we have

$$\frac{d^2(H)}{dx^2} = ja(L)\frac{d(I)}{dx} = ja(L)[ja(C)(H) + q]$$

$$= - a^2(L)(C)(H) + jaq(L) \quad \ldots \ldots \quad (13)$$

Neglecting friction and leakage, and putting

$$\mu = a\sqrt{LC}$$

$$\frac{d^2(H)}{dx^2} + \mu^2(H) - jaqL = 0 \quad \ldots \ldots \quad (14)$$

and by differentiating (11) and substituting $\frac{d(H)}{dx}$ from (12) we get

$$\frac{d^2(I)}{dx^2} + \mu^2(I) = 0 \quad \ldots \ldots \quad (15)$$

From (14), putting

$$(H) = Z + \frac{jaqL}{\mu^2} = Z + \frac{jaqL}{a^2CL} = Z + j\frac{q}{aC}$$

we have

$$\frac{d^2Z}{dx^2} + \mu^2Z = 0 \quad \ldots \ldots \quad (16)$$

The solution of this equation is

$$Z = A \cos \mu x + B \sin \mu x$$

So that we have

$$(H) = A \cos \mu x + B \sin \mu x + j\frac{q}{aC} \quad \ldots \ldots \quad (17)$$

and

$$(I) = A_1 \cos \mu x + B_1 \sin \mu x \quad \ldots \ldots \quad (18)$$

Take the origin of x at the receiver end, then, for $x = 0$, H and I are the hydromotive force and current at this end, so that

$$A = H - j\frac{q}{aC}$$

and

$$A_1 = I$$

differentiating (17) and (18), and considering (11) and (12), we have

$$\frac{d(H)}{dx} = jaL(I) = \mu(- A \sin \mu x + B \cos \mu x)$$

$$\frac{d(I)}{dx} = jaC(H) + q = \mu(- A_1 \sin \mu x + B_1 \cos \mu x)$$

putting

$$x = 0$$

we have

$$jaLI = B\mu = Ba\sqrt{LC}$$

$$jaCH + q = B_1\mu = B_1 a\sqrt{LC}$$

so that

$$B = jI\sqrt{\frac{L}{C}} \text{ and } B_1 = jH\sqrt{\frac{C}{L}} + \frac{q}{\mu}$$

Substituting in (17) (18) these values of A, B, A_1, B_1, we get for the hydromotive force and current at the generator end, at a distance l from the receiver,

$$(H_0) = H \cos \mu l + j\left[I\sqrt{\frac{L}{C}} \sin \mu l + \frac{q}{aC}(1 - \cos \mu l)\right]$$

$$(I_0) = I \cos \mu l + \frac{q}{\mu} \sin \mu l + j\sqrt{\frac{C}{L}} H \cos \mu l$$

If μl is very small we may put

$$\sin \mu l = \mu l$$

and

$$\cos \mu l = 1 - \frac{\mu^2 l^2}{2}$$

and we then have

$$\left.\begin{aligned} (H_0) &= H + j\left[I\sqrt{\frac{L}{C}}\mu l + \frac{q}{aC}\frac{\mu^2 l^2}{2}\right] \\ (I_0) &= I + ql + j\sqrt{\frac{C}{L}}H\mu l \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (19)$$

or

$$\left.\begin{aligned} (H_0) &= H + jaLl\left(I + q\frac{l}{2}\right) \\ (I_0) &= I + ql + jaClH \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

and

We may therefore consider the total amount of uniform current ql as divided into two equal parts $\frac{ql}{2}$, one half near the generator and the other half flowing into the receiver. This rule is easily remembered. Then the current in the line will be the current of the receiver $I + \frac{ql}{2}$ which will give a hydromotive force in quadrature equal to $aLl\left(I + \frac{ql}{2}\right)$, and the generator will have to give a total current equal to $I + \frac{ql}{2} + \frac{ql}{2}$

plus the current in quadrature $aClH$ due to the compression of the liquid. This rule is, however, accurate only if the line is very short relatively to the wave length.

Wave Transmission in Metallic Springs

In the discussion of condensers in the preceding chapters we have not considered the effect of the inertia of the springs required to give the necessary elasticity. In cases in which the frequency is high and in which elasticity is given by metallic springs it is obvious that such inertia will have material effect.

We may consider a metallic spring as a column having a given coefficient of inertia L and capacity C per unit length.

In liquid columns the coefficient L has the value $\frac{\gamma}{g\Omega}$ and C the value $\frac{\Omega}{E}$ where γ is the density of the liquid and E its coefficient of elasticity. If D is the diameter of the convolutions of the spring whose weight per unit length is q, we may consider the spring as a tube of this diameter filled with homogeneous matter, having a coefficient of elasticity given by the relation

$$E\frac{f}{l} = H\Omega = F$$

and density given by

$$\gamma = \frac{q}{\Omega}$$

where

F = the total force on the spring;
f = the compression of the spring in centimetres;
H = the pressure per square centimetre on the piston attached to the spring and assumed to have a sectional area Ω;
$\Omega = \frac{\pi D^2}{4}$ the section of the imaginary tube

and

q = the weight of the spring per unit length.

We may then apply to such a spring all the formulæ found for sonic transmission in liquid columns in pipes.

Thus, neglecting friction, an alternating movement applied at one end of the spring travels in waves having a velocity $v = \frac{1}{\sqrt{LC}}$; but we know that (see p. 18)

$$\frac{f}{F} = A$$

it follows that

$$E = \frac{l}{A}$$

l being the length of the spring,
so that

$$LC = \frac{\gamma}{g\Omega} \cdot \frac{\Omega}{E} = \frac{qA}{gl\Omega}$$

For a spring of circular section
we have

$$A = 8\frac{ND^3}{d^4G}$$

where d is the diameter of the wire of the spring, G is the coefficient of transverse elasticity, N the number of turns in the spring, and D the diameter of the coil,
so that

$$LC = \frac{q}{gl} \cdot \frac{8ND^3}{d^4G} \cdot \frac{4}{\pi D^2}$$

$$= \frac{32}{\pi} \cdot \frac{qND}{gld^4G}$$

but the weight $q = \pi D \times \frac{\pi d^2 N}{4l} \times \gamma_0$ where γ_0 is the density of the material from which the spring is made,
so that

$$LC = \frac{8\pi D^2N^2\gamma_0}{gGd^2l^2}$$

and

$$\sqrt{LC} = \frac{2DN}{dl}\sqrt{\frac{2\pi\gamma_0}{gG}}$$

$$\therefore \quad v = \frac{dl}{2DN}\sqrt{\frac{gG}{2\pi\gamma_0}}$$

Replacing G by the value 0·385E where E is the coefficient of longitudinal elasticity of the material of the spring,
we have

$$v = 0{\cdot}124 \times \frac{dl}{DN}\sqrt{\frac{gE}{\gamma_0}}$$

Note.—The quantity $\sqrt{\frac{gE}{\gamma_0}}$ is the velocity of sound in the material, for steel it is about 5000 metres per second.

If we denote by $b = \frac{\pi DN}{l}$ the length of wire of which a unit length (1 cm) of the spring is formed, we have for the velocity of the wave

$$v = 0{\cdot}39 \times 5000 \times \frac{d}{b} = 1950\frac{d}{b} \text{ metres per second.}$$

Thus, for example, for a spring where $d = 1$ cm. and $b = 10$ cm., we have

$$v = 195 \text{ metres per second.}$$

If this spring be subjected to a periodic pressure of 50 periods per second the wave length will be

$$\lambda = \frac{195}{50} = 3{\cdot}9 \text{ metres.}$$

As another example, let us take a spring having $D = 3$ cm., $\frac{N}{l} = 2$ and $d = 0{\cdot}4$; $b = \pi \times 3 \times 2 = 18{\cdot}8$ cm.

We have

$$v = 1950 \frac{0{\cdot}4}{18{\cdot}8} = 41{\cdot}3 \text{ metres per second.}$$

Suppose this spring is subjected to alternating pressure at a frequency of 50 periods per second. The wave length will be

$$\lambda = \frac{41{\cdot}3}{50} = 0{\cdot}826 \text{ metres}$$

and the half wave length

$$\frac{\lambda}{2} = 0{\cdot}413 \text{ metres.}$$

If the spring is of exactly this length, or an exact multiple of this length, it will be in resonance and may break under the high pressures which will accumulate in it, even if subjected to a small alternating force.

Capacity of Condensers taking into account the Inertia of the Spring

In practice the springs employed in condensers are so short relatively to the wave length that the transmission of motion along the springs may be regarded as instantaneous.

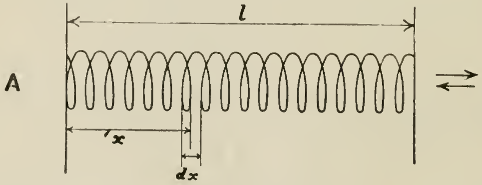

FIG. 29.

Let us consider the effect of the inertia due to the mass of metal of the spring itself. Assume that the spring (Fig. 29) is fixed at A, and consider the movement of a section of length dx at distance x from A. Across the section dx we shall have a difference of pressure dH given by

$$d(H) = jaIdL$$

dL being the coefficient of inertia of the section dx and being proportional to dx. But I is proportional to the movement of the section dx, we may then put

$$dH = k\frac{xdx}{l}L_0;$$

k being a constant and L_0 the coefficient of inertia of the spring considered as a rigid and entirely free body.

Let us denote by L the coefficient of inertia of a spring defined by the relation

$$H = aIL = klL$$

I being the current at the distance l, i. e. at the free end of the spring, and H being the hydromotive force given by

$$\mathrm{H} = \int_0^l dH$$

Then we have

$$H = k\frac{L_0}{l}\int_0^l xdx = k\frac{l}{2}L_0$$

so that

$$L = \frac{L_0}{2}$$

It follows that the effect of the mass of the spring is the same as that of two bodies each of half the mass concentrated at the two ends. This applies to cylindrical springs.

For conical springs a similar analysis gives the proportions of the mass to be concentrated at the ends to calculate the coefficient of inertia. As a rule for ordinary cylindrical springs, it may be said that the inertia due to the springs is equivalent to one-half of the mass of the spring acting at the end at which the alternating force is applied.

Thus, to calculate the capacity due to a heavy spring, particularly for high frequencies, we may proceed as follows. Let C' be the equivalent capacity which will give the same reactance as the spring and a the frequency. Let C be the capacity of the spring assumed to be without weight and let L be the coefficient of inertia.

We have

$$La - \frac{1}{Ca} = -\frac{1}{C'a}$$

then

$$\frac{1}{C'} = \frac{1}{C} - La^2 = \frac{1}{C}(1 - a^2LC)$$

If B is the volume and γ_0 the density of the metal of the spring, we have

$$L = \frac{\gamma_0 B}{2g\Omega^2}$$

Ω being the section of the piston in contact with the spring.

On the other hand, we have found in Chapter III

$$B = \frac{2G}{\sigma^2}Ff = 2\frac{G(f\Omega)^2}{\sigma^2 C}$$

Then

$$C = \frac{2G(f\Omega)^2}{\sigma^2 B}$$

We then have

$$a^2LC = \frac{\gamma_0}{g} \cdot \frac{G}{\sigma^2}(af)^2$$

If we replace af by $2V$, V being the maximum velocity of the oscillation, we have

$$a^2LC = 4\frac{\gamma_0 G}{g\sigma^2}V^2$$

It follows that

$$\frac{1}{C'} = \frac{1}{C}\left[1 - 4\frac{\gamma_0 V^2 G}{g\sigma^2}\right]$$

Wave Transmission in Fluids contained in Pipes of Non-uniform Section

The equations

$$\frac{d(H)}{dx} = ja(L)(I). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (21)$$

$$\frac{d(I)}{dx} = ja(C)(H) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (22)$$

found above at p. 45 are generally applicable. If, however, the pipe is of section varying throughout its length, the quantities (L) and (C) are functions of the section of the pipe and are no longer independent of x, as they would be in the case of a pipe of uniform section.

In a pipe of varying section, differentiating these equations and considering (L) and (C) as variables, we therefore get

$$\frac{d^2(H)}{dx^2} = ja(L)\frac{d(I)}{dx} + ja(I)\frac{d(L)}{dx} = -a^2(C)(L)(H) + ja(I)\frac{d(L)}{dx}$$

Equation (21) gives

$$ja(I) = \frac{1}{(L)} \cdot \frac{d(H)}{dx}$$

so that we have

$$\frac{d^2(H)}{dx^2} + a^2(C)(L)(H) - \frac{1}{(L)} \cdot \frac{d(L)}{dx}\frac{d(H)}{dx} = 0$$

For simplicity, assume that the friction and leakage are each zero; in this case the product $(C)(L)$ is independent of x, and putting $a\sqrt{CL} = \mu$ the equation becomes

$$\frac{d^2(H)}{dx^2} + \mu^2(H) - \frac{1}{L} \cdot \frac{d(L)}{dx} \cdot \frac{d(H)}{dx} = 0$$

but

$$L = \frac{\gamma}{g\Omega}$$

and

$$\frac{dL}{dx} = -\frac{\gamma}{g\Omega^2} \cdot \frac{d\Omega}{dx}$$

so that

$$\frac{dL}{dx} \cdot \frac{1}{L} = -\frac{1}{\Omega} \cdot \frac{d\Omega}{dx}$$

and we get

$$\frac{d^2(H)}{dx^2} + \mu^2(H) + \frac{1}{\Omega} \cdot \frac{d\Omega}{dx} \cdot \frac{d(H)}{dx} = 0 \quad . \quad . \quad . \quad . \quad (23)$$

Similarly, remembering that C is proportional to Ω,

$$\frac{d^2(I)}{dx^2} + \mu^2(I) - \frac{1}{\Omega} \cdot \frac{d\Omega}{dx} \cdot \frac{dI}{dx} = 0 \quad . \quad . \quad . \quad (24)$$

The solution of these equations is possible in several particular cases.*

Example I

Assume that the pipe is of conical form, then

$$x_0{}^2\Omega = \Omega_0 x^2$$

and we have

$$\frac{1}{\Omega} \cdot \frac{d\Omega}{dx} = \frac{2}{x}$$

so that the equations (23) and (24) become

$$\frac{d^2H}{dx^2} + \frac{2}{x} \cdot \frac{dH}{x} + \mu^2 H = 0$$

$$\frac{d^2I}{dx^2} - \frac{2}{x} \cdot \frac{dI}{dx} + \mu^2 I = 0$$

* For further developments see Chapter IX.

The general solution of these is

$$(H) = \frac{1}{x}(A \cos \mu x + B \sin \mu x)$$

$$(I) = A_1(\sin \mu x - \mu x \cos \mu x) + B_1(\cos \mu x + \mu x \sin \mu x)$$

Example II

Let the pipe be a horn such as that of a phonograph (Fig. 30) whose form is given by $r = r_0 e^{mx}$.

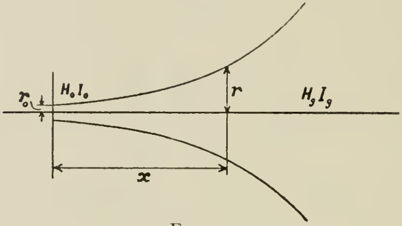

Fig. 30.

we have

$$\Omega = \pi r^2 = \pi r_0{}^2 e^{2mx}$$

so that

$$\frac{1}{\Omega} \cdot \frac{d\Omega}{dx} = 2m$$

The equations (23) and (24) therefore become

$$\frac{d^2H}{dx^2} + 2m\frac{dH}{dx} + \mu^2 H = 0$$

$$\frac{d^2I}{dx^2} - 2m\frac{dI}{dx} + \mu^2 I = 0$$

The solution of these is

$$H = e^{-mx}(A \sin x\sqrt{\mu^2 - m^2} + B \cos x\sqrt{\mu^2 - m^2})$$

$$I = e^{mx}(A_1 \sin x\sqrt{\mu^2 - m^2} + B_1 \cos x\sqrt{\mu^2 - m^2})$$

CHAPTER VI

ALTERNATING FLOW IN LONG PIPES ALLOWING FOR FRICTION

In the general equations found at page 46

$$(H_0) = H \cos \mu l + jI\sqrt{\frac{(L)}{(C)}} \sin \mu l \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$(I_0) = I \cos \mu l + jH\sqrt{\frac{(C)}{(L)}} \sin \mu l \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where

$$\mu = a\sqrt{(LC)},\ (L) = L - j\frac{R}{a},\ (C) = C - j\frac{S}{a}$$

Assume that $\frac{R}{a}$ is not negligible, and put

$$\mu = \alpha - j\beta$$

we have then

$$\sqrt{\frac{(L)}{(C)}} = \frac{1}{(C)}\sqrt{(LC)} = \frac{\mu}{a(C)} = \frac{\alpha - j\beta}{a(C)}$$

and

$$\sqrt{\frac{(C)}{(L)}} = \frac{C}{\sqrt{(LC)}} = \frac{a(C)}{\mu} = \frac{a(C)}{\alpha - j\beta} = \frac{a(C)}{\alpha^2 + \beta^2}(\alpha + j\beta)$$

If we neglect the loss due to leakage, we have

$$(C) = C,\ (L) = L - \frac{R}{a}$$

and

$$\cos \mu l = \cos (\alpha - j\beta)l = \cos \alpha l \cos j\beta l + \sin \alpha l \sin j\beta l$$
$$= \cos \alpha l \cosh \beta l + j \sin \alpha l \sinh \beta l$$
$$\sin \mu l = \sin (\alpha - j\beta l) = \sin \alpha l \cos j\beta l - \cos \alpha l \sin j\beta l$$
$$= \sin \alpha l \cosh \beta l - j \cos \alpha l \sinh \beta l$$

Substituting in equations (1) and (2), we get

$$(H_0) = H(\cos \alpha l \cosh \beta l + j \sin \alpha l \sinh \beta l)$$
$$+ j\frac{I}{aC}(\alpha - j\beta)(\sin \alpha l \cosh \beta l - j \cos \alpha l \sinh \beta l)$$

The second term on the right-hand side

$$= + j\frac{I}{aC}\begin{bmatrix}\alpha \sin al \cosh \beta l - j\alpha \cos al \sinh \beta l \\ - j\beta \sin al \cosh \beta l - \beta \cos al \sinh \beta l\end{bmatrix}$$

$$= + j\frac{I}{aC}\begin{bmatrix}\alpha \sin al \cosh \beta l - \beta \cos al \sinh \beta l \\ - j(\alpha \cos al \sinh \beta l + \beta \sin al \cosh \beta l)\end{bmatrix}$$

so that

$$(H_0) = H \cos al \cosh \beta l + \frac{I}{aC}(\alpha \cos al \sinh \beta l + \beta \sin al \cosh \beta l)$$

$$+ j\left[\frac{I}{aC}(\alpha \sin al \cosh \beta l - \beta \cos al \sinh \beta l) + H \sin al \sinh \beta l\right] \quad . \quad (3)$$

Similarly, for I_0 we have

$$(I_0) = I(\cos al \cosh \beta l + j \sin al \sinh \beta l)$$

$$+ jH\frac{aC}{\alpha^2 + \beta^2}(\alpha + j\beta)(\sin al \cosh \beta l - j \cos al \sinh \beta l)$$

from which

$$(I_0) = I \cos al \cosh \beta l + H\frac{aC}{\alpha^2 + \beta^2}(\alpha \cos al \sinh \beta l + \beta \sin al \cosh \beta l)$$

$$+ j\left[H\frac{aC}{\alpha^2 + \beta^2}(\alpha \sin al \cosh \beta l + \beta \cos al \sinh \beta l) + I \sin al \sinh \beta l\right] . \quad (4)$$

These formulæ are general. To determine the constants α and β, we have by definition

$$\mu = \alpha - j\beta = a\sqrt{C(L)} = a\sqrt{C}\sqrt{L - j\frac{R}{a}}$$

from which

$$\alpha^2 - \beta^2 - j2\alpha\beta = a^2CL - jaRC$$

equating horizontal and vertical components of these vectors

$$\alpha^2 - \beta^2 = a^2CL$$

$$2\alpha\beta = aRC$$

Further equating the moduli of the expression for μ

$$\alpha^2 + \beta^2 = a^2C\sqrt{L^2 + \frac{R^2}{a^2}} = aC\sqrt{a^2L^2 + R^2}$$

from which we get

$$\alpha = \sqrt{\frac{1}{2}aC(\sqrt{a^2L^2 + R^2} + aL)}$$

$$\beta = \sqrt{\frac{1}{2}aC(\sqrt{a^2L^2 + R^2} - aL)}$$

If the frequency is high the term $\left(\frac{R}{aL}\right)^2$ is small compared with unity, and we may with sufficient accuracy for practical purposes put

$$\alpha = a\sqrt{CL}$$

$$\beta = \frac{R}{2}\sqrt{\frac{C}{L}}$$

and

$$\alpha^2 + \beta^2 \sim a^2CL$$

The constant α is the value of μ on the assumption that the pipe is frictionless. This constant is then a function of the wave length λ only, and we have

$$\alpha = \frac{2\pi}{\lambda}$$

The constant β will be the *damping* constant, and as we have taken (*see* page 49)

$$\sqrt{\frac{C}{L}} = \psi$$

we shall have

$$\alpha = \frac{2\pi}{\lambda}$$

$$\beta = \frac{R}{2}\psi$$

These extremely simple formulæ are sufficiently accurate for practical calculations of long pipes in which there is no leakage.

We have found, Chapter III, that $R = kL$, thus we have

$$\beta = \frac{k}{2}\sqrt{CL}$$

If the pipe is an exact multiple of the wave length, we have simply

$$\cos \alpha l = 1$$

and

$$\sin \alpha l = 0$$

and the general formulæ (3) (4) become

$$(H_0) = H \cosh \beta l + \frac{I}{aC}\alpha \sinh \beta l - j\frac{I}{aC}\beta \sinh \beta l$$

$$(I_0) = I \cosh \beta l + \frac{H}{aL}\alpha \sinh \beta l + j\frac{H}{aL}\beta \sinh \beta l$$

or putting

$$\alpha = a\sqrt{CL} \qquad \beta = \frac{k}{2}\sqrt{CL}$$

$$(H_0) = H \cosh \beta l + I\sqrt{\frac{L}{C}} \sinh \beta l - jI\frac{k}{2a}\sqrt{\frac{L}{C}} \sinh \beta l \quad . \quad . \quad (5)$$

$$(I_0) = I \cosh \beta l + H\sqrt{\frac{C}{L}} \sinh \beta l + jH\frac{k}{2a}\sqrt{\frac{C}{L}} \sinh \beta l \quad . \quad . \quad (6)$$

The mechanical power given at the generator will be (*see* page 36)

$$W_0 = \frac{1}{2}\Big[(H \cosh \beta l + I\sqrt{\frac{L}{C}} \sinh \beta l)(I \cosh \beta l + H\sqrt{\frac{C}{L}} \sinh \beta l) + HI\frac{k^2}{4a^2} \sinh^2 \beta l\Big]$$

$$= \frac{1}{2}\Big[HI(\cosh^2 \beta l + \sinh^2 \beta l) + \Big(I^2\sqrt{\frac{L}{C}} + H^2\sqrt{\frac{C}{L}}\Big) \sinh \beta l \cosh \beta l + HI\Big(\frac{k}{2a} \sinh \beta l\Big)^2\Big]$$

The last term may be neglected for high frequency and pipes of large diameter, and we have in this case

$$W_0 = \frac{1}{2}\Big[HI \cosh 2\beta l + \frac{1}{2}\Big(I^2\sqrt{\frac{L}{C}} + H^2\sqrt{\frac{C}{L}}\Big) \sinh 2\beta l\Big]$$

We see that the power at the generator is greater than the power at the receiver

$$W = \frac{HI}{2}$$

The difference is the energy lost in heat in the pipe.

If l is sufficiently small we can put

$$\cosh 2\beta l = 1$$

and

$$\sinh 2\beta l = 2\beta l$$

in which case

$$W_0 \sim \frac{1}{2}\Big[HI + \frac{Rl}{2}\Big(I^2 + \frac{C}{L}H^2\Big)\Big]$$

The power required to run the generator when the receiver is not working is obtained by putting

$$I = 0$$

and we have for this power dissipated in the line

$$W_0' = \frac{H^2}{4} \psi \sinh 2\beta l$$

$$= \frac{\psi}{2} H_0^2 \tanh \beta l$$

If β is large we see that the loss of power is considerable, and it is, therefore, advisable to keep β as small as possible.

We have found

$$\beta = \frac{k}{2}\sqrt{CL}$$

but $\sqrt{CL}$ is the reciprocal of the speed of sound in the liquid, so that we have

$$\beta = \frac{k}{2v}$$

For water

$$\beta = \frac{k}{287000}$$

For practical calculation we can use the following formulæ. In the equations (5) (6), neglecting the last terms containing j, which are very small if the frequency is high, we have for pipes whose length is an exact multiple of wave lengths

$$H_0 = H \cosh \beta l + I\sqrt{\frac{L}{C}} \sinh \beta l \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$I_0 = I \cosh \beta l + H\sqrt{\frac{C}{L}} \sinh \beta l \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Note.—The equations (7) (8), using the proportional functions h and ι may be written

$$h_0 = h \cosh \beta l + \iota \sinh \beta l$$

$$\iota_0 = \iota \cosh \beta l + h \sinh \beta l$$

also

$$h = h_0 \cosh \beta l - \iota_0 \sinh \beta l$$

$$\iota = \iota_0 \cosh \beta l - h_0 \sinh \beta l$$

which can be deduced from (15) (16), changing l to $-l$.

For an infinitely long pipe we have

$$h = 0, \; i = 0$$

that is, since for $l = \infty$

$$\frac{\cosh \beta l}{\sinh \beta l} = 1$$

we must have

$$h_0 = i_0$$

or

$$H_0 = I_0 \sqrt{\frac{L}{C}}$$

so that in an infinitely long pipe the energy lost in friction is finite, and equal to

$$W = \frac{H_0 I_0}{2} = \frac{I_0^2}{2}\sqrt{\frac{L}{C}} = \frac{I_0^2}{2\psi}$$

The *efficiency* η will be

$$\eta = \frac{W}{W_0}$$

where W is the work done by the receiver

$$W = \frac{HI}{2}$$

so that

$$\eta = \frac{1}{\cosh 2\beta l + \frac{1}{2}\left(\frac{I}{H}\sqrt{\frac{L}{C}} + \frac{H}{I}\sqrt{\frac{C}{L}}\right)\sinh 2\beta l}.$$

This has its maximum value when the denominator is a minimum, which is the case when

$$\frac{I}{H}\sqrt{\frac{L}{C}} = 1$$

This relation shows that to obtain maximum efficiency in the transmission of energy we must have

$$\frac{LI^2}{2} = \frac{CH^2}{2}$$

that is to say, the kinetic energy and potential energy at the receiver should be equal. If this is the case, at each point in the transmission line this condition is fulfilled.

So that

$$\eta_{max.} = \frac{1}{\cosh 2\beta l + \sinh 2\beta l} = e^{-2\beta l}$$

and η has this value when

$$I\sqrt{L} = H\sqrt{C}$$

and we have,

$$H_0 = H\,(\cosh \beta l + \sinh \beta l) = He^{\beta l}$$
$$I_0 = I\,(\cosh \beta l + \sinh \beta l) = Ie^{\beta}$$

or

$$H = H_0 e^{-\beta l}$$
$$I = I_0 e^{-\beta l}$$

from which we get

$$\frac{I_0 - I}{I} = \frac{H_0 - H}{H} = 1 - e^{-\beta l}$$

The quantity

$$\epsilon = 1 - e^{-\beta l}$$

would be the *percentage drop of the line,* and for small values of βl is approximately

$$\epsilon \sim \beta l = \frac{kl}{2v}$$

We see, therefore, that in order to get high efficiency β should be kept as small as possible, and the receiver should be arranged so that the proportional pressure and proportional current at this point are equal.

This conclusion is correct in so far as β does not depend on the current I, but in fact β is a function of the mean effective velocity in the pipe, and thus depends on I.

It follows that the greater the hydromotive force for a given diameter of pipe the greater should be the current I and the effective velocity in order to obtain the maximum efficiency; at the same time we can keep β constant by increasing the diameter of the pipe, and large pipes transmitting large quantities of energy will give comparatively higher efficiency.

Example

Find the maximum efficiency of a pipe one kilometre in length transmitting 25 horse-power to a receiver at which the value of H is 30 kg/cm.²

$$W = 25 \text{ hp.} = 25 \times 7600 = 190000 \text{ kg.cm/sec.}$$

For water

$$I = H\sqrt{\frac{C}{L}} = 7\omega H$$

$$HI = 2W = 7\omega H^2 = 7\omega \times 30^2 = 2 \times 190000 = 380000$$

so that

$$\omega = \frac{380000}{7 \times 900} = 60{\cdot}5 \text{ cm.}^2$$

Thus the necessary diameter of the pipe would be

$$d = 8{\cdot}75 \text{ cm.}$$

The maximum velocity at the receiver in order to get maximum efficiency will be

$$V = 7H = 7 \times 30 = 210 \text{ cm./sec.}$$

and the effective velocity

$$= \frac{210}{\sqrt{2}} = 148 \text{ cm./sec.}$$

The value of k will be (*see* Chapter III)

$$k = \frac{1\cdot48}{8\cdot75} + \frac{0\cdot09}{8\cdot75}\sqrt{\frac{148}{8\cdot75}} = 0\cdot212$$

$$\beta = \frac{k}{2v} = \frac{0\cdot212}{2 \times 143500}$$

$$2\beta l = \frac{0\cdot212 \times 100000}{143500} = 0\cdot148$$

$$\cosh 0\cdot148 = 1\cdot011$$

$$\sinh 0\cdot148 = \frac{0\cdot150}{1\cdot161}$$

so that

$$\eta = \frac{1}{1\cdot161} = 0\cdot86$$

Thus the efficiency would be 86 per cent., and the power necessary at the generator will, therefore, be

$$\frac{25}{0\cdot86} = 29 \text{ h.p.}$$

The loss would, therefore, be

$$29 - 25 = 4 \text{ h.p.}$$

The power dissipated by the line when the receiver is stopped and the generator kept running would be

$$W_0' = \tfrac{1}{4} \times 30^2 \times 7 \times 60\cdot5 \times 0\cdot150 = 14300 \text{ kg. cm.}$$

or

$$W_0' = 1\cdot9 \text{ h.p.}$$

Effect of changes of frequency in long pipes.—If we take the general formulæ (3) and (4), found at p. 76—

$$(H_0) = H \cos al \cosh \beta l + \frac{I}{aC}\left(a \cos al \sinh \beta l + \beta \sin al \cosh \beta l\right)$$
$$+ j\left[\frac{I}{aC}\left(a \sin al \cosh \beta l - \beta \cos al \sinh \beta l\right) + H \sin al \sinh \beta l\right]$$
$$(I_0) = I \cos al \cosh \beta l + \frac{H}{aL}\left(a \cos al \sinh \beta l + \beta \sin al \cosh \beta l\right)$$
$$+ j\left[\frac{H}{aL}\left(a \sin al \cosh \beta l + \beta \cos al \sinh \beta l\right) + I \sin al \sinh \beta l\right]$$

Remembering that we have

$$a^2 + \beta^2 \sim a^2 CL$$

and

$$\alpha = a\sqrt{CL} = \frac{a}{v}$$

$$\beta = \frac{k}{2v}$$

where v is the velocity of sound as found above at p. 79, we see that in these formulæ only the quantity α is a function of the frequency n, or, since $a = 2\pi n$, of the pulsation a.

We shall now examine the effect produced on the generator by a change of frequency from a to $a + \Delta a$; Δa being a very small quantity compared with a.

Let us first consider the simple case in which there is no movement at the receiver end of the line.

In this case $I = 0$ and we have for H_0, I_0

$$(H_0) = H(\cos \alpha l \cosh \beta l + j \sin \alpha l \sinh \beta l)$$

$$(I_0) = \frac{H}{aL}\Big[\alpha \cos \alpha l \sinh \beta l + \beta \sin \alpha l \cosh \beta l + j(\alpha \sin \alpha l \cosh \beta l + \beta \cos \alpha l \sinh \beta l)\Big]$$

The work done by the generator would be

$$W = \frac{H^2}{2aL}\Big[\alpha \cos^2 \alpha l \sinh \beta l \cosh \beta l + \beta \cosh^2 \beta l \sin \alpha l \cos \alpha l + \alpha \sin^2 \alpha l \sinh \beta l \cosh \beta l + \beta \sin^2 \beta l \sin \alpha l \cos \alpha l\Big]$$

$$= \frac{H^2}{2aL}\left(\alpha \sinh \beta l \cosh \beta l + \frac{\beta}{2} \sin 2\alpha l \cosh 2\beta l\right)$$

$$= \frac{H^2}{4aL}\left(\alpha \sinh 2\beta l + \beta \cosh 2\beta l \sin 2\alpha l\right)$$

Suppose, now, that the frequency changes in such a way that the angular velocity a becomes $a + \Delta a$. The corresponding increase in the work W will be

$$\Delta W = \frac{H^2}{4aL}\left(\sinh 2\beta l + 2l\beta \cosh 2\beta l \cos 2\alpha l\right)\Delta\alpha$$

But we have

$$\alpha = \frac{a}{v}$$

so that

$$\Delta\alpha = \frac{\Delta a}{v}$$

and finally

$$\Delta W = \frac{H^2}{4aL}\Big[\sinh 2\beta l + 2\beta l \cosh 2\beta l \cos 2\alpha l\Big]\frac{\Delta a}{v}$$

If the line is a multiple of half wave lengths we have

$$\cos 2al = 1$$

The part in brackets in this case is positive, and we see that if the frequency tends to increase, the work done by the generator increases, with the result that the prime mover driving the generator is retarded.

If the frequency diminishes Δa is negative and ΔW is also negative, the work done by the generator decreases and an acceleration of the prime mover results. We see, therefore, that under these conditions the working of the generator and of the line tends to stability. This tendency is present for all values of a which satisfy the condition

$$\sinh 2\beta l + 2\beta l \cosh 2\beta l \cos 2al > 0$$

that is

$$\frac{\tanh 2\beta l}{2\beta l} + \cos 2al > 0$$

Some interesting conclusions may be drawn from this condition.

Suppose the pipe is of great length; the condition becomes

$$\cos 2al > 0$$

and maximum stability is reached when $\cos 2al$ is a maximum, *i.e.*, when

$$\cos 2al = 1$$

giving

$$2al = 2m\pi$$

or

$$al = m\pi$$

substituting

$$a = \frac{a}{v} = \frac{2\pi n}{v} = \frac{2\pi}{\lambda}$$

where λ is the wave length, we have

$$l = m\frac{\lambda}{2}$$

From this it follows that the frequency tends to adjust itself in such a manner that the line is divided up into an integral number of half wave lengths.

The *stability* may be measured by the ratio

$$\frac{\Delta W}{\Delta a} = \frac{H^2}{4a}\sqrt{\frac{L}{C}}\left(\sinh 2\beta l + 2\beta l \cosh 2\beta l \cos 2al\right)$$

The limit of stability is reached when

$$\frac{\Delta W}{\Delta a} = 0$$

i.e., when

$$\cos 2al = -\frac{\tanh \beta l}{2\beta l}$$

In the case of very long pipes this reduces to

$$\cos 2al = 0$$

or

$$al = \frac{2m + 1}{4}\pi = \frac{2\pi}{\lambda}l$$

which gives

$$l = \frac{2m + 1}{8}\lambda$$

In this case the equilibrium is indifferent.

When

$$\cos 2al = -1$$

we get the case of maximum *instability*, and instability persists for all values for which

$$\frac{\tanh 2\beta l}{2\beta l} + \cos 2al < 0$$

In the case of a very long pipe, the condition of maximum instability reduces to

$$l = \frac{2m + 1}{4}\lambda$$

This shows that it is not possible to keep running with a frequency which will produce an uneven number of quarter wave lengths in the line.

These considerations are of great practical importance in that in a long line the conditions for maximum stability are the best conditions for the transmission of energy, *i.e.*, when the two ends of the line are points of maximum pressure variation.

If the line is short we can put

$$\frac{\tanh 2\beta l}{2\beta l} = 1$$

and the condition for stability reduces to

$$1 + \cos 2al > 0$$

which is true for all values of a.

It follows, therefore, that for short lines any frequency is compatible with stable equilibrium. The best stability is, however, obtained when the ratio $\frac{\Delta W}{\Delta a}$ is a maximum numerically; that is, when

$$\cos 2al = 1$$

If

$$\cos 2al = -1$$

we have

$$\frac{\Delta W}{\Delta a} = 0$$

and we see that in a line of moderate length the equilibrium is indifferent when the line is divided into an odd number of quarter wave lengths.

In conclusion, it is evident that a line which is designed to be approximately a multiple of half wave lengths for the frequency considered is the best as regards the stability of the power generator.

A similar analysis can be worked out on the assumption that work is being performed at the receiver end, but it is unnecessary to labour the matter further. When there is a receiver working at the far end of the line, there will be a system of travelling waves superposed on a system of stationary waves in the line; and the general conclusions as regards stability of working are similar.

By an extension of the analysis it can be shown generally that any system which is in a state of *resonance* is also in a state of power equilibrium. Thus, if resonators are inserted at intervals in the line the stability is increased. This is also the case with *balanced* lines with condensers or inertias or large synchronous motors working on the line.

Note on the Law of Friction

In the analysis which has been given above it has been assumed that the coefficient of friction R and the coefficient of leakage S are constants; and the laws connecting the hydromotive force and current have been taken to be (*see* pp. 15, 24)

$$h = Ri \qquad (1)$$

and

$$i = Sh \qquad (2)$$

In the calculations we have taken a mean value for R and S assumed to be constant during the period of vibration. We will now consider how far the assumptions can be justified.

If we have a current i varying as a simple harmonic function we have

$$i = I \sin at$$

Assume that the coefficient of friction R is of the form

$$R = \beta_1 \pm \beta_2 i$$

The relation (1) gives

$$h = (\beta_1 \pm \beta_2 i) i$$

$$= \beta_1 i \pm \beta_2 i^2$$

so that

$$h = \beta_1 I \sin at \pm \beta_2 I^2 \sin^2 at \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The ambiguity of sign $\pm$ is due to the fact that in successive half periods in which the current is in opposite directions, the friction is reversed; that is, the coefficient of friction changes its sign but not its magnitude.

When the current i changes from positive to negative, both terms of the expression for hydromotive force h change their sign, although i^2 still remains positive.

We shall consider separately the term

$$\pm \beta_2 I^2 \sin^2 at = h_2$$

Put

$$h_2 = \beta_2 I^2 y$$

the function y being such that for phases between 0 and π we have

$$y = \sin^2 x$$

and for phases between π and 2π

$$y = -\sin^2 x$$

We can then expand y in a Fourier's series; thus

$$y = A_0 + A_1 \sin x + B_1 \cos x + A_2 \sin 2x + B_2 \cos 2x + .$$

The form of the terms A_m and B_m being given by the relations

$$A_m = \frac{2}{2\pi} \int_0^{2\pi} y \sin mx dx$$

$$B_m = \frac{2}{2\pi} \int_0^{2\pi} y \cos mx dx$$

We must divide these integrals into two periods; between 0 and π y has the value $+\sin^2 x$ and between π and 2π, y has the value $-\sin^2 x$.

Thus we have

$$A_m = \frac{1}{\pi}\left[\int_0^\pi \sin^2 x \sin mxdx - \int_\pi^{2\pi} \sin^2 x \sin mxdx\right]$$

$$B_m = \frac{1}{\pi}\left[\int_0^\pi \sin^2 x \cos mxdx - \int_\pi^{2\pi} \sin^2 x \cos mxdx\right]$$

Making the integrations, we get

$$A_0 = 0, \quad B_m = 0, \quad A_1 = \frac{8}{3\pi} \sin x$$

$$A_m = - \frac{8}{\pi(m - 2)m(m + 2)}$$

and finally

$$y = \frac{8}{\pi}\left[\frac{\sin x}{3} - \frac{\sin 3x}{1.3.5} - \frac{\sin 5x}{3.5.7} - \frac{\sin 7x}{5.7.9} \ldots\right]$$

This series represents the curve

$$y = \sin^2 x \text{ (for values of } x \text{ between 0 and } \pi)$$

and

$$y = - \sin^2 x \text{ (for values of } x \text{ between } \pi \text{ and } 2\pi)$$

It should be noted that the first term of the series is the most important, and if we neglect the remaining terms we can write approximately

$$y \sim \frac{8}{3\pi} \sin x$$

The exact value of the hydromotive force is given by the formula (3) above, which may be written

$$h = I\left[\beta_1 \sin x + \beta_2 I \frac{8}{\pi}\left(\frac{\sin x}{3} - \frac{\sin 3x}{1.3.5} \quad .\right)\right]$$

and approximately

$$h \sim I\left(\beta_1 + \frac{8\beta_2 I}{3\pi}\right) \sin x$$

In Fig. 31 the exact curve for y and the approximate curve taking into account the first term only of the series are drawn; and by inspection of the curves it is evident that for calculation in practice we may, without undue error, as a first approximation neglect all but the first term of the series.

Even in the worst case, when $R_0 = 0$, corresponding to turbulent flow through an aperture, the error is not great.

If greater accuracy is required the second harmonic may be taken into consideration and we get

$$y = \frac{8}{3\pi}\sin x - \frac{8}{15\pi}\sin 3x$$
$$= 0{\cdot}85 \sin x - 0{\cdot}17 \sin 3x$$

The mean energy lost in friction is

$$W = \frac{1}{2\pi}\int_0^{2\pi} hidt$$
$$= \frac{4}{\pi^2}\int_0^{2\pi}\left(\frac{\sin^2 x}{3}dx - \frac{\sin x \sin 3x}{1.3.5}dx \ . \ . \ .\right)$$

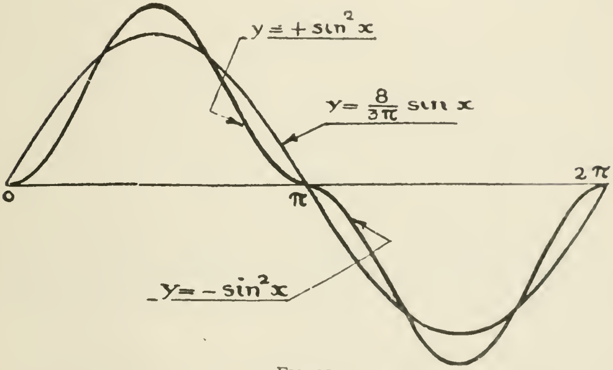

FIG. 31.

Since all the integrals of the form

$$\int_0^{2\pi} \sin x \sin mx dx = \frac{1}{2}\int_0^{2\pi}\left[\cos \overline{m-1}\, x dx - \cos \overline{m+1}\, x dx\right]$$
$$= \frac{1}{2}\left[\frac{\sin \overline{m-1}\, x}{m-1} - \frac{\sin \overline{m+1}\, x}{m+1}\right]_0^{2\pi} = 0$$

This reduces simply to

$$W = I^2\left(\beta_1 + \frac{8\beta_2 l}{3\pi}\right)\int_0^{\pi} \sin^2 dx$$

So that we have

$$W = \frac{I^2}{2}\left(\beta_1 + \frac{8\beta_2 l}{3\pi}\right)$$

It follows, therefore, that if we take only the first term of the series and neglect the remaining terms, no error is introduced into the value obtained for the mean work per second. On the other hand, it is quite possible that in special cases the influence of the higher harmonics of the series for h may produce important variations in the nature of the results in certain investigations.

In problems in which closer approximation is required in considering friction varying according to the square law, since the series of harmonics for h is very rapidly convergent, it is quite sufficient to take only two terms.

Thus, taking

$$h = \beta_2 I^2(0{\cdot}85 \sin x - 0{\cdot}17 \sin 3x)$$

for

$$x = \frac{\pi}{2}$$

We have

$$h_{\frac{\pi}{2}} = \beta_2 I^2(0{\cdot}85 + 0{\cdot}17) = 1{\cdot}02\beta_2 I^2$$

which only differs by 2 per cent. from the value given by

$$h_{\frac{\pi}{2}} = \beta_2 I^2 \sin^2 x = \beta_2 I^2$$

The analysis of cases in which a generator produces current absorbed in inertias, capacities and friction of the square law type, may be carried out by the method given above. If a first approximation only is required we may assume the friction to follow the simple proportional law

$$h = Ri$$

and take for R the value

$$R = 0{\cdot}85\beta_2 I$$

If closer approximation is required, we must introduce a supplementary hydromotive force at the ends of the friction device having three times the frequency and of amplitude equal to $0{\cdot}17\beta_2 I^2$.

If a different law of friction has to be investigated the method given can still be employed. For practical applications, however, it is simpler and sufficiently accurate to write the relation between hydromotive force and current in the form $h = Ri$ and to take for R a mean value over one period.

Generally R would be of the form

$$R = \phi(i)$$

We shall assume for R a constant mean value, putting for i a mean value such as the effective value of i

$$i = I_{\text{eff.}} = \frac{I}{\sqrt{2}}$$

but this method must be limited to cases in which resonance with the higher harmonics of the hydromotive force cannot occur. If such resonance effects are possible, the problem is a different one and the influence of the high frequency harmonics may be considerable. To illustrate this let us take the following problem

In Fig. 32 at G we have a generator producing a current

$$i = I \sin at$$

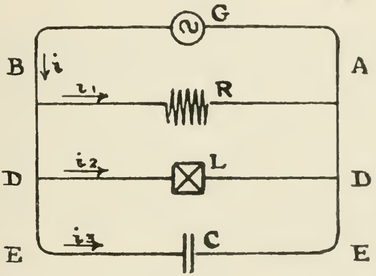

FIG. 32.

At R is a friction device following the square law, *e.g.*, a short pipe of small diameter compared with the feeding lines A, B. At D, E are branches in parallel containing respectively an inertia L and a capacity C; the hydromotive force is h and the currents in the three branches i_1, i_2, i_3 respectively.

We have

$$(I) = (I_1) + (I_2) + (I_3)$$

and if we assume for R a mean value we have

$$(H) = RI_1 = jLaI_2 = -j\frac{1}{Ca}I_3$$

Thus

$$(I) = H\left[\frac{1}{R} + j\left(Ca - \frac{1}{La}\right)\right]$$

Thus we have

$$H = \frac{IR}{\sqrt{1 + R^2\left(Ca - \frac{1}{La}\right)}}$$

$$I_2 = \frac{IR}{La\sqrt{1 + R^2\left(Ca - \frac{1}{La}\right)^2}}$$

These formulæ are obtained on the assumption that we retain only one term in the Fourier's series for h.

If we take into account the fact that h has a series of harmonics, it is quite possible that for one of these, say the mth harmonic, we may have

$$CLm^2a^2 = 1$$

For this frequency the inertia L and capacity C are in resonance and a current of m times the frequency would be set up through the inertia and condenser; and this current, if there is no dissipation of energy in the circuits of the inertia and condenser, may become infinite. Thus if $\frac{R}{La}$ is a very small fraction, nearly the whole current I will pass through the friction R and a considerable wattless current of frequency m times the original will flow in the resonator formed by the inertia and capacity for this higher harmonic.

The formulæ found above for I_2 and I_3 in this case would only give the current of the fundamental frequency which may be negligible; the current for the higher harmonics in the condenser would be considerably greater.

If, however, we have

$$LCa^2 > 1$$

it is impossible that resonance with the higher harmonics should occur.

Problem.—Find the coefficient of leakage S through an aperture of section ω in a partition in the path of the current.

If the hydromotive force is h and the current is i we have by definition (*see* p. 24)

$$i = Sh \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let v be the velocity of the liquid through the aperture and h the difference of pressure on the two sides of the aperture; we know from hydrodynamics that

$$v = \mu\sqrt{2g\frac{h}{\gamma}}$$

where

γ = the density of the fluid

μ = a constant; for an aperture in a thin wall equal to about 0·62.

Thus we have

$$i = \omega v = \mu\omega\sqrt{2g\frac{h}{\gamma}} = \mu\omega h\sqrt{\frac{2g}{\gamma h}} = Sh \quad . \quad . \quad . \quad . \quad (2)$$

so that

$$S = \mu\omega\sqrt{\frac{2g}{\gamma h}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

We see that S is not constant but is dependent on h.

The mean energy lost per second by friction if we assume a mean value for S would be

$$W = \frac{SH^2}{2} = \frac{I^2}{2S}$$

where H is the amplitude of h.

We can also obtain the value of this energy by the formula

$$W = \frac{1}{T}\int_0^T ihdt$$

But we have found

$$i = \mu\omega\sqrt{2g\frac{h}{\gamma}}$$

from which

$$h = \frac{\gamma i^2}{2\mu^2\omega^2 g}$$

and we get

$$W = \frac{\gamma}{2Tg\mu^2\omega^2}\int_0^T i^3dt$$

If i is a simple harmonic function

$$i = I \sin at$$

we have

$$W = \frac{\gamma I^3}{2Tg\mu^2\omega^2}\int_0^T \sin^3 atdt$$

Substituting x for at, we get

$$W = \frac{\gamma I^3}{4\pi g\mu^2\omega^2}\int_0^{2\pi} \sin^3 xdx$$

But

$$\int_0^{2\pi} \sin^3 xdx = \int_0^{2\pi} (\cos^2 x - 1)d(\cos x)$$

$$= \left[\frac{\cos^3 x}{3} - \cos x\right]_0^{2\pi} = 0$$

This result, taking the integral over the range 0 to 2π is incorrect, for since the work is always positive, the sum of the work over one period necessarily differs from zero. The error arises from the fact that on reversal of the flow, the hydromotive force is also reversed, so that the integral is of opposite sign for the two half-periods. It is, therefore, necessary to find the work during one half-period $\frac{T}{2}$ and double it to get the work done during the complete period.

Thus we get

$$W = \frac{\gamma I^3}{2\pi g \mu^2 \omega^2} \int_0^\pi \sin^3 x dx$$

But

$$\int_0^\pi \sin^3 x dx = \frac{4}{3}$$

so that we get finally

$$W = \frac{2\gamma I^3}{3\pi g \mu^2 \omega^2}$$

Comparing this value of W with that obtained above, namely

$$W = \frac{I^2}{2S}$$

we find

$$S = \frac{3\pi g \mu^2 \omega^2}{4\gamma I}$$

and putting

$$I = SH$$

we get

$$S = \mu\omega\sqrt{\frac{3\pi g}{4\gamma H}}$$

For water we have

$$S = 1540 \frac{\mu\omega}{\sqrt{H}}$$

where μ is the contraction coefficient which varies from 0·5 to 1·1 according to the form of the nozzle or orifice through which the flow takes place.

An average value for water would be

$$S = 1200 \frac{\omega}{\sqrt{H}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Numerical Example.

Find the power lost in a cock which does not close perfectly, the size of the aperture ω left open being one-tenth of a square centimetre and the hydromotive force $H = 100$ kg./cm.^{2}

The energy lost per second would be

$$W = \frac{SH^2}{2}$$

$$S = 1200 \frac{0 \cdot 1}{\sqrt{100}} = 12$$

so that

$$W = \frac{12 \times 100^2}{2} = 60000 \text{ kg. cm.}$$

and the horse-power lost

$$= 7 \cdot 9 \text{ h.p.}$$

It is evident, therefore, that a leak of comparatively small area gives rise to considerable loss of power. The current through the leak is given by the equation

$$W = \frac{HI}{2}$$

$$I = \frac{2W}{H} = \frac{2 \times 60000}{100} = 1200 \text{ cm.}^3\text{/sec.}$$

The amplitude of the velocity will be

$$V = \frac{I}{\omega} = \frac{1200}{0 \cdot 1} = 12000 \text{ cm./sec.}$$

This is an extremely high velocity, and the effect of the water moving in contact with the metal at such a velocity will soon produce overheating and erosion of the metal. It is clear, therefore, that the greatest care must be taken to make the joints in such cases perfectly tight in order to avoid serious trouble.

CHAPTER VII

THEORY OF DISPLACEMENTS—MOTORS

DISPLACEMENTS

It will be convenient, before proceeding to the mathematical discussion of the various types of rotary motors operated by alternating fluid currents, to establish certain relations with regard to the movements imparted to fluid by the relative movements of pistons and cylinders.

We shall define by the term *displacement* of a piston or of a current the quantity $\Delta = \int i dt$, where i is the current.

Remembering that

$$i = v\Omega = \Omega\frac{dr}{dt}$$

where v is the velocity of the current at the time t at a point at a distance r from mean position of the piston, we have

$$\Delta = \int \Omega dr = \Omega r + C \quad . \quad . \quad . \quad . \quad . \quad (1)$$

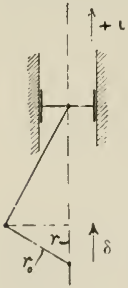

FIG. 33.

If r is a sinusoidal function, and if we take the origin of time at the instant when the piston is at the out-centre, we have

$$\Delta = \Omega r_0 \cos at \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From the above definition we have

$$i = \frac{d\Delta}{dt}$$

In the application of the theory to motors we shall have to consider two cases—

(1) Displacement in a fixed cylinder by a moving piston attached to a rotating crank or eccentric;

(2) Displacement due to the rotation of a cylinder round a fixed crank or eccentric.

The rotation of a crank connected to a piston by a long connecting rod gives a sinusoidal reciprocation of the piston in the direction of its axis.

If r_0 is the amplitude of the crank and a the angular velocity, we have

$$r = r_0 \cos at$$

The quantity r would be called the *crank* or the *eccentric* of the displacement. We shall have to deal later with *rotating cranks* and *alternating cranks*. Thus the definition of crank or eccentric is extended and includes a crank which is variable not only in position but also in magnitude. The relation between the displacement and the crank will, however, always be

$$\Delta = \Omega r$$

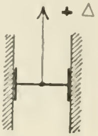

FIG. 34.

If the cylinder rotates instead of the crank, the direction of rotation being the same, the current will have the same value, but will be of opposite sign. We can represent an alternating displacement by a vector whose value is $r_0\Omega$ rotating at the same angular velocity as the crank.

We can then employ the symbolic notation. If the displacement is represented by a line, since we have

$$i = \frac{d\Delta}{dt}$$

and the variation is sinusoidal, the current will be represented by a line at right angles to it. Fig. 34 shows the displacement due to the rotation of a crank actuating a piston in a fixed cylinder. The current produced will be

$$i = \frac{d\Delta}{dt}$$

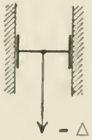

FIG. 35.

If the cylinder were rotated and the crank fixed we should arrive at the condition shown in Fig. 35, and should then have

$$i = -\frac{d\Delta}{dt}$$

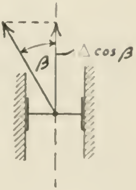

FIG. 36.

In the general case where the direction of the reciprocating movement of the piston is different from the direction of the displacement, as shown in Fig. 36, we have for the current

$$i = \frac{d(\Delta \cos \beta)}{dt}$$

$$= \cos \beta \frac{d\Delta}{dt}$$

The current produced in this manner by a reciprocating piston in a fixed cylinder will be termed *static current,* due to an alternating displacement parallel to the current.

If the cylinder is rotating at uniform angular velocity a_1, and the piston is connected to a fixed crank, a current is also produced which will be termed *dynamic current.* The velocity of this current depends on the velocity a_1, and is a maximum when the crank is at right angles to the axis of the piston. At this moment its value will be

$$\dot{i}_1 = v_1\Omega = ra_1\Omega = a_1\Delta$$

This current has a negative value if the crank is on the right-hand side (Fig. 37) and the cylinder is rotating in the direction shown by the arrows. In a machine in which the cylinders move we can consider the cylinders as turning about a fixed centre O and the piston as turning about another fixed centre O'.

The instantaneous value of the *static current* will be

$$\dot{i}_1 = \frac{d\Delta_1}{dt}$$

where Δ_1 is the displacement in the direction of the axis of the cylinder and the instantaneous value of the *dynamic current* will be

$$\dot{i}_2 = -a_1\Delta_2$$

The sign of this current is positive when the cylinder moves towards the displacement, and negative when the cylinder moves away from it. Δ_2 is the displacement in the direction at right angles to the axis of the cylinder. These displacements are proportional to the actual movement of the point O', which is the centre of rotation of the piston relative to the point O which is the centre of rotation of the cylinders. So if the centre O is given a displacement Δ at an angle α with the axis of the piston, the total current in the cylinder supposed rotated with a uniform angular velocity a_1 in the direction of the arrow (Fig. 37) will be

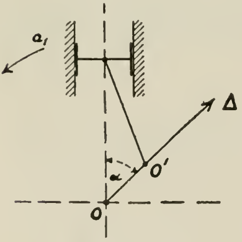

FIG. 37.

$$\dot{i} = \cos\alpha\frac{d\Delta}{dt} - a_1\Delta\sin\alpha$$

so that if

$$\Delta = r\Omega\cos at$$

$$\dot{i} = r\Omega[-a\cos\alpha\sin at - a_1\sin\alpha\cos at]$$

and since

$$a = a_1 t$$

$$= r\Omega[-a \cos a_1 t \sin at - a_1 \sin a_1 t \cos at]$$

This case is equivalent to a displacement Δ, whose direction is unaltered while the cylinder rotates with uniform angular velocity a_1.

But we know that

$$\sin at \cos a_1 t = \frac{1}{2}\left[\sin \overline{a + a_1}\, t - \sin \overline{a_1 - a}\, t\right]$$

and

$$\cos at \sin a_1 t = \frac{1}{2}\left[\sin \overline{a + a_1}\, t + \sin \overline{a_1 - a}\, t\right]$$

so that

$$i = -\frac{r\Omega}{2}\left[(a_1 - a) \sin \overline{a_1 - a}\, t + (a + a_1) \sin \overline{a + a_1}\, t\right] \quad . \quad . \quad . \quad (3)$$

We have seen above that the displacement due to the rotation of a crank is of the form

$$\Omega r = \Omega r_0 \cos at$$

and the corresponding static current is

$$i = -r_0 \Omega a \sin at$$

This shows that the current i given by formula (3) is equivalent to two superposed currents produced by rotating cranks of length $\frac{r}{2}$ turning with angular velocities of $a_1 - a$ and $a_1 + a$ respectively. Thus the displacement Δ can be considered as equivalent to two displacements rotating in opposite directions at the same angular velocity a.

Motors

General Considerations

In Chapter II we have shown from elementary principles that if alternating pressures are impressed on a liquid column, the train of waves produced, travelling along the liquid column, can give up the whole of its energy to a piston situated at any point in the liquid column, provided this piston is moving at the same speed as and in phase with the layer of liquid in immediate contact with it. This condition is fulfilled by a receiver consisting of a piston, connecting-rod and crank similar to the piston, connecting-rod and crank of the generator, provided that the cranks of the generator and receiver are rotating at the same angular velocity, and in a certain phase relationship depending on the relation between the distance from one piston to the other and the wave length of the train of waves travelling in the liquid column. It is, therefore, clear that a receiver or motor of this type can take up energy from the wave train only if it is running at the same speed as the generator. Such a motor we shall term a *synchronous motor.*

It will be obvious that a motor with a single piston, connecting-rod and crank, such as we have been considering, will exert a torque which varies during a revolution, and if a continuous rotation against a load is required it is necessary to use a flywheel. Unless, however, such a motor has been started and has attained the same speed as the generator it will not be able to overcome the inertia of the flywheel or to develop torque. It is, therefore, necessary to give such a motor an angular speed very near to that of the generator and to clutch it at the proper phase in order to develop power.

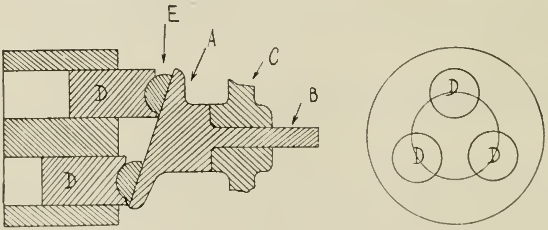

FIG. 38.

If in a synchronous motor instead of a single piston we employ three pistons acting on the same shaft and moving along axes inclined at 120° actuated by three liquid columns in which wave trains differing in phase by 120° are travelling, it will be readily seen that at the synchronous speed a uniform torque will be developed, so that a flywheel or other inertia is unnecessary; the starting torque, however, will be very small, and the motor will be unable to start against a load. In order that the motor may start, the inertia must be such that the acceleration is large enough for the rotor to reach synchronous speed in one revolution. Such a motor will be termed a *synchronous three-phase motor.*

A simple example of such a motor is illustrated at Fig. 38. In this motor the rotor consists of an inclined disc A fixed to a shaft B which rotates in a bearing C; the *stator* is formed by three cylinders uniformly distributed around the disc, and three trains of waves differing in phase by 120° act on pistons D in the three cylinders, pressing these pistons against the inclined plate. It will be evident that such a motor will operate as described above; that is, it would run only at a speed *synchronous* with the generator, or otherwise not rotate at all; no variation of speed is therefore possible with such a motor.

Suppose, now, that instead of the plate A, whose inclination to the shaft B is fixed, we have a plate or segment T, as shown in Fig. 39, resting on a bearing in a spherical cup R fixed to the shaft B symmetrically. Assume that there is no friction either between the pistons D and the plate T nor between the plate T and the cup; and let the pistons be acted on by three wave trains as above described in the case of the synchronous motor illustrated in Fig. 38.

It will readily be seen that in this case no rotation about the

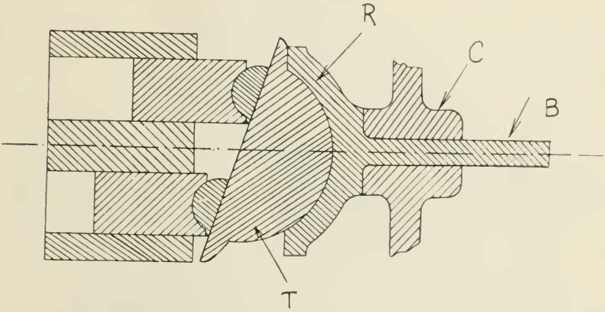

FIG. 39.

horizontal axis will take place either of the segment T, which we may call the transmitter, or of the rotor R. All that will occur will be that the transmitter T will rock in the cup in such a manner that the line of greatest inclination of its plane face will generate a cone and will move on the surface of this cone at the synchronous speed.

If there is friction between the transmitter T and the rotor R, and no resistance to rotation, both transmitter and rotor will rotate at the synchronous speed, and the combination will be equivalent to the synchronous motor shown in Fig. 38, the transmitter maintaining a constant inclination to the horizontal axis. If, now, resistance be opposed to the rotation of the rotor by the application of a load, both the rotor and transmitter will slow down by nearly the same amount. The motor will, however, still develop torque. It will be seen that the *rocking movement* of the transmitter relative to the rotor is proportional to the difference between the angular speed of the rotor and the synchronous speed, while there is loss of energy due to the rocking friction between the transmitter and the rotor. Since motors of this class have a starting torque and are able to develop power at any speed up to the synchronous speed we shall term them *asynchronous polyphase motors.*

Another form of motor of this type is shown at Fig. 40. The motor comprises a circular ring *D*, which forms the transmitter, supported by pistons *A*, *B*, *C*, uniformly distributed around its circumference; a second ring *E* is provided within the ring *D*, and ball

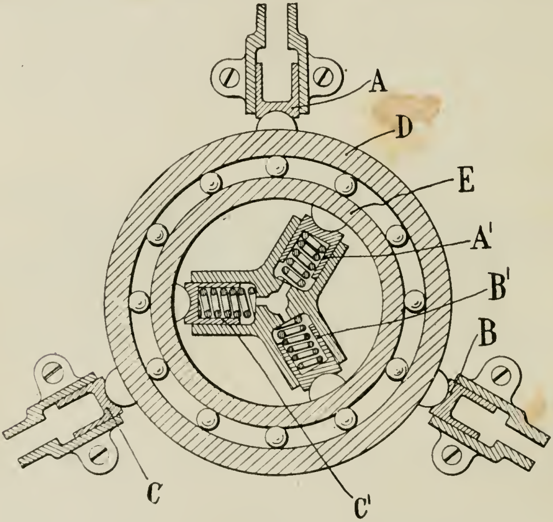

FIG. 40.

bearings are fitted between the two rings. The rotor consists of three cylinders provided with pistons *A′*, *B′*, *C′* bearing against the ring *E*.

The cylinders of the rotor are in communication with each other through small apertures. The pistons *A*, *B*, *C* of the stator work in fixed cylinders in communication with three pipes containing liquid subjected to alternating pressures differing in phase by 120°. That is to say, the pulsations of liquid in the three feed-pipes are produced by a generator formed by three pistons at 120° phase difference. The concentric arrangement shown in the figure is merely diagrammatic, and in practice the stator and rotor may be side by side. The rings

D and *E* are rigid as regards the movement of their common centre *O*, but can rotate independently about this centre, the slip being facilitated by a ball-bearing between them, or other well-lubricated anti-friction device. This motor is exactly similar in principle to that shown at Fig. 39, the frictional resistance between the transmitter and the rotor being replaced by the resistance to the flow of liquid between the different cylinders of the rotor.

Another type of motor is illustrated at Fig. 41. In this motor we have two pistons *A*, *B*, producing an alternating pressure acting

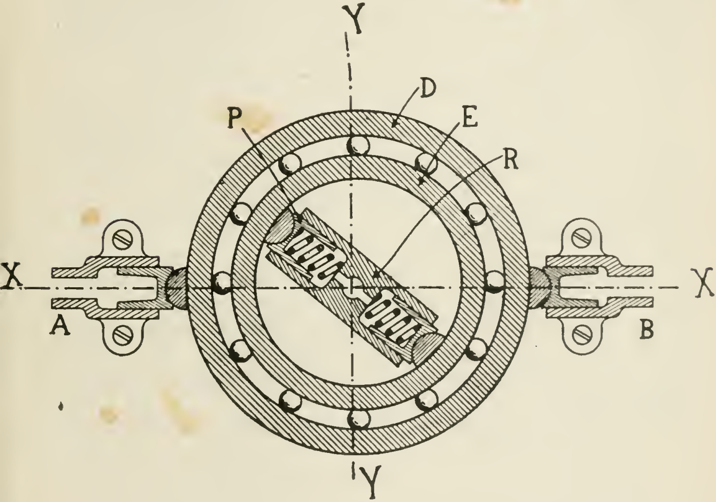

FIG. 41.

on the ring *D* in the direction *XX*. The ring *D* is concentric with the ring *E*, which presses on and moves with the piston *P* of the rotor. The cylinders in which the pistons *P* work are connected through small apertures as shown.

We have seen that an alternating crank acting in a given direction such as we have assumed can be replaced by two alternating cranks of half the amplitude whose vectors are assumed to be rotating uniformly with the same angular velocity in opposite directions. If we have a suitable rotor the resulting rotating displacements would tend to produce rotation. We know that the effect of inertia is greater the higher the frequency of the alternating currents in the circuit in which it is placed. If, therefore, we have a rotor connected as shown in Fig. 41, and consider the effect on it of the two rotating displacements due to the stator, we see that as by rotating the rotor the angular velocities of the two rotating displacements relatively to the rotor will be made to differ from one another, we can by rotating the rotor cause a certain asymmetry in the internal forces acting on the rotor owing to the greater effect of the inertia on the alternating currents of higher relative frequency. Asymmetry can also be produced by inserting leakage, or resistance. Motors giving torque at all speeds can thus be constructed. Such motors will be termed *asynchronous monophase motors.* It is characteristic of these motors that relative motion of the rotor and stator is necessary before torque is produced, and that the torque is not uniform during the revolution.

In this type of motor also it is obvious that the stator and rotor may be arranged side by side instead of concentrically. In the motors above described the maximum torque is that which is obtained at nearly the synchronous speed, and it is not possible with these constructions to obtain a higher torque on starting.

We can now proceed with the mathematical investigation of the types of motors described above.

Synchronous Motors. — We may consider a polyphase synchronous motor as formed by a number of monophase synchronous motors coupled on the same shaft but at different angles of phase. To find the torque of such a motor we will consider the case of a simple motor consisting of a single piston with its crank and a long connecting-rod.

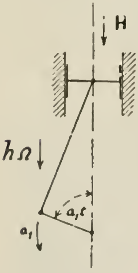

FIG. 42.

Acting on the piston we have an alternating pressure

$$h = H \sin (at + \phi)$$

This pressure produces an instantaneous torque

$$M = rh\Omega \sin (a_1 t + \psi)$$

a_1 being the angular velocity of the crank, then we have

$$M = rH\Omega \sin (a_1 t + \psi) \sin (at + \phi)$$

but

$$\sin \alpha \sin \beta = \frac{1}{2}\Big[\cos (\alpha - \beta) - \cos (\alpha + \beta)\Big]$$

so that

$$M = \frac{rH\Omega}{2}\Big[\cos (\overline{a_1 - a}\, t + \psi - \phi) - \cos (\overline{a_1 + a}\, t + \psi + \phi)\Big]$$

We see that this torque is pulsating, and its mean value over a complete period will be

$$M_m = \frac{1}{T}\int_0^T M dt$$

T being the time of one period of the pulsation, so that we have

$$M_m = \frac{rH\Omega}{2T}\left[\frac{\sin(\overline{a_1 - a}\,t + \psi - \phi)}{a_1 - a} - \frac{\sin(\overline{a_1 + a}\,t + \psi + \phi)}{a_1 + a}\right]_0^T$$

We see that if a_1 is not equal to a, this torque is zero, because both terms are periodic functions of time.

When, however, $a_1 = a$, we have

$$M = \frac{rH\Omega}{2}\Big[\cos(\psi - \phi) - \cos(2at + \psi + \phi)\Big]$$

the mean value of which is not zero but is given by

$$M_m = \frac{rH\Omega}{2}\cos(\psi - \phi)$$

If there are a number of cylinders around the shaft the torque will be proportional to this number. Moreover, if the cylinders are at the angles of a regular polygon the torque will be constant, for the sum

$$\sum_{\psi + \phi}^{2\pi + \psi + \phi} \cos(2at + \psi + \phi)$$

becomes zero for values of ψ and ϕ differing by the same angle, it being assumed that each cylinder is supplied with pressure whose phase differs from that of the next by the angle between the axes of the cylinders.

The displacement Δ at each instant is

$$\Delta = r\Omega\cos(a_1 t + \psi)$$

The static current will be

$$i = \frac{d\Delta}{dt} = -a_1 r\Omega \sin(a_1 t + \psi)$$

The work done at any instant $= hidt$;
and this must be equal to the resisting couple $= -Ma_1 dt$;
so we have

$$M = rh\Omega\sin(a_1 t + \psi)$$

The same formula we have found on p. 104.

We can thus simplify the proof for the case of polyphase currents. To find the torque produced by a polyphase motor we need only calculate the work per second for each phase, take the sum and divide by the speed of the rotor.

Suppose, then, we have a motor of m phases and we wish to calculate the constant torque corresponding to the synchronous speed. Let I be the maximum current in one phase and H the maximum hydromotive force. We have for the mechanical work of one phase

$$W_1 = \frac{1}{2} HI \cos \phi$$

ϕ being the phase angle between the current and the hydromotive force. The torque of the motor will be

$$M = \frac{m}{2a} HI \cos \phi$$

the maximum value is

$$M = \frac{m}{2a} HI = \frac{mr\Omega H}{2} = \frac{m}{2} \triangle H$$

where

r = the length of the crank,
Ω = the section of each piston,
m = the number of phases,
$\triangle = r\Omega$, the amplitude of the displacement in one cylinder.

If the energy absorbed by the motor is W and the angular velocity is a, we have also

$$M = \frac{W}{a}$$

Similarly, if we consider a generator having m phases, the torque which must be applied to the rotor to produce m currents differing in phase by $\frac{2\pi}{m}$ will be

$$M = m\frac{W_1}{a}$$

where W_1 is the energy spent in each phase.

If the phases are doing work in a circuit having resistance R, inertia L, and capacity C, the energy dissipated will be

$$W_1 = \frac{1}{2} RI^2$$

where I is the current in one phase.

The inertia L and capacity C have no effect on the value of the torque, as they do not cause any loss of energy.

Example

Find the length of crank required in a three-phase motor having cylinders of 2·5 cm. diameter required to develop 20 hp. at a speed of 600 revs. per minute, the hydromotive force H being 100 kg./cm.^{2}

The torque necessary to obtain 20 hp. at the frequency of 10 per second will be

$$M = \frac{20 \times 76 \times 100}{2\pi \times 10} = 2420 \text{ kg.cm.}$$

and we have

$$2420 = \frac{m}{2} \triangle H = \frac{3}{2} \times 100 \times \triangle$$

so that $\triangle = 16{\cdot}1$ cm.^{3} for each phase. If, therefore, we use pistons of 2·5 cm. diameter the length of the crank r will be

$$r = \frac{4\triangle}{\pi \times 2{\cdot}5^2} = 3{\cdot}28 \text{ cm.}$$

We have seen that synchronous motors can only develop torque if their speed corresponds exactly with the synchronous speed.

For small temporary variations of speed they develop a positive or negative supplementary torque which tends to correct the variation of speed; when the speed is increasing this torque tends to reduce the speed; similarly, a drop of speed introduces a torque which tends to accelerate the motor.

From this it is evident that synchronous motors can take considerable variation of load without changing their speed. If, however, the load increases beyond a certain limit the motor stops abruptly.

Synchronous motors should be started at no load, for their starting torque is very small, particularly if the motor has much inertia.

Let A be the moment of inertia of the rotor; we shall have for the equation of motion

$$A\frac{da_1}{dt} = M = rH\Omega \sin (a_1 t + \psi) \sin (at + \phi)$$

$$= \frac{rH\Omega}{2}\left[\cos (\overline{a_1 - a}\, t + \psi - \phi) - \cos (\overline{a + a_1}\, t + \psi + \phi)\right]$$

This equation is of the form

$$\frac{dy}{dt} = B \cos (\overline{y + a}\, t + b) + D \cos (\overline{y - a}\, t + c)$$

It is difficult to integrate this equation, but an approximate solution for polyphase motors can be found; for such motors

$$M = m\frac{rH\Omega}{2} \cos (\overline{a_1 - a}\, t + \psi - \phi) = A\frac{da_1}{dt}$$

At the time $t = 0$ we have $a_1 = 0$. After a time θ we shall have

$$a_1 = a$$

we can then put for the period of starting, if this period of time is very short,

$$a_1 \sim a\frac{t}{\theta}$$

Then

$$(a_1 - a)t = a(\frac{t}{\theta} - 1)t$$

The motor will start if θ is small enough, but it must be less than the time-period T.

Assume this condition fulfilled.

Then we have

$$\frac{da_1}{dt} = \frac{mrH\Omega}{2A} \cos \gamma$$

The quantity

$$\cos \gamma = \cos (\overline{a_1 - a}\, t + \psi - \phi)$$

has a value between zero and unity. If the motor takes a very short time to start, so that during this time we can consider this quantity constant, putting

$$k = \cos (\overline{a_1 - a}\, t + \psi - \phi)$$

we get simply

$$a_1 = kt\frac{mrH\Omega}{2A}$$

or if

$$a_1 = a$$

and

$$t = \theta$$

$$a = k\theta\frac{mrH\Omega}{2A}$$

The constant k may be given any value between zero and unity. The condition most favourable to starting will be

$$k = 1$$

Let

$$\theta = \frac{T}{q} = \frac{1}{nq} = \frac{2\pi}{aq}$$

assuming that

$$k = 1$$

$$\therefore\ qa^2 = \frac{\pi mrH\Omega}{A}$$

In order, therefore, that it may be possible to start it is necessary that A should be less than or equal to

$$A = \frac{\pi m r H \Omega}{q a^2}$$

The less the value of q the more easy it is to start the motor, $\frac{1}{q}$ representing the fraction of the period of one revolution in which the rotor attains the synchronous speed.

We see, therefore, that the moment of inertia of the moving parts should be less than a definite limit.

Let W be the mechanical power of the motor.

We have

$$W = \frac{m}{2} r a H \Omega$$

so that for starting A must be less than or equal to

$$A = \frac{2\pi W}{q a^3}$$

If J is the volumetric moment of inertia of the rotor and its flywheel about the axis expressed in centimetres[5] we shall have

$$A = \frac{\gamma}{g} J = \frac{2\pi W}{q a^3}$$

γ being the specific gravity of the metal forming the rotor, so that it is evident that the motor will not start if a has any considerable value.

Example

Take a 20 hp. motor whose rotor weighs 20 kg. What will be the radius of gyration of the rotor in order that the motor may start in a quarter of a revolution ($q = 4$), the synchronous speed being $a = 100$?

Let x be the maximum radius of gyration.

Then

$$A = \frac{20x^2}{981} = \frac{2 \times 20 \times 7600 \times \pi}{q a^3}$$

from which

$$x = 3{\cdot}4 \text{ cm.}$$

If a were 50 instead of 100 we should get

$$x = 9{\cdot}7 \text{ cm.}$$

We see, therefore, that synchronous motors are difficult to start against inertia, and it would be impossible to start such a motor against the inertia of a flywheel, even of small dimensions. Only at very low frequencies is it possible to start without special devices. In such motors it is advisable to use very light pulleys, which should preferably be of small diameter.

Asynchronous Polyphase Motors.—It is possible to obtain a starting torque with asynchronous polyphase motors. Suppose that, instead of applying a sonic current to an ordinary crank or eccentric, we construct a motor as illustrated in Fig. 40, described above.

Let the currents at any moment in the three cylinders A, B, C of the stator be

$$i_1 = I \sin at;\ i_2 = I \sin\left(at + \frac{2\pi}{3}\right);\ i_3 = I \sin\left(at - \frac{2\pi}{3}\right)$$

components of a displacement $\triangle$.

To these currents there will correspond displacements given by

$$\triangle_1 = \int i_1 dt = -\frac{I}{a}\cos at$$

$$\triangle_2 = -\frac{I}{a}\cos\left(at + \frac{2\pi}{3}\right)$$

and

$$\triangle_3 = -\frac{I}{a}\cos\left(at - \frac{2\pi}{3}\right)$$

We see, then, that $\triangle$ is a vector whose numerical value is $\frac{I}{a}$ turning with constant velocity about a fixed point.

The movement of the point O, therefore, takes place in a circle with a constant angular velocity corresponding to the synchronous speed.

This movement of the centre of the ring E will produce in the cylinders of the rotor, currents j_1, j_2, j_3, whose value we will determine.

Let a_1 be the angular velocity of the rotor. The relative velocity between the displacement $\triangle$ and the pistons of the rotor will then be $a - a_1$. The phenomena will occur exactly as if we supposed the rotor fixed and the displacement $\triangle$ rotating with a velocity $a - a_1$.

The actual movement or the crank of the point O is $\frac{\triangle}{\Omega}$; Ω being the section of a piston of the stator. The rotor, therefore, is under the action of a crank of length

$$r = \frac{\triangle}{\Omega} = \frac{I}{a\Omega}$$

rotating with angular velocity $a - a_1$. Under these conditions the current in each cylinder of the rotor will be of the form

$$j_1 = -J \sin (a - a_1 t)$$

$$j_2 = -J \sin\left(a - a_1 t + \frac{2\pi}{3}\right)$$

$$j_3 = -J \sin\left(a - a_1 t - \frac{2\pi}{3}\right)$$

where

$$J = \frac{I\Omega_1}{\Omega}\left(1 - \frac{a_1}{a}\right)$$

Ω being the sectional area of a piston of the rotor.

Let M be the resisting torque and H_1 the hydromotive force in the rotor. The rotor is acted on by a virtual crank $r = \frac{\triangle}{\Omega}$, and we can consider it as a generator having a crank r rotating with velocity $a - a_1$ and developing a torque M between the crank and the opposing resistance. In order to find the necessary torque to rotate a generator it is sufficient to know the work done by the currents produced.

Let S be the leakage of the current in one phase of the rotor. The mean work done per second in the rotor will be given by

$$W_1 = \frac{3SH_1^2}{2} \text{*}$$

and, on the other hand, the torque M multiplied by the angular velocity $a - a_1$ gives

$$W_1 = M(a - a_1)$$

Let us put

$$a - a_1 = \alpha$$

then

$$M = \frac{3SH_1^2}{2(a - a_1)} = \frac{3SH_1^2}{2\alpha}$$

If J_1 is the current in the rotor and S, C the leakage and capacity in series in each phase we have in symbolic notation

$$(J_1) = [S + jC\alpha]H_1$$

and numerically

$$H_1^2 = \frac{J_1^2}{S^2 + C^2\alpha^2}$$

On the other hand, if H is the hydromotive force in the stator, observing that as regards hydromotive force, action and reaction are equal, the total force on the pistons of the stator must be equal to the total force on the pistons of the rotor, so that we have

$$H\Omega = H_1\Omega_1$$

so that

$$H_1^2 = \frac{J_1^2}{S^2 + C^2\alpha^2} = \left(H\frac{\Omega}{\Omega_1}\right)^2$$

* The energy per second given to the motor will be

$$Ma_1 + W_1 = Ma_1 + M(a - a_1) = Ma.$$

and

$$M = \frac{3}{2} \frac{SJ_1^2}{(S^2 + C^2\alpha^2)a}$$

Also we have

$$J_1 = \frac{d\Delta_1}{dt}$$

Δ_1 being the displacement in one phase of the motor. But Δ_1 is a periodic function having a pulsation $\alpha = a - a_1$; thus we have in symbolic notation

$$(J_1) = j\alpha\Delta_1$$

and for the amplitude

$$J_1 = \alpha\Delta_1$$

The expression for M would then be

$$M = \frac{3}{2} \frac{S\alpha\Delta_1^2}{S^2 + C^2\alpha^2}$$

The maximum value of M for a pulsation α is given by the relation $S = C\alpha$, therefore

$$M_{\text{max.}} = \frac{3}{4} \frac{\Delta_1^2}{C} = \frac{3}{2} CH_1^2 = \frac{3}{2} C\left(\frac{H\Omega}{\Omega_1}\right)^2$$

The starting torque is

$$M_0 = \frac{3}{2} \frac{SH_1^2}{a},$$

the maximum being

$$M_{\text{max.}} = \frac{3}{2} \frac{SH_1^2}{\alpha}$$

thus

$$\frac{M_0}{M_{\text{max.}}} = \frac{\alpha}{a} = 1 - \frac{a_1}{a} = s,$$

s being the *slip* of the motor.

The energy given to the motor being Ma and the energy given out Ma_1, the efficiency η is given by

$$\eta = \frac{Ma_1}{Ma} = 1 - \frac{a - a_1}{a} = 1 - s$$

Thus in order to get high efficiencies the slip s must be kept as small as possible. For example, if a motor is designed to develop its maximum torque with a slip of 5 per cent. the maximum efficiency possible is 95 per cent. The starting torque would be 5 per cent. of the maximum torque; if it is desired to get the maximum torque on starting S must be increased by some means to the value

$$S = aC$$

and as soon as the motor runs up to speed the leakage S should be reduced to the value

$$S = (a - a_1)C = saC$$

in order to obtain the maximum torque near the synchronous speed.

Asynchronous Monophase Motors.—In the motor illustrated at Fig. 41, let a_1 be the speed of rotation of the rotor and I the maximum current in the stator.

The displacement Δ will be

$$\Delta = -\frac{I}{a}\cos at = -\Delta\cos at$$

the current in the stator being $i = I\sin at$.

But the alternating displacement can be considered as the resultant of two displacements of half the amplitude rotating in opposite directions with angular velocity a (*see* p. 99).

The value of each component will be

$$\frac{I}{2a}\cos at$$

while the amplitude of the relative displacements will be

$$\frac{I}{2a}$$

and will correspond to virtual cranks of length

$$\frac{I}{2a\Omega} = r$$

The current J produced in the rotor will then be the sum of the two currents due to the motion of the rotor relative to the two rotating displacements considered.

The velocities of the rotor relative to these two displacements will be

$$a_1 = a - a_1$$

and

$$a_2 = a + a_1$$

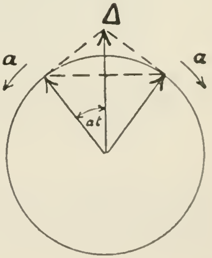

FIG. 43.

We have found at p. 112 that the mean torque due to a displacement rotating with angular velocity a in an asynchronous motor with three pistons is

$$M = \frac{3}{2}\frac{Sa\Delta_1^2}{S^2 + C^2a^2}$$

The motor we are now considering may be treated as the resultant of two asynchronous motors, one having a relative slip $a - a_1$ with a current J_1 and the other a relative slip $a + a_1$ with a current J_2, the two motors driving in opposite directions.

The resultant torque would then be the difference

$$M = S\triangle^2\left[\frac{a_1}{S^2 + C^2a_1^{\,2}} - \frac{a_2}{S^2 + C^2a_2^{\,2}}\right]$$

In this formula we have multiplied by the factor 2 instead of 3, as the motor we are considering has two pistons instead of three.

At starting $a_1 = a = a$, and we have $M_0 = 0$. It follows that the motor is not self-starting, and will require an impulse to set it rotating.

Put

$$a_1 = a - a_1 = sa$$

then

$$a_2 = a(1 - s) + a \sim 2a$$

and if s is small, we have

$$M = \frac{a\triangle^2}{S}\left[\frac{s}{1 + \left(\frac{Cas}{S}\right)^2} - \frac{2}{1 + \left(\frac{2Ca}{S}\right)^2}\right]$$

The torque is practically a maximum when

$$S = Cas$$

and we see that for this value the second term in the bracket becomes

$$\frac{2}{1 + \frac{4}{s^2}} \sim \frac{s^2}{2}$$

which is negligible.

The maximum torque is, therefore, about

$$M_{\text{max.}} = \frac{as\triangle^2}{2S} = \frac{\triangle^2}{2C}$$

which is similar to the value found for three-phase motors, having the factor 2 instead of 3 as multiplier of the torque

$$M_1 = \frac{\triangle^2}{4C}$$

due to a single phase in the rotor.

Collector Motors. — The motors described above develop a comparatively small starting torque, which can only be increased by devices involving a loss of energy. Such motors are not adapted to cases in which a variable speed is required without loss of power. In order, therefore, to obtain high starting torques it is necessary to employ a different construction.

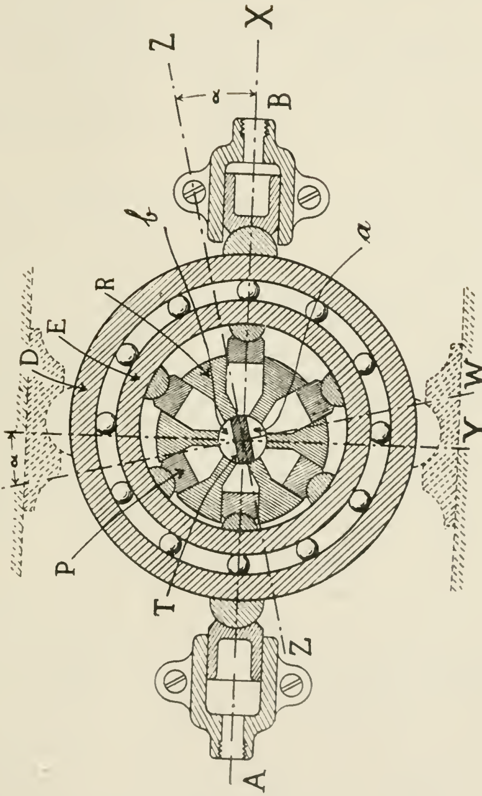

Fig. 44.

The characteristic feature of this type of motor is that the currents produced by the pistons of the rotor are collected and suitably distributed by a fixed member, which we shall call a collector, or distributer. To this type of motor we shall give the general designation *collector motors.* The rotor is formed of a number of cylinders, uniformly spaced, carrying pistons and capable of rotating about the fixed collector, or distributer. A member which we have above termed a transmitter is interposed between the pistons of the stator and the rotor as in the various types of asynchronous motors above described.

One form of collector motor is shown at Fig. 44. The essential parts are shown in full lines. The rotor R is formed by a number of cylinders carrying pistons P uniformly distributed around a circle; all the cylinders communicate with a cylindrical space divided by a partition T into two equal parts ab. The partition T is concentric with the rotor and allows the rotor to rotate freely, so that the cylinders of the rotor are brought successively into communication with the two compartments a and b. The two compartments a, b are extended outside the rotor by two fixed outlet pipes, which we shall also designate by the letters a, b. The stator comprises two cylinders on the axis XX having outlets A and B. Surrounding the rotor there is a ring E in contact with the pistons P of the rotor through suitable sliding connections. By means of the ring E we can impress on all the pistons of the rotor a certain displacement through a second ring D acted on by the pistons of the stator and actuating the ring E through a ball-bearing, or other suitable anti-friction device. The cylinders of the rotor are assumed to be filled with liquid.

If an alternating movement is given by the stator to the ring D, this movement will result in the production of certain currents in the cylinders of the rotor, and these currents can be collected in the outlets a and b of the rotor.

Before proceeding with the theoretical investigation of this type of motor we shall show that the rotor with a number of pistons, as illustrated, is equivalent to a rotor with two pistons of a certain section Ω whose axes are on the line WW at right angles to the line ZZ in the plane of the partition T, the axis WW remaining fixed in space during the displacement of the ring E, while the rotor R rotates about the centre of the fixed partition T, this centre coinciding with the centre of the stator which is the point of intersection of the lines XX and YY.

Let us suppose that a certain displacement has taken place in any direction through the ring E. This displacement Δ will be the resultant of a number of elementary displacements in the direction of the different pistons of the rotor. Let the ring E, Fig. 45, move through a distance r in the direction WW, and let β be the angle between the axis of any one of the pistons and the line ZZ.

The component of the movement of the piston considered in the direction β will be $r \sin \beta$. Let ω be the section of the piston.

The displacement of liquid by this piston will be

$$\delta = \omega r \sin \beta$$

The total displacement of liquid in the space a will, therefore, be

$$\triangle = \Sigma_0^\pi \omega r \sin \beta$$

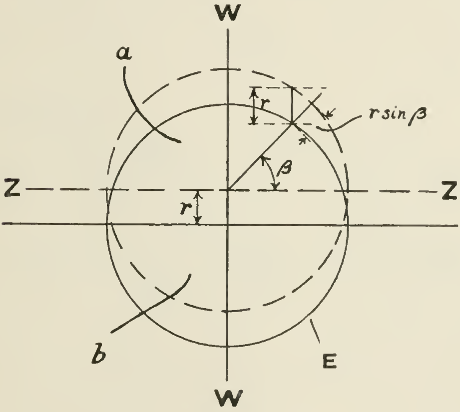

FIG. 45.

Representing by Ω the section of an equivalent piston which by a movement through a distance r in the direction WW would produce the same displacement of liquid, we should have

$$\triangle = \Omega r = \Sigma_0^\pi \omega r \sin \beta$$

or

$$\Omega = \Sigma_0^\pi \omega \sin \beta$$

If p is the total number of pistons in the rotor we shall have for the different angles β which enter into this sum

$$\beta_1 = \frac{2\pi}{p}, \quad \beta_2 = 2\frac{2\pi}{p}, \quad \beta_3 = 3\frac{2\pi}{p}$$

and

$$\Omega = \omega \sum_{k=1}^{k=\frac{p}{2}} \sin k \frac{2\pi}{p}$$

Taking the sum, we get

$$\Omega = \omega \cot \frac{\pi}{p}$$

If the arc $\frac{\pi}{p}$ is very small we can take simply

$$\Omega = \frac{p\omega}{\pi}$$

With this value of Ω, if we replace all the pistons on one side of the axis ZZ by a single piston of section Ω having its axis on the line WW, the displacement in the direction WW will be the same.

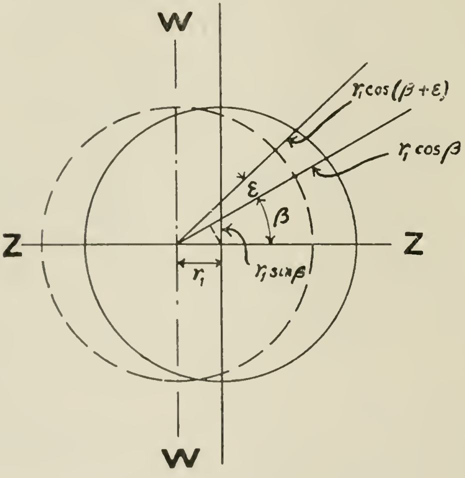

FIG. 46.

It is now necessary to show that for an instantaneous rotation of the rotor, with the ring E displaced through any distance r_1 in the direction ZZ, the displacement of the liquid is still the same in the two systems.

Referring to Fig. 46 for the piston whose axis is inclined at an angle β with the axis ZZ, the variation of displacement due to rotation of the rotor will be

$$\delta_1 = \omega r_1 [\cos \beta - \cos (\beta + \epsilon)] = \epsilon \omega r_1 \sin \beta$$

ϵ being the instantaneous angle of rotation.

We see, then, that the total displacement of liquid in this case will be

$$\triangle_1 = \epsilon\Sigma_0^\pi \omega r_1 \sin \beta$$

If the equivalent piston has sectional area Ω

$$\triangle_1 = \epsilon\Omega r_1$$

so that

$$\epsilon\Omega r_1 = \epsilon\Sigma_0^\pi \omega r_1 \sin \beta$$

and

$$\Omega = \Sigma_0^\pi \sin \beta$$

the same relation as obtained above.

We can now investigate certain cases of collector motors, utilising the relations established to simplify the analysis by replacing any system of pistons, such as that illustrated in Fig. 44, by a single equivalent piston.

Collector Motor with Connections in Parallel.—A collector motor of the type shown in Fig. 44 may be connected to a generator, as illustrated as Fig. 47. For simplicity the rotor is represented by its equivalent piston; *a*, *b* are the fixed outlets from the collector which communicates with the rotor, and *A*, *B* the stator connections. The two pipes leading from the stator and collector at *A* and *b* are in communication through a common pipe with one of the poles of a two-phase generator *G*, while the other outlets *B* and *a* are connected to the other pole of the generator. The generator may consist of a single double-acting piston, one side of which is in communication with *A* and *b*, while the other side is in communication with *B* and *a*.

Assume that the rings of the transmitter are restrained by parallel guides *G*, as shown in dotted lines in Fig. 44, so that no displacement can take place in the direction $a - b$ or $b - a$.

Suppose that the sectional area of the piston of the stator is equal to the sectional area Ω of the equivalent piston of the rotor. If the current produced by the generator is i, this will divide into two currents i_1, i_2, one flowing to the stator and the other flowing to the rotor; and we shall have at any instant

$$i = i_1 + i_2$$

The only displacement which can be communicated to the transmitter is a displacement in the direction AB; let this displacement be $\triangle$. Let a and a_1 be the angular velocities of the generator and rotor respectively. By reason of this rotation there will be produced in the rotor a dynamic current

$$i_2 = a_1 \triangle_1$$

If we neglect leakage and frictional losses, the current i_1 will be in quadrature with the hydromotive force H; and since the current i_1 is also in quadrature with the displacement Δ_1, it follows that Δ_1 and H are in phase. Since Δ_1 is a periodic function of a and we know that

$$i_1 = \frac{d\Delta_1}{dt}$$

we may write in symbolic notation

$$(I_1) = ja\Delta$$

where Δ is the amplitude of the displacement Δ_1.

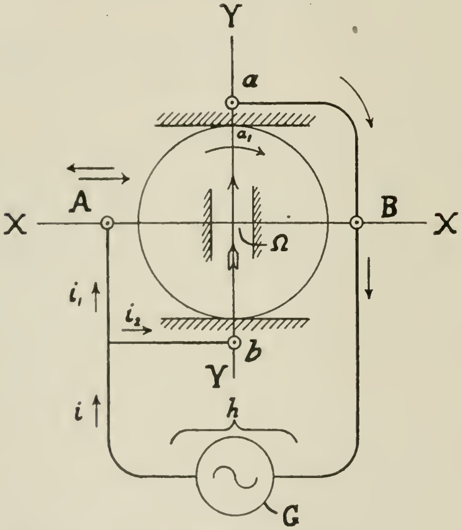

FIG. 47.

We then have, in symbolic notation

$$(I) = (I_1) + (I_2)$$

$$\therefore \quad (I) = ja\Delta_1 + a_1\Delta_1$$

and we have for the amplitude I of the current at the generator

$$I^2 = \Delta^2(a_1{}^2 + a^2)$$

so that

$$\Delta = \frac{I}{\sqrt{a_1{}^2 + a^2}}$$

The work done by the rotor is

$$Ma_1 = hi_2 = a_1 \triangle_1 h$$

The instantaneous torque, therefore, is

$$M = \triangle_1 h$$

We have found above that $\triangle_1$ and h are in phase, so that if we denote by $\triangle$ and H the amplitudes of these two quantities, we can put

$$h = H \sin at$$

$$\triangle_1 = \triangle \sin at$$

and we shall have

$$M = H\triangle \sin^2 at$$

The torque, therefore, is pulsating, and its mean value will be

$$M = \frac{H\triangle}{2}$$

Let C be the capacity in the circuit in which the current i_1 flows; we have arithmetically

$$I_1 = aCH = a\triangle$$

so that

$$\triangle = CH$$

and

$$M = \frac{\triangle^2}{2C}$$

Substituting for $\triangle$ its value found above, we get

$$M = \frac{I^2}{2(a^2 + a_1^2)C}$$

Let $\triangle_0$ be the displacement in the generator; we have arithmetically

$$I = a\triangle_0$$

so that

$$M = \frac{\triangle_0^2}{2C\left[1 + \left(\frac{a_1}{a}\right)^2\right]}$$

We see that the torque is a maximum when $a_1 = 0$, that is, at starting. If we choose for the normal speed $a_1 = 2a$, i.e., the rotor running at double the synchronous speed, the starting torque will be five times the normal torque.

We see, therefore, that this type of motor is very suitable for purposes of traction and cases in which self-regulation of power is required. We can find the hydromotive force as follows—

It has been found that

$$\triangle^2 = \frac{I^2}{a^2 + a_1^2}$$

Since $I = a\triangle_0$ we have

$$\triangle = \frac{\triangle_0}{\sqrt{1 + \left(\frac{a_1}{a}\right)^2}}$$

and further

$$H = \frac{\triangle}{C} = \frac{\triangle_0}{C\sqrt{1 + \left(\frac{a_1}{a}\right)^2}}$$

We see, then, that H is also a maximum at starting and decreases as the speed increases. The mechanical power will be

$$W = Ma_1 = \frac{a_1\triangle_0^2}{2C\left[1 + \left(\frac{a_1}{a}\right)^2\right]}$$

which is a maximum for the value of a_2 at which the expression

$$\frac{1}{a_1} + \frac{a_1}{a^2}$$

is a minimum. This occurs when $a_1 = a$, i.e., at the synchronous speed.

If $\cos\phi$ represents the power factor, we have

$$\frac{HI}{2}\cos\phi = W$$

and

$$\cos\phi = \frac{a_1}{\sqrt{a_1^2 + a^2}}$$

The value of this is zero at starting and approaches unity as a_1 approaches infinity. It is, therefore, advantageous to run the motor at the highest possible speed.

At the synchronous speed

$$\cos\phi = \frac{1}{\sqrt{2}} = 0{\cdot}71$$

at twice the synchronous speed

$$\cos\phi = \frac{2}{\sqrt{5}} = 0{\cdot}89$$

In the above investigation we have neglected losses; there is little difficulty in the more complete analysis taking account of losses; it is only necessary to follow a procedure analogous to that given above for the case of asynchronous motors without collectors.

CHAPTER VIII

THEORY OF RESONATORS

If we have a weight supported between two springs and under the influence of a periodic force, acting along the axis of the springs, the weight will commence to oscillate according to a certain law. If, however, the relative dimensions of the springs and weight are so chosen that the *natural period of vibration* of the weight (left free to oscillate after the periodic force has ceased to act) coincides with the period of the impressed force, the system formed by the weight and springs is called a *resonator*.

Theoretically such a resonator is capable of increasing the amplitude of its oscillations to infinity if there is no friction. If there is friction, however, this limits the amplitude at a certain point. Suppose now that, instead of dissipating the energy in friction, the energy is dissipated in a number of blows, as, for instance, if the weight is arrested in its course by an obstacle on each stroke, so that the amplitude of the motion of the weight is limited. In this case a discharge of energy will take place abruptly at each shock. It is readily seen that the energy of the shock depends on the velocity of the weight at the moment the blow is struck and is proportional to the mass and the square of this velocity. It is obvious, further, that the most powerful blow will be obtained when the resonator is so built that the velocity of striking is a maximum.

It can be shown by mathematical analysis that if the periodic force acting on the weight is due to a series of longitudinal waves travelling in the pipe and acting on a piston attached to the weight, the condition which gives the maximum blow will be that in which the weight is in equilibrium in contact with the obstacle to be hit, under the pressure exerted by the springs and the mean pressure in the pipe line.

Another necessary condition for the maximum blow is that the supporting springs should be designed so that the system is a resonator for the particular periodicity of the line. From these two conditions the relative size of the springs of the resonator can be readily calculated. If, instead of arranging the resonator for the maximum blow, we increase the amplitude of the movement of the resonator by moving the obstacle, the energy of the blow diminishes continuously until it vanishes completely when the amplitude is reached, at which further increase is prevented by the internal friction of the resonator. We shall define the amplitude of maximum blow as the *amplitude optima*. If we diminish the amplitude below this value the energy of the blow again diminishes until it obviously vanishes when the amplitude becomes zero, as the weight is then at rest.

In industrial applications of resonators, where a tool is required to give the maximum shock or hitting power with a minimum weight, the above condition of amplitude optima fulfils the requirement.

There are, however, in practice, cases in which it is required to build, say, a heavy hammer of very high lift or amplitude, but required to deliver light blows. In this case the desired result may be obtained by arresting the mass of the resonator nearer to the amplitude maxima and not at the amplitude optima.

It should be observed that if a resonator discharges its energy by successive blows, as above described, the curve representing the velocity or the current at the piston of the resonator is not a continuous curve, but is discontinuous, because at the moment of shock the velocity passes abruptly from a definite value to zero. This discontinuity of the current may be termed the *deformation* of the current supplied to the resonator.

It will be seen that the deformation is more pronounced for the conditions of amplitude optima and vanishes when the amplitude maxima is reached. In the latter case the curve of velocity of the current is a true harmonic curve. Thus the term *deformation* indicates the variation in shape of the current curve from the true harmonic curve. We see that in order to obtain negligible deformation of the current an amplitude in the resonator near the amplitude maxima must be adopted.

The effect of the deformation of current by a resonator is to introduce higher harmonics in the transmission line. In the case in which we have several small tools worked from a main transmission line these deformations do not give rise to difficulty, but in the case of heavy resonators, such as forging hammers, the deformation of the current may give rise to trouble in the working of other tools operated simultaneously from the same transmission line. Generally, small tools of a portable character should be designed to give *maximum work with a minimum of weight*, and thus work with the amplitude optima. Forging hammers or like heavy instruments should be designed to *give a small deformation of the current.* The calculations as to discharge of energy in the form of blows is applicable also to the cases in which the energy is discharged in any other unsymmetrical way; for example, in the case of a single-acting water pump. In such cases other conditions than amplitude optima are required.

We will first investigate the case in which the energy is discharged in blows, as a limiting case. Other cases are intermediate between this and the other limiting case, in which the energy is delivered uniformly, as in an ordinary friction device.

We shall see that even in the limiting case of a hammer giving powerful blows, we can simplify considerably the calculation of the energy delivered in the case of amplitude optima by assuming that no deformation takes place and that the current is truly harmonic, with the hydromotive force and current in phase. In the case of hammers of the second type—that is, with small deformation of the current—we can apply the symbolic analysis, and the case again corresponds with that of the hydromotive force and current in phase.

We have seen that if a body whose coefficient of inertia is L and capacity C be placed in a liquid column which is pulsating under a hydromotive force h given by $h = H \sin (at + \phi)$, and if we have $LCa^2 = 1$, the body is in resonance. If in such a case the resistance is negligible, we have

$$h = L\frac{di}{dt} + \frac{1}{C}\int idt = L\frac{di}{dt} + \frac{\Omega}{C}y \quad . \quad . \quad . \quad . \quad (1)$$

where Ω is the section of the piston of the resonator and y is the distance which the piston has travelled.

Put

$$at = x$$

then we have

$$h = H \sin (x + \phi)$$

Differentiating (1) with regard to x we get

$$i + LCa^2\frac{d^2i}{dx^2} = aC\frac{dh}{dx} = aCH \cos (x + \phi)$$

and since

$$LCa^2 = 1$$

$$i + \frac{d^2i}{dx^2} = aCH \cos (x + \phi) \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If we assume that

$$i_0 = 0$$

when

$$x = 0$$

the solution of the equation is

$$i = A \sin x + Bx \sin (x + \phi) \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Taking as the origin of time the instant at which $i = 0$ and $y = 0$ the equation (1) gives

$$h_0 = L\left(\frac{di}{dt}\right)_0 = La\left(\frac{di}{dx}\right)_0$$

Differentiating (3) and substituting we get

$$A = \frac{H}{2}Ca \sin \phi$$

and finally

$$i = \frac{1}{2}aCH\left(\sin x \sin \phi + x \sin \overline{x + \phi}\right) \quad . \quad . \quad . \quad . \quad (4)$$

and since

$$\Omega y = \frac{1}{a}\int i dx$$

we get

$$y = \frac{CH}{2\Omega}\left(\sin x \cos \phi - x \cos \overline{x + \phi}\right) . \quad . \quad . \quad . \quad . \quad (5)$$

The energy stored at any instant will be

$$E = \int hidt = \frac{Li^2}{2} + \frac{\Omega}{C}\int yidt$$

but since

$$\int idt = \Omega y$$

we get

$$E = L\frac{i^2}{2} + \frac{\Omega y}{2C} + K$$

K being a constant.

Assume that E is zero when

$$i = 0$$

and

$$y = 0,$$

then

$$E = \frac{1}{2}\left(Li^2 + \frac{\Omega^2}{C}y^2\right)$$

substituting for i and y we get

$$E = \frac{1}{8}CH^2\left(x^2 + \sin x - 2x \sin x \cos \overline{x + 2\phi}\right) \quad . \quad . \quad . \quad (6)$$

the value of which is independent of ϕ at any instant given by

$$x = k\pi$$

It follows that the energy absorbed by a resonator is proportional to the square of the time.

The mean energy at a given instant will be

$$E = \frac{1}{8} Ca^2H^2t^2 = \frac{H^2t^2}{8L} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

We see, therefore, that a resonator is a very effective accumulator of energy.

The mechanical power at any instant is given by

$$\frac{dE}{dt} = \frac{H^2t}{4L} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If the instant considered is at a sufficiently long period of time from the commencement of the oscillations of the resonator the formulæ (4) and (5) may be simplified by neglecting the first terms in the brackets; and we have approximately

$$i = \frac{1}{2}aCHx \sin (x + \phi)$$

$$y = - \frac{CH}{2\Omega} x \cos (x + \phi)$$

and the energy stored is given by

$$E = \frac{CH^2}{8} x^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

If we write the expression for i in the form

$$i = I \sin (x + \phi)$$

in which I is the amplitude of i at a given instant, we have

$$I = \frac{aCH}{2} x$$

and, substituting in the expression for E, we get

$$E = \frac{I^2}{2a^2C}$$

Remembering that

$$LCa^2 = 1$$

we get finally

$$E = \frac{LI^2}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

This formula shows that the energy stored in a resonator is equal to the maximum kinetic energy of the mass forming the resonator. In the above analysis we have neglected the friction in the circuit. If the resonator is employed in conjunction with a machine performing mechanical work, or if there is friction in the circuit, the conditions are somewhat different.

If the coefficient of friction is R, the hydromotive force is given by the equation

$$h = Ri + L\frac{di}{dt} + \frac{1}{C}\int idt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

If h is a harmonic function of the form

$$h = H \sin at$$

the equation gives on differentiation

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = Ha \cos at$$

Putting

$$x = at$$

and multiplying by C, we get

$$LCa^2\frac{d^2i}{dx^2} + RCa\frac{di}{dx} + i = CHa \cos x$$

If we have the condition of resonance

$$LCa^2 = 1$$

the equation becomes

$$\frac{d^2i}{dx^2} + RCa\frac{di}{dx} + i = CHa \cos x$$

The solution of this equation is

$$i = Ae^{-\beta x} \cos (x + \psi) + \frac{H}{R} \sin x$$

subject to the condition that the quantity

$$\beta = \frac{RCa}{2} = \frac{R}{2aL}$$

is very small in comparison with unity, as is generally the case in practical problems.

Free oscillations.—Suppose that the hydromotive force is zero, and we have the resonator to *oscillate freely*, then if I is the amplitude of the current at time t and I_0 the amplitude of the current when $t = 0$, we have

$$I_t = I_0 e^{-\frac{R}{2L}t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

We see that the amplitude decreases gradually with the time; and the formula also shows that an infinite time elapses before the oscillations cease, *i.e.*, before $I_t = 0$. In actual practice this is not the case; a resonator put in oscillation comes to rest after a finite time, owing to the fact that the true law of friction is somewhat different from that assumed in the analysis given above.

Decrement.—The variation $\triangle I$ of the amplitude I_t during a time period $\triangle t = T$ is given by

$$\triangle I = -\frac{R}{2L} I_0 e^{-\frac{R}{2L}t} \triangle t$$

$$= -\frac{RT}{2L} I$$

Remembering that if n is the number of periods per second we have

$$nT = 1$$

we get

$$\frac{\triangle I}{I} = -\frac{R}{2nL}$$

This quantity we shall term *the decrement of the resonator.* It will be denoted by δ, so that we have

$$\delta = \frac{R}{2nL}$$

Thus, if we measure time by the number of periods, putting

$$t = mT = \frac{m}{n}$$

we can find the number of periods m at which a given amplitude is reached.

If we consider that the amplitude is negligible when I falls to one hundredth of its initial value I_0, the relation

$$\log\left(\frac{I_t}{I_0}\right) = -\frac{R}{2L} t$$

gives

$$4 \cdot 6 = \frac{R}{2L} t = \frac{Rm}{2nL} = \delta m$$

and we have

$$m = \frac{4 \cdot 6}{\delta}$$

For example, if a resonator has a decrement $\delta = 0 \cdot 1$, the number of periods before the oscillation is extinguished would be

$$m = \frac{4 \cdot 6}{0 \cdot 1} = 46$$

and if the frequency is, say, 23 per second, the time elapsed would be

$$t = \frac{m}{n} = \frac{46}{23} = 2 \text{ seconds}$$

Case of continuous oscillations with amplitude maxima.—The mathematical investigation of resonators when friction or leakage is taken into account may be simplified by making use of the symbolic notation.

The symbolic form of the general equation (11) is

$$H = RI + j\left(La - \frac{1}{Ca}\right)I$$

If, therefore,

$$La - \frac{1}{Ca} = 0$$

we have simply

$$H = RI \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

The energy supplied to the resonator is employed, first, in starting and gradually increasing the motion until the amplitude of the current reaches the maximum value given by equation (13). The hydromotive force H and current I are then in phase and the mechanical power supplied is

$$W = \frac{HI}{2} = \frac{RI^2}{2}$$

If the resonator is attached to a tool performing work at a certain average rate, we must add to the friction R a virtual friction given by the relation

$$R' = \frac{2W}{I^2}$$

The equation (13) will then become

$$H = (R + R')I = RI + \frac{2W}{I}$$

and we get

$$RI^2 - HI + 2W = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

from which

$$I = \frac{H \pm \sqrt{H^2 - 8RW}}{2R}$$

From this it is evident that the limiting value obtainable for work done is

$$W = \frac{H^2}{8R}$$

In the formula for I, of the positive and negative signs we must take only the positive, since for $W = 0$ we must have

$$H = RI$$

as shown by relation (14), so that we get definitely

$$I = \frac{H}{2R} + \sqrt{\left(\frac{H}{2R}\right)^2 - 2\frac{W}{R}} \quad . \quad . \quad . \quad . \quad . \quad (15)$$

We see that for the limiting value of work done the amplitude is reduced to half the amplitude obtained when no work is being done.

In the case in which R is negligible we have from (14)

$$I \sim \frac{2W}{H} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

We have found that the energy stored in a resonator at a given time is expressed by the relation

$$E = \frac{LI^2}{2}$$

Suppose that the work done takes the form of a *shock* at the end of the stroke of the resonator. It is clear that at the instant of the shock a sudden change takes place in the energy stored and the diminution of the kinetic energy would be approximately

$$\triangle E = LI\triangle I \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Supposing this variation of energy to be small in comparison with E, the mechanical energy to be supplied to the resonator during one period T before the next shock must be

$$\triangle E = WT = \frac{W}{n}$$

and therefore from (17) we get

$$\triangle I = \frac{W}{LIn}$$

Dividing by I, we have

$$\delta_1 = \frac{\triangle I}{I} = \frac{W}{nLI^2} = \frac{W}{2nE} = \frac{\pi W}{aE}$$

The quantity δ_1 we shall term the *deformation* of the current produced by the resonator when attached to a percussive tool.

If e represents the energy delivered in a single blow we have

$$\delta_1 = \frac{e}{2E} = \frac{\pi W}{aE} = \frac{W}{nLI^2}$$

It is useful to remember that the *deformation* and the *decrement* are similar quantities and the deformation is identical with the decrement of the resonator if it be assumed that the friction R which occurs in the expression for the decrement is due to the average work done by the resonator on the supposition that no work is supplied to the resonator to make up the deficiency in energy.

We have seen that the decrement has the value

$$\delta = \frac{R}{2nL}$$

If we replace R by the equivalent friction R' due to the work done by the resonator

$$R' = \frac{2W}{I^2}$$

we get

$$\delta_1 = \frac{W}{nLI^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

which is identical with that found above. We shall use the word *decrement* to denote the number δ due to internal friction in the resonator and the word *deformation* to denote the number δ_1 due to the work done by the resonator.

The deformation δ_1 indicates the changes produced in the current by percussive tools, or tools acting by impulses at each period such as forging hammers or pumps.

Application of the Theory to Hammers

Suppose we have a hammer consisting of a heavy body carried by a piston which is supported in a mean position by springs and subjected on one side to the action of a single alternating liquid column whose period of pulsation is equal to the natural time period of vibration of the hammer, i.e., $LCa^2 = 1$.

The springs of the condenser should be such that the mean pressure in the liquid column is balanced. Let H_0 be the mean pressure; this will produce a travel of the piston y_0 and the equation of motion will be

$$H_0 + h = L\frac{di}{dt} + \frac{\Omega}{C}(y - y_0)$$

but if the mean pressure is statically balanced

$$H_0 = \frac{\Omega y_0}{C}$$

so that we get

$$h = L\frac{di}{dt} + \frac{\Omega}{C}y$$

which is identical with the equation we have considered above, and is applicable to the case we are considering, provided we take as origin of y the position which the piston of the hammer takes up under the action of the mean pressure.

To obtain the *greatest effect* from the blow of the hammer it is necessary that at the instant the hammer starts upwards from its position of rest the hydromotive force h shall be negative in order that the hammer may be raised by the fall of pressure below the mean.

This is the case referred to above of a hammer working with amplitude optima.

In the general equations found above, make

$$\phi = -\pi$$

Then we get

$$i = -\frac{1}{2}HCax \sin x = -\frac{I}{2}x \sin x$$

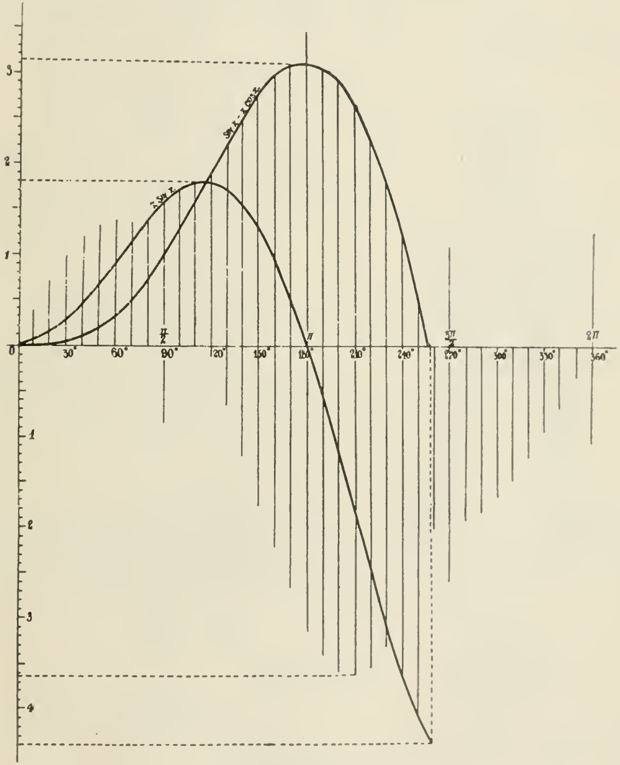

FIG. 48.

$$y = +\frac{HC}{2\Omega}(-\sin x + x \cos x)$$

$$h = -H \sin x$$

The values of y and i are plotted in the curves (Fig. 48). From these curves it is evident that y is a maximum when $x = \pi$ and vanishes after a time given by

$$\tan x = x$$

i.e., when

$$x = \pi + 77^\circ\, 25' = 4{\cdot}49$$

The value of i at this instant, which is the instant of striking the blow, is $2{\cdot}2\, HCa$.

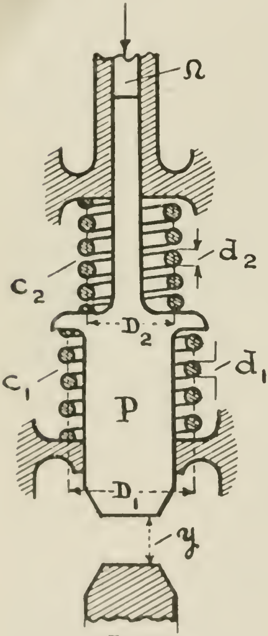

FIG. 49.

The maximum value of y will be

$$f = \frac{\pi HC}{2\Omega}$$

The energy of the hammer at the moment of striking is got by putting

$$x = \pi + 77^\circ\, 25'$$

and

$$\phi = -\pi$$

in the expression for E, and we get

$$E = 2{\cdot}4CH^2$$

If the two springs of the resonator have capacities C_1, C_2, and are arranged as in Fig. 49, we have

$$\frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{C}$$

On the other hand, the spring C_1 should be compressed by a distance f under a pressure H_0. In practice H_0 is very close to H and we may write

$$H\Omega = \frac{f\Omega^2}{C_1}$$

from which

$$\frac{1}{C_1} = \frac{H}{f\Omega}$$

But we have found above

$$\Omega f = \frac{\pi}{2} HC$$

so that we get

$$\frac{1}{C_1} = \frac{2}{\pi} \cdot \frac{1}{C}$$

If B_1, B_2, are the volumes of metal in the springs

$$B_1C_1 = B_2C_2 = BC$$

so that

$$B_1 = \frac{2}{\pi}B = 0{\cdot}638B$$

and since

$$B_1 + B_2 = B$$

we get

$$B_2 = \left(1 - \frac{2}{\pi}\right)B = 0{\cdot}362B$$

From the formula (13) obtained in Chapter III for springs we have

$$B = \alpha\frac{1}{C}(f\Omega)^2 = \alpha\frac{\pi^2}{4}H^2C$$

Putting E for $2{\cdot}4CH^2$ we get with sufficient accuracy for practical application

$$B = \alpha E$$

This shows that the volume of steel in the springs is proportional only to the energy delivered in one blow and to the stress allowed in the springs, and quite independent of other variables.

The value of α is given in Chapter III (page 19) for different stresses allowed in the springs.

The force necessary to move the piston of a condenser through a stroke f given by equation (12), p. 18, is—

$$F = \frac{f\Omega^2}{C} = \frac{\pi}{2}H\Omega = 1{\cdot}57H\Omega$$

Since we have two springs in the resonator and consider that each spring is in turn completely released for a stroke equal to f; if we denote by F_1, F_2 the forces necessary to compress each of these springs to the extent f, the above relation shows that

$$FC = F_1C_1 = F_2C_2$$

or,

$$\frac{F}{B} = \frac{F_1}{B_1} + \frac{F_2}{B_2}$$

and finally, since

$$\begin{aligned} B &= B_1 + B_2 \\ F &= F_1 + F_2 \\ F_1 &= 0{\cdot}638F = \mathrm{H}\Omega \\ F_2 &= 0{\cdot}362F = 0{\cdot}57\mathrm{H}\Omega \end{aligned}$$

For practical calculation of a hammer working under the conditions discussed above, that is, at amplitude optima, it is generally advisable to proceed as follows—

We have given the quantities

E = energy of each blow in kilogram-centimetres.
H = hydromotive force in the wave transmission line in kg./cm.^{2}
a = the pulsation of the wave in radians per sec.
P = the weight of the hammer in kilograms.
D_1, D_2 = the diameters of the front and back springs respectively.

From the relation

$$E = 2{\cdot}4CH^2$$

we calculate the capacity

$$C = \frac{E}{2{\cdot}4H^2}$$

Then from the resonance condition

$$LCa^2 = 1$$

we get

$$\Omega = \frac{a}{31{\cdot}3}\sqrt{PC}$$

The stroke is then obtained from the relation

$$f = \frac{\pi HC}{2\Omega}$$

The maximum force on the springs will be

$$F_1 = H\Omega$$
$$F_2 = 0{\cdot}57H\Omega$$

and the volumes of the springs will be

$$B_1 = \alpha F_1 f$$
$$B_2 = \alpha F_2 f$$

The diameters of the wire of the two springs are readily calculated from the formulæ in Chapter III.

$$d_1 = \beta\sqrt[3]{F_1D_1}$$
$$d_2 = \beta\sqrt[3]{F_2D_2}$$

Thus all the elements of the hammer are found.

Note.—Consider the hammer as a receiver whose power factor is unity (*i.e.*, the pressure and current in phase) and that the current taken is of sinusoidal form having a maximum value $I = ra\Omega$. The power taken in this case will be $\frac{HI}{2}$ and the work done during a single oscillation will be

$$E = \frac{HI}{2n} = \pi\frac{HI}{a} = \pi Hr\Omega$$

But we have found above

$$E = 2{\cdot}4CH^2 = \frac{4{\cdot}8}{\pi}Hf\Omega$$

so that we have

$$r = \frac{4{\cdot}8}{\pi^2}f$$

or approximately

$$r = \frac{f}{2}$$

We may, therefore, draw the following conclusion: The *maximum useful effect* of a hammer having a given stroke may be calculated simply by the formula $W = \frac{HI}{2}$; H and I being the maximum hydromotive force and the maximum current given by the equation $I = ra\Omega$ as though the current were of simple sine form and in phase with H. This maximum effect is only obtainable if the hammer is constructed so that with its condenser it is in resonance with the impressed hydromotive force, and is in equilibrium at the point of its stroke at which the blow is struck, under the action of the springs of the condenser and the mean pressure in the line supposed to act statically.

On the other hand, these conditions of a hammer working with amplitude optima at the same time produce the greatest deformation in the current, and such hammers should be of comparatively small dimensions so that they absorb a relatively small current from the main transmission line.

In cases in which a small deformation is required the formulæ given for the case of amplitude maxima can be applied. For example, to calculate a forging hammer, which generally requires a fairly high lift of a weight and a relatively small quantity of energy per second, we proceed as follows: In order to get considerable movement it is advisable to make use of a resonator consisting of the mass of the hammer of considerable inertia and two supporting springs or capacities. Instead of attempting to get the greatest effect from the blow, we impose the condition that the deformation of the current feeding the hammer is to have a given small value. Assume that the deformation δ_1 and the mechanical power W to be absorbed are given; and the hydromotive force H, frequency n and stroke f are known; we can determine the dimensions of the hammer.

We have the formulæ

$$I = \frac{2W}{H}$$

$$\delta_1 = \frac{W}{nLI^2}$$

Thus we get

$$L = \frac{H^2}{4n\delta_1 W}$$

But

$$L = \frac{P}{g\omega^2}$$

and

$$I = \frac{fa\omega}{2} = \pi n f \omega$$

$$P = 100 \frac{W}{f^2 n^3 \delta_1}$$

$$\omega = \frac{2W}{\pi H n f}$$

Thus the weight of the hammer and the section of the piston are determined.

CHAPTER IX

HIGH-FREQUENCY CURRENTS

Consideration of transmission lines of variable section.—Taking the equations established in Chapter V relating to pipes of variable section

$$\left.\begin{aligned}\frac{d(H)}{dx} &= ja(L)(I)\\ \frac{d(I)}{dx} &= ja(C)(H)\end{aligned}\right\} \quad \ldots \ldots \quad (1)$$

in which

$$(L) = L - j\frac{R}{a}$$

and

$$(C) = C - j\frac{S}{a}$$

We have found (*see* p. 15) for R when $l = 1$ cm.

$$R = k\frac{\gamma}{g\omega}$$

We have also

$$L = \frac{\gamma}{g\omega}$$

so that when

$$l = 1$$

we have

$$(L) = \frac{\gamma}{g\omega}\left[1 - j\frac{k}{a}\right]$$

Similarly we can write

$$S = k_1 C$$

with

$$C = \frac{\omega}{E}$$

and

$$(C) = \frac{\omega}{E}\left[1 - j\frac{k_1}{a}\right]$$

k and k_1 are constants whose values depend, the former on the *viscosity* and the latter on the *hysteresis* or plasticity of the fluid.

We may define the *viscosity* of any particular form of matter by the coefficient R or k, so that the energy lost owing to internal forces is of the form

$$W = \frac{RI^2}{2} = k\frac{LI^2}{2}$$

that is, a loss of kinetic energy.

Hysteresis may be defined by the coefficient S or k_1, so that the resulting loss of energy is

$$W_1 = \frac{SH^2}{2} = k_1\frac{CH^2}{2}$$

that is, a loss of potential energy.

It is to be noticed that k and k_1 are simple numerical coefficients representing the proportion of energy, kinetic or potential, which is transformed into heat or other form of energy and so disappears from the wave considered. These are two constants which, with the mass and the coefficient of elasticity, completely define any type of matter.

A species of matter is perfectly elastic when S or k_1 are zero and perfectly fluid if R and k are zero.

For ordinary materials none of the constants L, C, R, S are of zero value; and the great variety of different materials found in nature is due to the differing values of these quantities in different materials.

It should be observed that the constants are not absolutely independent of the values of H and I. For example, in the case of a gas the capacity C diminishes with the pressure, while the coefficient of inertia increases. In liquids also variations occur; viscosity varies with temperature as well as with the current. The plasticity or hysteresis should also vary with these conditions.

The property which is of great importance is, however, that these values may be considered as functions of the mean values of H and I, or of the temperature, and that we can treat them as constants over the cycle of vibratory movement provided the frequency is sufficiently high and the amplitude of the variations of H and I and the temperature very small in comparison with their absolute values.

We may, therefore, include solids among the different kinds of matter which we are studying, and the same equations are applicable. Thus, a metallic rod or a wire under high tension and subjected to longitudinal vibrations may be compared with a liquid column in longitudinal vibration, the same equations being applicable to the two cases. We shall thus have to consider the variation of *tension* or *compression* per unit of surface in the rod supposed uniform over the transverse section; it corresponds to H and is measured in kg./cm.^{2} We have also to consider the *current* in the rod—that is to say, the speed of any particle multiplied by the transverse sectional area.

From experiments we know that no metal is perfectly elastic, and if we submit a rod of metal to tension, and then relieve the tension, the rod does not return to exactly the same length as before. This property corresponds with the phenomenon of *leakage, plasticity* or *hysteresis*, and the loss of current due to it is represented by SH as in a column of liquid.

We shall, therefore, give a more general meaning to the definitions; *friction*, represented by the constant R, will include any retarding force in phase with the current due to movements in the body; and *hysteresis*, represented by the constant S, will include all loss of movement in phase with the pressure and due to the pressure.

The most general case of practical transmission of energy by longitudinal waves will be that in which the energy has to travel along a column of variable section. Let us study this case more closely.

We have found above

$$(L) = \frac{\gamma}{g\omega}\left(1 - j\frac{k}{a}\right)$$

$$(C) = \frac{\omega}{E}\left(1 - j\frac{k_1}{a}\right)$$

from which we get

$$(C)(L) = \frac{\gamma}{gE}\left(1 - j\frac{k}{a}\right)\left(1 - j\frac{k_1}{a}\right)$$

$$= \frac{\gamma}{gE}\left(1 + \frac{kk_1}{a^2} - j\frac{k + k_1}{a}\right)$$

We see, then, that the product $(C)(L)$ is independent of the sectional area of the transmission line.

We have found (*see* p. 73) the general equation

$$\frac{d^2(H)}{dx^2} + a^2(C)(L)(H) - \frac{1}{(L)}\frac{d(L)}{dx}\frac{d(H)}{dx} = 0$$

and an analogous equation for (I)

$$\frac{d^2(I)}{dx^2} + a^2(C)(L)(I) - \frac{1}{(C)}\frac{d(C)}{dx}\frac{d(I)}{dx} = 0$$

Taking into account the values of (L) and (C), we get

$$\frac{d^2(H)}{dx^2} + a^2(C)(L)(H) + \frac{1}{\omega}\frac{d\omega}{dx}\frac{d(H)}{dx} = 0$$

$$\frac{d^2(I)}{dx^2} + a^2(C)(L)(I) - \frac{1}{\omega}\frac{d\omega}{dx}\frac{d(I)}{dx} = 0$$

These equations are general and are identical with the equations (23) and (24) (*see* p. 73), with the difference only that the value

$$(\mu_1) = a\sqrt{(C)(L)}$$

is symbolic and is equal to

$$a\sqrt{CL}\sqrt{1 + \frac{kk_1}{a^2} - j\frac{k + k_1}{a}}$$

since

$$\mu = a\sqrt{CL}$$

we have then

$$(\mu_1) = \mu\sqrt{1 + \frac{kk_1}{a^2} - j\frac{k + k_1}{a}}$$

For *high frequency* the term $\frac{kk_1}{a^2}$ becomes very small and we can take simply

$$\mu_1 = \mu\sqrt{1 - j\frac{k + k_1}{a}}$$

This expression can be further simplified still, on the hypothesis that the frequency is high, by taking

$$\mu_1 = \mu\left(1 - j\frac{k + k}{2a}\right)$$

Let us now consider a pipe of conical form; in this case the section ω can be expressed by the relation

$$\omega = qx^2$$

where x is the distance of the section from the vertex of the cone, b being a numerical constant. This relation gives

$$\frac{1}{\omega}\frac{d\omega}{dx} = \frac{2}{x}$$

and the general equations become

$$\frac{d^2(H)}{dx^2} + \mu_1^2(H) + \frac{2}{x}\frac{d(H)}{dx} = 0$$

$$\frac{d^2(I)}{dx^2} + \mu_1^2(I) - \frac{2}{x}\frac{d(I}{dx} = 0$$

It should be noted that in these equations the constant q does not appear. It follows from this that if a number of conical pipes are supplied from the same source, they can be superposed and their walls suppressed; the results will be the same. We can then consider a vibratory "ray" formed by any virtual conical pipe having the angle of the cone as small as desired, and to investigate the phenomena in any conical pipe we have only to add the results obtained from the consideration of all the rays making up the cone.

Let us consider, then, what takes place in such a ray, and put

$$a = \mu_1 x$$

Substituting in the equations (1), we get

$$\left.\begin{aligned} \frac{d^2(H)}{da^2} + \frac{2}{a}\frac{d(H)}{da} + (H) = 0 \\ \frac{d^2(I)}{da^2} - \frac{2}{a}\frac{d(I)}{da} + (I) = 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The general solution of these equations is

$$\left.\begin{aligned} (H) &= \frac{1}{a}(A \cos a + B \sin a) \\ (I) &= A_1 \operatorname{Cor} a + B_1 \operatorname{Sir} a \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

in which

$$\operatorname{Cor} a = \cos a + a \sin a$$

and

$$\operatorname{Sir} a = \sin a - a \cos a$$

The functions Cor a and Sir a have interesting properties, some of which are

$$\operatorname{Sir}^2 a + \operatorname{Cor}^2 a = 1 + a^2$$

$$\sin a \operatorname{Cor} a - \cos a \operatorname{Sir} a = a$$

$$\cos a \operatorname{Cor} a + \sin a \operatorname{Sir} a = 1$$

Also between two variables a and β

$$\sin a \operatorname{Cor} \beta + \cos a \operatorname{Sir} \beta = \sin (a + \beta) - \beta \cos (a + \beta)$$

$$\cos a \operatorname{Cor} \beta - \sin a \operatorname{Sir} \beta = \cos (a + \beta) + \beta \sin (a + \beta)$$

$$\operatorname{Cor} a \operatorname{Cor} \beta - \operatorname{Sir} a \operatorname{Sir} \beta = \operatorname{Cor} (a + \beta) - a\beta \cos (a + \beta)$$

$$\operatorname{Sir} a \operatorname{Cor} \beta + \operatorname{Cor} a \operatorname{Sir} \beta = \operatorname{Sir} (a + \beta) - a\beta \sin (a + \beta)$$

Another series of properties of these functions is

$$\operatorname{Sir} (-a) = -\operatorname{Sir} a$$

$$\operatorname{Cor} (-a) = +\operatorname{Cor} a$$

$$\frac{d}{da} (\operatorname{Sir} a) = a \sin a$$

$$\frac{d}{da} (\operatorname{Cor} a) = a \cos a$$

$$\frac{d^2}{da^2} (\operatorname{Sir} a) = \operatorname{Sir} a + 2a \cos a$$

$$\frac{d^2}{da^2} (\operatorname{Cor} a) = \operatorname{Cor} a + 2a \sin a$$

The quantities a or β may be real numerical values or complex quantities.

Substituting $a = \mu_1 x$ in the equations (1), we get

$$\left.\begin{aligned}\frac{d(H)}{da} &= j(I)\sqrt{\frac{(L)}{(C)}}\\ \frac{d(I)}{da} &= j(H)\sqrt{\frac{(C)}{(L)}}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

We thus have to consider the complex quantity $\sqrt{\frac{(C)}{(L)}}$ which we will write (ψ); we shall then have, replacing (C) and (L) by their values found above,

$$(\psi) = \omega\sqrt{\frac{g}{\gamma E}}\sqrt{\frac{1 - j\frac{k_1}{a}}{1 - j\frac{k}{a}}}$$

But we have found (*see* p. 49) that

$$\psi = \omega\sqrt{\frac{g}{\gamma E}}$$

so that

$$(\psi) = \psi\sqrt{\frac{1 - j\frac{k_1}{a}}{1 - j\frac{k}{a}}}$$

Expanding the radical in powers of $\frac{k_1}{a}$ and $\frac{k}{a}$ we get in the case of high frequency in which these fractions are very small

$$(\psi) = \psi\left[1 - j\frac{k_1 - k}{2a}\right]$$

The value

$$\frac{1}{(\psi)} = \sqrt{\frac{(L)}{(C)}}$$

will be given by

$$\frac{1}{(\psi)} = \psi\left(1 + j\frac{k_1 - k}{2a}\right)$$

Differentiating the equations (3), we find

$$\frac{d(H)}{da} = -\frac{1}{a^2}(A \operatorname{Cor} a + B \operatorname{Sir} a)$$

$$\frac{d(I)}{do} = a(A_1 \cos a + B_1 \sin a)$$

Considering these equations with the equations (4), we have

$$\left.\begin{aligned} \frac{1}{a^2}(A \operatorname{Cor} a + B \operatorname{Sir} a) &= -j\frac{(I)}{(\psi)} \\ a(A_1 \cos a + B_1 \sin a) &= j(\psi)(H) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Let us represent by β the value of the angle a at the point at which the receiver is situated, the source or generator being at a point defined by the quantity a; we shall have for the point β

$$\left.\begin{aligned} A \operatorname{Cor} \beta + B \operatorname{Sir} \beta &= -j\beta^2\frac{(I)}{(\psi_\beta)} \\ A \cos \beta + B \sin \beta &= \beta(H) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The first of these equations is deduced from the equations (5) and the second from the equations (3).

From equations (6) we get directly

$$\left.\begin{aligned} A &= -\left[(H) \operatorname{Sir} \beta + j\beta\frac{(I)}{(\psi_\beta)} \sin \beta\right] \\ B &= (H) \operatorname{Cor} \beta + j\beta\frac{(I)}{(\psi_\beta)} \cos \beta \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

(ψ_β) represents the value of (ψ) at the point β, (ψ) being a quantity variable with x and also variable with β.

For the point β we may also write the equations

$$\left.\begin{aligned} A_1 \cos \beta + B_1 \sin \beta &= j\frac{(\psi_\beta)(H)}{\beta} \\ A_1 \operatorname{Cor} \beta + B_1 \operatorname{Sir} \beta &= (I) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

From these equations we get

$$\left.\begin{aligned} A_1 &= \frac{(I)}{\beta} \sin \beta - j\frac{(\psi_\beta)(H)}{\beta^2} \operatorname{Sir} \beta \\ B_1 &= -\frac{(I)}{\beta} \cos \beta + j\frac{(\psi_\beta)(H)}{\beta^2} \operatorname{Cor} \beta \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Substituting in the equations (3) the values of A, B, A_1, B_1 given by the equations (7) and (9), we have, finally

$$\left.\begin{aligned} (H_a) &= \frac{(H_\beta)}{(a)}\left[(\sin \overline{(a) - (\beta)} + (\beta) \cos \overline{(a) - (\beta)}\right] \\ &\qquad + j\frac{(\beta)}{(a)}\frac{(I_\beta)}{(\psi_\beta)} \sin \overline{(a) - (\beta)} \\ (I_a) &= \frac{(I_\beta)}{(\beta)}\left[(a) \cos \overline{(a) - (\beta)} - \sin \overline{(a) - (\beta)}\right] \\ &\qquad + j\frac{(\psi_\beta)(H_\beta)}{(\beta)^2}\left[\operatorname{Sir} \overline{(a) - (\beta)} + (a)(\beta) \sin \overline{(a) - (\beta)}\right] \end{aligned}\right\} \quad . \quad (10)$$

The indices α and β applied to the different values of $(I)(H)$ and (ψ) indicate the values of these quantities at the points defined by the angles α and β respectively. The formulæ (10) give the complete solution of the problem.

If we take friction and hysteresis to be zero, we can replace (H), (I) and (ψ) in these formulæ by H, I and ψ respectively. If this is not the case all the quantities α, β and ψ are complex and it is necessary to separate the terms containing j to arrive at explicit formulæ.

We have found above that α is of the form

$$(\alpha) = \mu_1 x = \mu x\left(1 - j\frac{k + k_1}{2a}\right)$$

Let us represent by α and β without brackets the numerical values of these quantities (their moduli neglecting the square of the quantity $\frac{k + k_1}{2a}$ compared with unity). To simplify the proof let us study the particular case in which the receiver and generator are at a considerable distance from the vertex of the cone, so that we can retain only terms containing α and β multiplying the circular functions.

The formulæ (10) are simplified and become

$$\left.\begin{aligned} H_\alpha &= \frac{\beta}{\alpha}\left[(H_\beta)\cos\overline{(\alpha) - (\beta)} + j\frac{(I_\beta)}{(\psi_\beta)}\sin\overline{(\alpha) - (\beta)}\right] \\ I_\alpha &= \frac{\alpha}{\beta}\left[(I_\beta)\cos\overline{(\alpha) - (\beta)} + j(H_\beta)(\psi_\beta)\sin\overline{(\alpha) - (\beta)}\right] \end{aligned}\right\} \quad . \quad (11)$$

But we know that if we have the complex angle

$$(\alpha) = \alpha_1 + j\alpha_2$$

we have the relations

$$\sin(\alpha) = \sin\alpha_1\cosh\alpha_2 - j\cos\alpha_1\sinh\alpha_2$$

$$\cos(\alpha) = \cos\alpha_1\cosh\alpha_2 + j\sin\alpha_1\sinh\alpha_2$$

the sinh and cosh being the well-known hyperbolic functions. If the points α, β are such that the distance between them is an integral number of wave lengths, and if we put

$$(\alpha) - (\beta) = (\alpha - \beta)\left(1 - j\frac{k + k_1}{2a}\right)$$

and if we have also

$$\alpha - \beta = 2m\pi$$

we get

$$\sin\overline{(\alpha) - (\beta)} = -j\sinh(\alpha - \beta)\frac{k + k_1}{2a}$$

$$\cos\overline{(\alpha) - (\beta)} = \cosh(\alpha - \beta)\frac{k + k_1}{2a}$$

Using these relations, the formulæ (11) become

$$\left.\begin{aligned}(H_\alpha) &= \frac{\beta}{\alpha}\left[(H_\beta)\cosh(\alpha-\beta)\frac{k+k_1}{2a} + \frac{(I_\beta)}{(\psi_\beta)}\sinh(\alpha-\beta)\frac{k+k_1}{2a}\right] \\ (I_\alpha) &= \frac{\alpha}{\beta}\left[(I_\beta)\cosh(\alpha-\beta)\frac{k+k_1}{2a} + (H_\beta)(\psi_\beta)\sinh(\alpha-\beta)\frac{k+k_1}{2a}\right]\end{aligned}\right\} \quad (12)$$

It is to be noticed that we can replace (ψ_β) by ψ, the term containing j in the expression for (ψ_β) being negligible.

Representing the number of wave lengths between the receiver and the generator by m and the frequency by n, we have in the case in which H_β and I_β are in phase

$$\left.\begin{aligned}H_\alpha &= \frac{\beta}{\alpha}\left[H_\beta\cosh\gamma + \frac{I_\beta}{\psi_\beta}\sinh\gamma\right] \\ I_\alpha &= \frac{\alpha}{\beta}\left[I_\beta\cosh\gamma + \psi_\beta H_\beta\sin\gamma\right]\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (13)$$

in which

$$\gamma = \frac{m(k+k_1)}{2n}$$

Noticing that $m\lambda = l =$ the distance between the receiver and the generator, and, moreover, that

$$n\lambda = v = \text{the velocity of sound}$$

we have

$$\gamma = \frac{l}{v}(k+k_1)$$

Comparing the formulæ (13) with the formulæ (7, 8) (p. 79) for cylindrical pipes, we see that they only differ by the factors $\frac{\beta}{\alpha}$ and $\frac{\alpha}{\beta}$.

Infinitely long pipes.—If, in the preceding formulæ, we put $H_\beta = 0$ and $l = 8$, the equations become indeterminate. We will treat this case in a different manner.

Let us suppose that the distance l between the generator and the receiver is so great that we can put

$$\cosh\gamma = \sinh\gamma = \frac{1}{2}e^\gamma$$

we have

$$H_\alpha = \frac{\beta}{2\alpha}e^\gamma\left[H_\beta + \frac{I_\beta}{\psi_\beta}\right]$$

$$I_\alpha = \frac{\alpha}{2\beta}e^\gamma\left[I_\beta + \psi_\beta H_\beta\right]$$

From these two equations we get

$$\psi_\beta \frac{H_\alpha}{I} = \left(\frac{\beta}{\alpha}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Let us represent by V_α the amplitude of the velocity at the point α, and by ω_α and ω_β the sectional areas of the pipe at the same two points.

We shall have

$$\psi_\beta = \omega_\beta \sqrt{\frac{g}{\gamma E}}$$

and

$$I_\alpha = V_\alpha \omega_\alpha$$

Substituting in (14), we get

$$\frac{H_\alpha}{V_\alpha} = \frac{\omega_\alpha}{\omega_\beta} \cdot \left(\frac{\beta}{\alpha}\right)^2 \sqrt{\frac{\gamma E}{g}}$$

But, on the other hand, we have

$$\frac{\omega_\alpha}{\omega_\beta} = \frac{\alpha^2}{\beta^2}$$

We get then the important relation

$$V_\alpha = \frac{H_\alpha}{\sqrt{\frac{\gamma E}{g}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

This relation shows that the amplitude of the velocity is simply proportional to the hydromotive force when the alternating wave travels to infinity without interference of reflection towards the source. This equation also allows us to calculate the maximum energy which can be emitted by a vibrating surface. Leaving on one side the indices α, let H, I be the hydromotive force and current at a point in the path of a conical vibratory ray. Let ω be the area of the vibrating surface supposed placed at a considerable distance from the vertex of the cone.

The energy per second emitted by this surface towards infinity will be

$$W = \frac{HI}{2} = \omega \frac{HV}{2}$$

But we have found that

$$V = \frac{H}{\sqrt{\frac{\gamma E}{g}}}$$

We therefore get

$$W = \frac{\omega}{2} H^2 \frac{I}{\sqrt{\frac{\gamma E}{g}}}$$

which is a finite quantity.

Example I

It is required to find the maximum vibratory energy which can be emitted by a vibrating surface of one 1 sq. cm. area immersed in water, the mean pressure of the water being 1 kg. per square centimetre.

We have for water

$$\sqrt{\frac{\gamma E}{g}} = \frac{1}{7}$$

As the pressure in the water must not become negative, we have for the maximum value of H which is possible

$$H = 1 \text{ kg./cm.}^2$$

so that

$$W = 3{\cdot}5 \text{ kg./cm.}$$

It follows that a vibrating surface of 1 sq. metre area cannot transmit in water more energy than is given by

$$W = 3{\cdot}5 \times 10^4 \text{ kg./cm.} = 4{\cdot}6 \text{ hp.}$$

This result is independent of the frequency if this is sufficiently high.

Example II

In the preceding example find the amplitude r of the displacement of the vibrating surface when transmitting the maximum possible vibratory energy, the frequency being $n = 100$ per second.

We have

$$V = ra = 2\pi rn$$

But

$$V = \frac{H}{\sqrt{\frac{\gamma E}{g}}}$$

so that

$$r = \frac{H}{2\pi n} \frac{1}{\sqrt{\frac{\gamma E}{g}}} = \frac{7}{2\pi \times 100} = 0{\cdot}011 \text{ cm.}$$

With a frequency of 1000 per second the amplitude will be about one hundredth of a millimetre. We see, therefore, that it is useless to produce large displacements of surfaces vibrating under water to obtain strong radiation of energy To produce under water sound waves of great energy, ***a large radiating surface and not a large amplitude is required.*** In air the phenomenon is analogous with the difference that the amplitude admissible to obtain the maximum radiated energy from a given vibrating surface in air is much greater than in water.

If we consider the motion quite close to the source of a conical ray, we see from the formulæ established above that certain peculiarities appear. We have discussed above the case of infinite pipes where β becomes infinite; the complete discussion of the equation (10) would show what takes place if the point β is quite close to the source, but this discussion would occupy too much space and is beyond the ambit of this work.

Conical pipes of infinite length with the generator near the apex.—Going back to the equations (3)

$$\left.\begin{aligned}(H) &= \frac{1}{a}\left(A \cos a + B \sin a\right)\\ (I) &= A_1 \operatorname{Cor} a + B_1 \operatorname{Sir} a\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (16)$$

In the case of an infinitely long pipe we can determine the constants A, B, A_1, B_1 from the consideration that, when

$$a = \infty$$

we have

$$(H) = 0$$

and the velocity

$$V = \frac{I}{\omega} = 0$$

which is equivalent to the relation

$$\left(\frac{I}{a^2}\right) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

We are justified in assuming these conditions because, owing to the dissipation of energy along the length of the pipe, there will be no variation of pressure and no movement of liquid at the point $a = \infty$.

Substituting in (1) the exponential values for cos a and sin a, *i. e.*

$$\cos a = \frac{e^{ja} + e^{-ja}}{2}$$

and

$$\sin a = \frac{e^{ja} - e^{-ja}}{2j} = -j\frac{e^{ja} - e^{-ja}}{2}$$

the equations become

$$\left.\begin{aligned}(H) &= \frac{1}{a}\left(De^{ja} + Ee^{-ja}\right)\\ (I) &= \frac{A_1}{2}\left[e^{ja} + e^{-ja} - ja(e^{ja} - e^{-ja})\right]\\ &\quad - \frac{B_1}{2}\left[a(e^{ja} + e^{-ja}) + j(e^{ja} - e^{-ja})\right]\end{aligned}\right\} \quad . \quad (18)$$

where D and E are arbitrary constants.

In the equations (18), putting $a = \infty$ and remembering the relation (17), we get

$$D = 0$$

The expression for (I), when a is infinite, becomes

$$(I)_{a=\infty} = \frac{1}{2}e^{ja}\left[A_1(1 - ja) - B_1(a + j)\right]$$

In order that the condition (17) may be satisfied when $\alpha = \infty$ we must have

$$\left[\frac{(I)}{\alpha^2}\right]_{\alpha=\infty} = \frac{e^{j\alpha}}{2\alpha}\left[A_1\left(\frac{1}{\alpha} - j\right) - B_1\left(1 + \frac{j}{\alpha}\right)\right] = 0$$

or, since $\frac{e^{j\alpha}}{\alpha}$ becomes infinite when α is infinite, we must have

$$-jA_1 - B_1 = 0$$

or

$$A_1 = jB_1$$

We see, then, that in the case of an infinite conical pipe the expressions for (H) and (I) become

$$\left.\begin{aligned}(H) &= \frac{1}{\alpha}Ee^{-j\alpha} = \frac{1}{\alpha}E(\cos\alpha - j\sin\alpha)\\ (I) &= B_1(\text{Sir}\,\alpha + j\,\text{Cor}\,\alpha)\end{aligned}\right\} \quad . \quad . \quad . \quad (19)$$

The constants E, B_1, will be determined by the values of (H) and (I) at the generator; denoting by (H_0), (I_0), α_0 the values of the corresponding quantities at the generator we have for a point defined by the quantity α

$$\left.\begin{aligned}E &= \frac{(H)\alpha}{\cos\alpha - j\sin\alpha} = \frac{(H_0)\alpha_0}{\cos\alpha_0 - j\sin\alpha_0}\\ B_1 &= \frac{(I)}{\text{Sir}\,\alpha + j\,\text{Cor}\,\alpha} = \frac{(I_0)}{\text{Sir}\,\alpha_0 + j\,\text{Cor}\,\alpha_0}\end{aligned}\right\} \quad . \quad . \quad . \quad (20)$$

An interesting relation may be obtained by differentiating the equations (19), namely

$$\frac{d(H)}{d\alpha} = -\frac{E}{\alpha^2}\left(\text{Cor}\,\alpha - j\,\text{Sir}\,\alpha\right) = j\frac{E}{\alpha^2}\left(\text{Sir}\,\alpha + j\,\text{Cor}\,\alpha\right) = j\frac{E}{\alpha^2}\frac{(I)}{B_1}$$

But we have found at p. 144 the formula

$$\frac{d(H)}{d\alpha} = j(I)\sqrt{\frac{(L)}{(C)}}$$

so that we get

$$\frac{E}{B_1} = \alpha^2\sqrt{\frac{(L)}{(C)}}$$

so that from the equations (20) we get

$$\frac{(H)}{(I)} = \alpha\sqrt{\frac{(L)}{(C)}} \cdot \frac{\cos\alpha - j\sin\alpha}{\text{Sir}\,\alpha + j\,\text{Cor}\,\alpha}$$

For high frequencies we can replace (L) and (C) by L and C, and we get simply

$$\frac{(H)}{(I)} = a\sqrt{\frac{L}{C}} \cdot \frac{\cos a - j\sin a}{\operatorname{Sir} a + j\operatorname{Cor} a} \quad . \quad . \quad . \quad . \quad . \quad (21)$$

and for arithmetical values

$$H\sqrt{\left[(\operatorname{Sir} a)^2 + (\operatorname{Cor} a)^2\right]} = Ia\sqrt{\frac{L}{C}} = H\sqrt{(1 + a^2)}$$

It should be noticed that the product $a\sqrt{\frac{L}{C}}$ is independent of the position of the point a along the length of the pipe, and we can then write

$$I = \psi H\sqrt{\frac{1 + a^2}{a^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

We see, therefore, that a very simple relation connects I and H.

This relation is important and allows us to calculate the maximum energy which can be radiated from a source in a conical pipe.

The relation between the vectors (H) and (I) found above at (21) may be written

$$(I) = \frac{\psi}{a}H(\operatorname{Sir} a + j\operatorname{Cor} a)(\cos a + j\sin a)$$

$$= \frac{\psi}{a}H\left[\operatorname{Sir} a\cos a - \operatorname{Cor} a\sin a + j(\operatorname{Cor} a\cos a + \operatorname{Sir} a\sin a)\right]$$

$$= \frac{\psi}{a}H(j - a)$$

If ϕ is the phase angle between I_0 and H_0, we see that

$$\tan\phi = \frac{1}{a}$$

$$\cos\phi = \frac{a}{\sqrt{1 + a^2}}$$

The mechanical power developed at the source will then be

$$W = \frac{HI}{2}\cos\phi = \frac{\psi}{2}H^2$$

The following table shows the power factors for different values of α.

α	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$\cos\phi$	0	0·62	0·85	0·93	0·95	0·97	0·98	0·99	1·0

We see that this factor is very near the limit even for $\alpha = \frac{3\pi}{4}$; showing that, as regards the formulæ to be employed, a generator situated half a wave length from the apex of the cone may be considered as a generator at a very considerable distance from the apex.

The relation (22)

$$I = \psi H \sqrt{\left(\frac{1}{\alpha^2} + 1\right)}$$

shows that even for the value of $\alpha_0 = \pi$ we shall have

$$I = 1{\cdot}05\psi H$$

which differs by only 5 per cent. from the value

$$I = \psi H$$

found by considering the generator situated at a very great distance from the apex.

It follows that the approximation given above can be applied without appreciable error if the generator or receiver is situated at a distance of not less than a half wave length from the apex.

If the distance is less than this, we can make use of the correction, which is comparatively simple.

In the analysis we have given above conical pipes have been dealt with, as these are of most practical importance; the analysis may, however, be adapted to pipes whose section variation follows a different law. In practice it is difficult to obtain a continuous variation of section, but the results obtained are practically the same in lines built up of uniform segments whose diameter increases or decreases progressively in steps, provided the length of the segments is small in comparison with the wave length.

An analogous case of discontinuity is investigated in the following chapter.

CHAPTER X

CHARGED LINES

In the previous chapters we have dealt with alternating fluid currents in pipes of constant or variable section without interruptions or discontinuities. In the present chapter we shall consider special types of transmission line which are not continuous but are made up of a series of lines interrupted at intervals by condensers, inertias, or leakage devices inserted at intervals. These condensers, inertias, and leakages will be regarded not, as heretofore, as single pieces of apparatus, but as forming part of a single system which consists of a pipe line with these devices inserted at regular intervals according to a definite law.

Before proceeding with the mathematical analysis we may state that by certain distribution of these pieces of apparatus along a transmission line, we can obtain a composite transmission line along which waves of a certain frequency will travel without interruption in a similar manner to that in which waves travel along a continuous line in which the capacity, inertia, friction and leakage are uniformly distributed, despite the fact that the line is built up of discontinuous collected capacities, inertias, friction and leakage devices.

It is not self-evident that such will be the case, and at first sight this may seem improbable; the object of the following analysis is to show that if such apparatus, which may be referred to generally as "*discontinuities*," be inserted in a line, there is a certain arrangement or distribution of these discontinuities for which their introduction in the line produces the same effect as capacity, inertia, friction and leakage uniformly distributed. In other words, a transmission line having a certain distribution of discontinuities is equivalent to a continuous line having a certain capacity, inertia, friction and leakage per unit length uniformly distributed.

This is a fact of importance and leads to very important practical consequences. It provides a method of building up transmission lines in which the coefficients C, L, R and S per unit length *may have any value which we desire to give them.*

Far-reaching consequences of this are that we can increase or decrease the velocity of the waves, vary the power lost in friction, obtain any desired wave length for a given frequency and fluid used, and generally vary the constants of a transmission line.

It has hitherto been assumed that the velocity of the waves travelling along the liquid in a pipe cannot exceed the velocity of sound in the liquid; we can, however, cause waves of a given frequency to travel at any velocity by a suitable distribution of discontinuities in the line.

To demonstrate this we shall consider a discontinuity inserted *in series* in the pipe as an *inertia* and represent it by (L_1). Thus, if we have an inertia L_1, a condenser of capacity C_1, a friction R_1, and a leakage S_1 in series, the drop of hydromotive force through the discontinuity would be (*see* p. 31)—

$$(H) = \left(R_1 + \frac{1}{S_1}\right)I + j\left(L_1 a - \frac{1}{C_1 a}\right)I$$

putting

$$(L_1) = L_1 - \frac{1}{C_1 a^2} - j\left(\frac{R_1}{a} + \frac{1}{S_1 a}\right)$$

we get, simply

$$(H) = j(L_1)aI \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In the same way, if at the discontinuity we have capacity, inertia, friction and leakage *in parallel* the drop of current will be

$$(I) = j(C_2)aH \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

denoting by (C_2) a symbolic capacity embodying all the discontinuities present at the point considered. The statements made at p. 33 relative to the interpretation of the symbolic forms of inertia are analogous to the above.

If we adopt the convention that the letters H, L_1, I, C_2, in the formulæ (1) and (2) all represent symbolic values, bearing in mind what has been said above, we may dispense with the brackets and write simply

$$H = jL_1 aI$$

$$I = jC_2 aH$$

and thus come to the conclusion that if we prove our proposition to be true for ordinary inertias in series or capacities in parallel, we may take it to be true for another form of discontinuity.

Suppose we have a line built up as shown in Fig. 50, having inertias L_1 in series and capacities C_2 in parallel.

We can write the following equations connecting three consecutive elements—

$$\left.\begin{aligned} H_{m-1} - H_m &= jL_1 a I_m \\ H_m - H_{m+1} &= jLa_1 I_{m+1} \end{aligned}\right\} \quad \ldots \ldots \quad (3)$$

Subtracting, we get

$$H_{m-1} - 2H_m + H_{m+1} = jL_1 a(I_m - I_{m+1}) \quad \ldots \quad (4)$$

But we have also

$$I_m - I_{m+1} = jC_2 a H_m$$

Substituting in (4), we get

$$H_{m-1} - 2H_m + H_{m+1} = -L_1 C_2 a^2 H_m$$

Putting

$$a\sqrt{L_1 C_2} = 2 \sin \frac{\theta}{2}$$

we get simply

$$H_{m-1} - 2H_m \cos \theta + H_{m+1} = 0 \quad \ldots \ldots \quad (5)$$

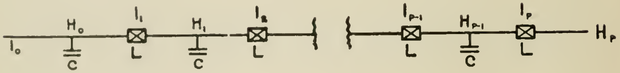

FIG. 50.

The equation (5) is satisfied if we put

$$H_m = A \cos \overline{p - m}\,\theta + B \sin \overline{p - m}\,\theta \quad \ldots \quad (6)$$

A and B being two arbitrary constants which can be determined as follows—

If we put $m = o$ and $m = p$ successively in equation (6) we get

and

$$\left.\begin{aligned} H_o &= A \cos p\theta + B \sin p\theta \\ H_p &= A \end{aligned}\right\} \quad \ldots \ldots \quad (7)$$

On the other hand, we can write successively

$$\left.\begin{aligned} H_{m-1} &= A \cos \overline{p - m + 1}\,\theta + B \sin \overline{p - m + 1}\,\theta \\ H_m &= A \cos \overline{p - m}\,\theta + B \sin \overline{p - m}\,\theta \end{aligned}\right\}$$

From which we get

$$H_{m-1} - H_m = A\,[\cos \overline{p - m + 1}\,\theta - \cos \overline{p - m}\,\theta] + B\,[\sin \overline{p - m + 1}\,\theta - \sin \overline{p - m}\,\theta]$$

In this equation, if we put $m = p$ we get

$$H_{p-1} - H_p = A(\cos \theta - 1) + B \sin \theta \quad \ldots \quad (8)$$

But from the equations (3), putting $m = p$, we get

$$H_{p-1} - H_p = jL_1 a I_p$$

Substituting in (8), we get

$$A(\cos\theta - 1) + B\sin\theta = jL_1 a I_p$$

from which

$$B = jI_p \frac{L_1 a}{\sin\theta} + H_p \tan\frac{\theta}{2}$$

Introducing these values of A and B in equations (7), we get

$$H_o = H_p\left(\cos p\theta + \sin p\theta \tan\frac{\theta}{2}\right) + jI_p \frac{L_1 a}{\sin\theta}\sin p\theta$$

Remembering that

$$2\sin\frac{\theta}{2} = a\sqrt{L_1 C_2}$$

this equation becomes

$$H_o = H_p \frac{\cos \overline{p - \frac{1}{2}}\,\theta}{\cos\frac{\theta}{2}} + jI_p\sqrt{\frac{L_1}{C_2}}\frac{\sin p\theta}{\cos\frac{\theta}{2}} \quad . \quad . \quad . \quad (9)$$

This equation gives us the relations which connect H_o, H_p, I_p; that is, the hydromotive force at the ends of the line and the current at one end, namely, the receiver end.

A similar equation can be obtained giving the relations between I_o, I_p, and H_p by a similar analysis starting from the equivalent equations

$$I_{m-1} - I_m = jC_2 a H_{m-1}$$

$$I_m - I_{m+1} = jC_2 a H_m$$

from which we get

$$I_{m-1} - 2I_m + I_{m+1} = jC_2 a(H_{m-1} - H_m)$$

but

$$H_{m-1} - H_m = jLaI_m$$

and we get

$$I_{m-1} - 2I_m\cos\theta + I_{m+1} = 0$$

which is analogous to equation (5), the only difference being that I is substituted for H, so that we get

$$I_o = I_p \frac{\cos p - \frac{1}{2}\theta}{\cos\frac{\theta}{2}} + jH_p\sqrt{\frac{C_2}{L_1}}\frac{\sin p\theta}{\cos\frac{\theta}{2}}$$

Suppose now we consider an imaginary line which we shall call the *equivalent uniform line* having a length l and uniformly distributed inertia L' per unit length and also uniformly distributed capacity, friction and leakage C'', R' and S'' respectively per unit length. If we assume this line to have the same length as the discontinuous line above considered, and if we assume the discontinuities in the discontinuous line to be uniformly distributed, the equivalent line would have as inertia and capacity, friction and leakage per unit length

$$\left.\begin{aligned} L' &= \frac{pL_1}{l} \qquad & R' &= \frac{pR_1}{l} \\ C'' &= \frac{pC_2}{l} \qquad & S'' &= \frac{pS_2}{l} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

But we have found (*see* p. 47) that in a uniform line we have

$$H_o = H_p \cos \mu l + jI_p\sqrt{\frac{L'}{C''}} \sin \mu l$$

and

$$\mu = a\sqrt{L'C''}$$

Substituting for L', C'' from (10), we get

$$H_o = H_p \cos \mu l + jI_p\sqrt{\frac{L_1}{C_2}} \sin \mu l \quad . \quad . \quad . \quad . \quad (11)$$

and

$$\mu = \frac{ap}{l}\sqrt{L_1C_2} = 2\frac{p}{l}\sin\frac{\theta}{2}$$

In order that the equations (9), (10) and (11) may express the same relations we must have

$$\cos \mu l = \frac{\cos \overline{p - \frac{1}{2}}\,\theta}{\cos\frac{\theta}{2}} = \cos\left(2p \sin\frac{\theta}{2}\right)$$

$$\sin \mu l = \frac{\sin p\theta}{\cos\frac{\theta}{2}} = \sin\left(2p \sin\frac{\theta}{2}\right)$$

which may be written

$$\left.\begin{aligned} \frac{\cos \overline{2p - \frac{\theta}{2}}}{\cos\frac{\theta}{2}} &= \cos\left(2p \sin\frac{\theta}{2}\right) \\ \frac{\sin p\theta}{\cos\frac{\theta}{2}} &= \sin\left(2p \sin\frac{\theta}{2}\right) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (12)$$

θ being an angle less than 2π, the relation (12) cannot be rigorously satisfied unless

$$\theta = 0$$

But it should be noted that if θ is so small that we can put

$$\sin\frac{\theta}{2} = \frac{\theta}{2}$$

and

$$\cos\frac{\theta}{2} = 1$$

the conditions (12) reduce to

$$\cos 2p - \frac{\theta}{2} = \cos p\theta$$

$$\sin p\theta = \sin p\theta$$

The second of these equations is an identity; the first reduces to

$$\cos p\theta \cos\frac{\theta}{2} + \sin p\theta \sin\frac{\theta}{2} = \cos p\theta$$

which, if θ is a very small angle, reduces to

$$\theta \sin p\theta = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

or simply to

$$\sin p\theta = 0$$

If the friction and leakage in the discontinuities are negligible, this reduces to

$$\theta = \pm\frac{k\pi}{p}$$

We have found above

$$\mu l = 2p\sin\frac{\theta}{2}$$

If θ is a very small angle this reduces to

$$\mu l = p\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

At p. 47 we have found

$$\mu = \frac{2\pi}{\lambda}$$

λ being the wave length of the equivalent uniform line; substituting in (13), we get

$$2\pi\frac{l}{\lambda} = p\theta = \pm k\pi$$

or

$$\frac{2l}{\lambda} = \pm k$$

and

$$\frac{\theta}{2} = \frac{\pi l}{p\lambda}$$

and finally

$$l = \frac{k\lambda}{2}$$

It follows, therefore, that the condition necessary in order that a discontinuous line in which friction and leakage are negligible may be equivalent to a uniform line in which inertia and capacity are uniformly distributed is that the angle $\frac{\theta}{2}$ should be so small that we may take for the value of its sine the angle itself, and that the length of the line shall be an exact multiple of half wave lengths of the equivalent uniform line.

If s be the number of discontinuities in one wave length, we have

$$ks = 2p$$

$$l = \frac{p}{s}\lambda$$

$$\theta = \frac{1}{2}\frac{2\pi}{s}$$

The conditions for equivalence, therefore, reduce to the following practical rule:—

If we imagine a circle divided into a number of parts equal to the number of discontinuities in one wave length of the equivalent line obtained, then for equivalence the ratio of the arc to the chord of a subdivision must be approximately equal to unity, and the length of the line must be an exact number of half wave lengths of the equivalent uniform line.

As an example, suppose we have to calculate a charged line in which inertias in parallel and condensers in series are introduced as discontinuities in such a way that in the charged line $C = 0$ and $L = 0$.

Referring to the formulæ (3), (4), p. 46, we get

$$\left.\begin{aligned} H_o &= H \cos \mu l + jI\sqrt{\frac{R}{S}} \sin \mu l \\ I_o &= I \cos \mu l + jH\sqrt{\frac{S}{R}} \sin \mu l \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (15)$$

and we have for this case

$$\mu = a - j\beta$$

$$a = 0$$

$$\beta = \sqrt{RS}$$

Thus we get

$$\cos \mu l = \cos(-j\beta) = \cosh \beta$$
$$\sin \mu l = \sin(-j\beta) = -j \sinh \beta$$

and finally

$$\left.\begin{aligned} H_o &= H \cosh \beta l + I\sqrt{\frac{R}{S}} \sinh \beta l \\ I &= I \cosh \beta l + H\sqrt{\frac{S}{R}} \sinh \beta l \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (16)$$

The mechanical power supplied to the generator would be

$$W_o = \frac{H_o I_o}{2} = \frac{1}{2}\left[HI \cosh 2\beta l + \frac{H^2 S + I^2 R}{2\sqrt{RS}} \sinh 2\beta l\right]$$

The efficiency of the transmission would be

$$\eta = \frac{HI}{H_o I_o} = \frac{1}{\cosh 2\beta l + \frac{1}{2}\left(\frac{H}{I}\sqrt{\frac{S}{R}} + \frac{I}{H}\sqrt{\frac{R}{S}}\right)\sinh 2\beta l} \quad . \quad (17)$$

The maximum efficiency is obtained when

$$\frac{H^2 S}{2} = \frac{I^2 R}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

i.e., when the power losses are equally divided between friction and leakage.

In this case the maximum efficiency would be

$$\eta = e^{-2\beta l}$$

the value of β being equal to

$$\beta = \sqrt{RS}$$

We see from this that if we can keep the leakage and friction small the efficiency can be kept as high as we like.

In order to calculate a charged line made up of an ordinary uniform line in which suitable discontinuities have been inserted—

Let L, C, R, and S represent the resultant uniform characteristics of the charged line;

L_1, C_1, R_1, and S_1 the concentrated discontinuities *in series* in the line;

L_2, C_2, R_2, and S_2 the concentrated discontinuities *in parallel* on the line;

and L', C', R', and S' the uniform characteristics of the original continuous line not charged with discontinuities;

we have from the above the following relations—

$$L = L' + \frac{p}{l}\left(L_1 - \frac{1}{C_1 a^2}\right)$$

$$R = R' + \frac{p}{l}\left(R_1 + \frac{1}{S_1}\right)$$

$$C = C' + \frac{p}{l}\left(C_2 - \frac{1}{L_2 a^2}\right)$$

$$S = S' + \frac{p}{l}\left(S_2 + \frac{1}{R_2}\right)$$

We can calculate the charged line exactly as if it were a simple uniform line having characteristics L, R, C, S, provided the law of distribution given above at p. 160 is followed.

It follows that if it is desired to obtain a charged line in which the wave velocity is infinite we must have either L or C zero. In the example given above we have taken a case in which

$$L = 0 \text{ and } C = 0$$

this would be obtained if

$$L' = \frac{p}{l}\left(\frac{1}{C_1 a^2} - L_1\right)$$

and

$$C' = \frac{p}{l}\left(\frac{1}{L_2 a^2} - C_2\right)$$

that is, we must introduce as discontinuities condensers in series having capacity C_1 and inertia L_1; and inertias in parallel having inertia L_2 and capacity C_2.

We can take $C_2 = 0$ and allow for L_1 only the unavoidable inertia of the oscillating parts of the condensers.

If we neglect L_1 by building the condensers in series with very light oscillating parts we have simply

$$C_1 = \frac{p}{lL'a^2}$$

$$L_2 = \frac{p}{lC'a^2}$$

These equations determine the value of the capacity and inertias required for the discontinuities.

Examples

Problem I.—It is proposed to transmit 2000 hp. through a two-phase water wave transmission line to a distance of 10 kilometres with an efficiency of 90 per cent. Find the hydromotive force to be employed and the section of the pipes required with ordinary non-charged lines.

The line is made up of two parallel pipes each transmitting 1000 hp.

At p. 80 we have found that the maximum efficiency in a transmission line is given by the relation

$$\eta = e^{-2\beta l} \quad \ldots\ldots\ldots\ldots \quad (1)$$

In the case we are considering

$$l = 1000000 \text{ cm. and } \eta = 0{\cdot}9$$

We get from (1)

$$2\beta l = \log 1{\cdot}11 = 0{\cdot}104$$

We have found at p. 79 for water

$$\beta = \frac{k}{287000}$$

so that we have

$$k = 0{\cdot}015$$

Let ω be the section of one of the pipes transmitting 1000 hp.; the work at the generator end would be

$$W = \frac{HI}{2}$$

and since for maximum efficiency

$$I = H\sqrt{\frac{C}{L}} = \psi H$$

we get

$$W = \frac{HI}{2} = \psi\frac{H^2}{2} = 3{\cdot}5\omega H^2$$

If d is the internal diameter of the pipe we have further

$$W = 3{\cdot}5\frac{\pi}{4}(Hd)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Let v_e be the effective velocity in the pipe, and V the amplitude of the velocity, we have

$$I = \omega V = \psi H = 7\omega H$$

from which

$$V = 7H$$

$$v_e = \frac{V}{\sqrt{2}} = 4{\cdot}95H$$

Substituting in equation (2), we get

$$W = 1000 \times 76 \times 100 = 0{\cdot}112(v_e d)^2$$

and

$$v_e d = 8250 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

We have previously found at p. 15

$$100k = \frac{v_e}{d}\left(1 + \frac{9}{\sqrt{v_e d}}\right)$$

giving k and $v_e d$ their values found above, we get

$$\frac{v_e}{d} = 1{\cdot}36 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

From (3) and (4) we get

$$v_e = 105{\cdot}5 \text{ cm./sec.}$$

$$d = 78 \text{ cm.}$$

and further

$$V = v_e\sqrt{2} = 150 \text{ cm./sec.}$$

$$H = \frac{V}{7} = 21{\cdot}5 \text{ kg./cm.}^2$$

We see, therefore, that to transmit 2000 hp. over a distance of 10 kilometres with a loss of 10 per cent. the two pipes must be of 78 cm. diameter and a low hydromotive force of only 21·5 kg. cm.^{2} must be employed. Such a line would be cumbersome and expensive.

Problem II.—The same power is to be transmitted over the same distance as in *Problem I* with the same efficiency; but instead of an ordinary line, a charged line having inertias in parallel distributed at adequate intervals is to be employed.

This problem can be solved by successive trials as follows—

We have for the work in one pipe

$$W = \frac{HI}{2} = \psi\frac{H^2}{2} = 1000 \times 76 \times 100 \text{ kg./cm.}$$

Taking an arbitrary value for H say,

$$H = 500 \text{ kg./cm.}^2$$

we get

$$\psi = 61$$

Then we have

$$I = \psi H = 61 \times 500 = 3050 \text{ cm.}^3\text{/sec.}$$

Suppose we select a pipe having an internal diameter $d = 10$ cm., then

$$\omega = \frac{\pi}{4}d^2 = 78{\cdot}5 \text{ cm.}^2$$

The amplitude of the current would be

$$V = \frac{I}{\omega} = 390 \text{ cm./sec.}$$

and the effective velocity

$$v_e = \frac{390}{\sqrt{2}} = 276 \text{ cm./sec.}$$

The coefficient of friction R would be (*see* p. 15)

$$R = \frac{\gamma}{g\omega}\left(0{\cdot}01\frac{v_e}{d} + \frac{0{\cdot}09}{d}\sqrt{\frac{v_e}{d}}\right)$$

and (*see* pp. 77 and 80)

$$2\beta l = Rl\psi = 0{\cdot}256$$

The efficiency in this case would be

$$\eta = e^{-0{\cdot}256} = 0{\cdot}775$$

We see that the efficiency is too low; we must therefore use a larger size of pipe or choose a higher hydromotive force.

Suppose we take a pipe having an internal diameter of 12 cm.

We get

$$\omega = \frac{\pi}{4}12^2 = 113 \text{ cm.}^2$$

$$V = \frac{I}{\omega} = 270 \text{ cm./sec.}$$

$$v_e = \frac{270}{\sqrt{2}} = 191 \text{ cm./sec.}$$

$$2\beta l = Rl\psi = 0{\cdot}104$$

and the efficiency would be

$$\eta = e^{-0{\cdot}104} = 0{\cdot}9$$

We see, therefore, that the diameter of 12 cm. is correct. Knowing the diameter of the pipe and the value of ψ, we can calculate the capacity necessary in the pipe to get the required efficiency.

We have found

$$\psi = \sqrt{\frac{C}{L}} = 61$$

This gives

$$C = 61^2 L = 3721 L$$

but (from p. 23)

$$L = \frac{\gamma}{g\omega} = \frac{0{\cdot}001}{981 \times 113} = \frac{1}{1{\cdot}1 \times 10^8}$$

so that

$$C = \frac{3{\cdot}721}{981 \times 113} = \frac{1}{30000}$$

The uniform capacity of unit length of the line is

$$C' = \frac{\omega}{E} = \frac{113}{20000} = \frac{1}{177}$$

E being the coefficient of elasticity of water.

The inertias L_2 required to be inserted in parallel (see p. 162) would be given by the relation

$$C = C' - \frac{p}{a^2 l L_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

In order to determine p we must find the wave length of the charged line.

The velocity of the waves would be

$$v = \frac{1}{\sqrt{LC}} = 1820000 \text{ cm./sec.}$$

Suppose we choose a frequency of $n = 10$ cycles per second; the wave length would be

$$\lambda = \frac{1820000}{10} = 182000 \text{ cm.}$$

and the number of wave lengths in the total length of the charged line would be

$$q = \frac{1000000}{182000} = 5{\cdot}5$$

The current across the inertia would be

$$(I_2) = -j \frac{H}{L_2 a^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and we have numerically (see formula (5) above)

$$I_2 = H(C' - C)\frac{l}{p}$$

Suppose we put in the whole 10 kilometres of line 200 inertias distributed at intervals of 50 metres; then we have $p = 200$ and putting the numerical values in the formula (6) we get

$$I_2 = 500\left(\frac{1}{177} - \frac{1}{30000}\right)\frac{10^6}{200} = 14000 \text{ cm.}^3\text{/sec.}$$

In order to find the weight and displacement Δ_2 of the inertia, we have

$$I_2 = a\Delta_2$$

where

$$a = 2\pi n = 62{\cdot}8$$

Thus we get

$$\Delta_2 = \frac{14000}{62{\cdot}8} = 223 \text{ cm.}^3$$

Suppose we choose for the stroke $2r$ of the inertia piston

$$2r = 10 \text{ cm.}$$

we get for the section of this piston

$$\Omega = \frac{\Delta_2}{r} = 44{\cdot}6 \text{ cm.}^2$$

A piston of 7·5 cm. diameter would be about right.

The weight P of the inertia would be given by the known relation

$$L_2 = \frac{P}{g\Omega^2}$$

and from formula (6) we know that

$$L_2 = \frac{H}{a^2 I_2} = \frac{500}{62{\cdot}8^2 \times 14000} = \frac{P}{981 \times 44{\cdot}6^2}$$

so that

$$P = 17{\cdot}7 \text{ kg.}$$

As we have two lines differing in phase by 180° and the inertia is across the two lines, the weight of the piston necessary to charge both lines would be $2 \times 17{\cdot}7 = 35{\cdot}4$ kg.

We have thus determined all the elements necessary for the construction of the charged line. The above calculation is a simple case given merely as an example of the general method of determining long-distance high-pressure lines. For practical cases the analysis should be more complete, taking into account the losses in the inertias. These can be ascertained by considering the charging inertia across the two lines in series with a friction, including in this friction the losses due to friction of the piston and the drop of kinetic energy due to the sudden change of section of the branches to the main lines, which would necessarily be at right angles. The loss due to leakage past the pistons should also be taken into consideration. The complete analysis is somewhat longer, but presents no special difficulty.

It follows from the examples given above that in long lines we must employ high hydromotive forces in order to be able to use reasonable sizes of pipes. On the other hand, difficulties arise in the direct employment of these high hydromotive forces in the distribution of energy for practical purposes; and we are therefore compelled to use special apparatus designed to reduce the hydromotive force to practicable values in order to avoid excessive pressures in the distribution of power at the points at which it is to be utilised. Such instruments we shall call *transformers*; they consist broadly of differential pistons; the high pressure main transmission line is connected to a cylinder in which the liquid acts on a small piston which in turn actuates a larger piston working in a low-pressure cylinder. In this arrangement we get at the same time a transformation of the pressure and of the current. In the following chapter the theory of these instruments is discussed.

CHAPTER XI

TRANSFORMERS

In previous chapters we have discussed condensers and inertias subjected to the action of the same hydromotive force on both sides. We will now deal with apparatus designed to directly transform the magnitude of the hydromotive force. Such an apparatus we term generally a *transformer*. The simplest form of transformer consists of a differential piston in contact with the liquid in the transmission line on its two sides and free to oscillate about a mean position. The piston may, however, be acted on by springs, as illustrated at Fig. 51.

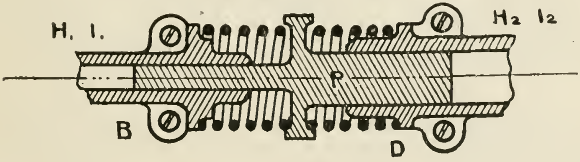

FIG. 51.

In this type of transformer the liquid in the transmission line B acts on a piston of section Ω_1 rigidly connected to a piston of section Ω_2 acting on the liquid in the pipe line D. The body of the piston is assumed to have a weight P and as illustrated is supported by a spring device or condenser having a characteristic A (see p. 18).

If we take an instrument exactly similar to that shown with the exception that the diameters of the cylinders B and D are equal, we can write (see p. 31)

$$H_1 - H_2' = j\left(L_1 a - \frac{1}{C_1 a}\right)I_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where

$$L_1 = \frac{P}{g\Omega_1^2}$$

$$C_1 = A\Omega_1^2$$

and H_2' is an equivalent hydromotive force so chosen that at any instant we have

$$H_2'\Omega_1 = H_2\Omega_2$$

That is, the force exerted by the liquid in the cylinder D on the piston P is equal in both instruments.

Under these conditions the movements of the pistons in both instruments are identical, and it follows that if we substitute for H_2' in the equation (1) we get the equation of movement of the transformer illustrated.

We therefore have

$$H_1 - H_2\frac{\Omega_2}{\Omega_1} = j\left(\frac{Pa}{g\Omega_1^2} - \frac{1}{aA\Omega_1^2}\right)I_1$$

This may be written

$$H_1\Omega_1 - H_2\Omega_2 = j\left(\frac{Pa}{g\Omega_1\Omega_2} - \frac{1}{aA\Omega_1\Omega_2}\right)\Omega_2 I_1$$

and, finally, putting

$$L = \frac{P}{g\Omega_1\Omega_2}$$

$$C = A\Omega_1\Omega_2$$

Since Ω_1 and Ω_2 are proportional to the currents $I_1 I_2$ in the two cylinders B and D, we have

$$H_1 I_1 - H_2 I_2 = j\left(La - \frac{1}{Ca}\right)I_1 I_2 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This equation is general if the friction and leakage in the transformer are neglected. L and C are respectively the *mutual coefficient of inertia* and *capacity* of the transformer.

It should be noted that if L_1, C_1, and L_2, C_2, represent the inertia of the piston and capacity of the springs in relation to the respective cylinders B and D, we have for the mutual coefficient of inertia and mutual capacity of the transformer

$$L = \sqrt{L_1 L_2}$$

$$C = \sqrt{C_1 C_2}$$

This may be easily verified, since by definition

$$L_1 = \frac{P}{g\Omega_1^2}$$

$$L_2 = \frac{P}{g\Omega_2^2}$$

and we get

$$\sqrt{L_1 L_2} = \frac{P}{g\Omega_1\Omega_2} = L$$

and further

$$C_1 = A\Omega_1^2$$
$$C_2 = A\Omega_2^2$$
$$\sqrt{C_1C_2} = A\Omega_1\Omega_2 = C$$

The currents I_1, I_2 are always in the same phase as the transformer piston moves in one piece from one side to the other, so that the velocities must be the same at both ends.

The equation (2) can be written

$$(H_1)I_1 = H_2I_2 + j\left(La - \frac{1}{Ca}\right)I_1I_2 \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and numerically

$$H_1I_1 = I_2\sqrt{H_2^2 + \left(La - \frac{1}{Ca}\right)I_1^2}$$

Examples

Problem I.—A transformer with a piston of weight P and sections Ω_1 and Ω_2 is placed on a branch of a primary transmission line in which the hydromotive force is H_1. No springs are used in the transformer. The secondary line is connected to an infinitely long transmission line of section ω. Find the primary and secondary currents I_1, I_2 and the secondary hydromotive force H_2.

In an infinitely long pipe we have (see p. 80)

$$I_2 = \psi H_2$$

From equation (3)

$$(H_1)I_1 = H_2I_2 + jLaI_1I_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and further we have

$$\frac{I_1}{I_2} = \frac{\Omega_1}{\Omega_2} = m$$

m being the ratio of transformation and

$$L = \frac{P}{g\Omega_1\Omega_2}$$

from (4) we get numerically

$$H_1 = \frac{I_2}{I_1}\sqrt{H_2^2 + (LaI_1)^2}$$

from which we get

$$I_1 = \frac{m^2\psi H_1}{\sqrt{1 + (\psi maL)^2}}$$

so that

$$I_2 = \frac{I_1}{m} = \frac{m\psi H_1}{\sqrt{1 + (\psi maL)^2}}$$

$$H_2 = \frac{I_2}{\psi} = \frac{I_1}{m\psi} = \frac{mH_1}{\sqrt{1 + (\psi maL)^2}}$$

If L is very small, as will be the case if a very light piston is used or if the sections Ω_1 and Ω_2 are large, these formulæ reduce to

$$I_1 \sim m^2\psi H_1[1 - \tfrac{1}{2}(\psi maL)^2]$$
$$H_2 \sim mH_1[1 - \tfrac{1}{2}(\psi maL)^2]$$

and for a first approximation

$$I_1 \sim m^2\psi H_1$$
$$H_2 \sim mH_1$$

Problem II.—The transformer of the previous example is connected to a resonator which absorbs no work or to an infinite capacity. Find the primary current.

In this case we have $H_2 = 0$ and the equation (4) of the previous example gives

$$(H_1) = jLaI_2 = j\frac{LaI_1}{m}$$

And we have numerically

$$I_1 = \frac{mH_1}{La}$$

It is readily seen from this that if the piston of the transformer is very light or L is very small, the current I_1 absorbed may become very great. From this it follows that it is advisable to provide sufficient inertia in the transformer in order to keep down the current absorbed when, for instance, the load is removed suddenly from the secondary line supposed connected to a free circuit without reactance or friction.

If we design a transformer with sufficient inertia L, we can use transformers in installations for the distribution of power to limit the current supplied to a given machine to any desired value.

Note.—A transformer with the ratio $\frac{\Omega_1}{\Omega_2} = 1$ reduces to an ordinary inertia or capacity according as inertia or elasticity predominates in the value of the total reactance.

APPENDIX

TABLE I

TABLE II

TABLE III

TABLE IV

TABLE V

TABLE I

WAVE CONSTANTS FOR WATER AT DIFFERENT TEMPERATURES

t = temperature in degrees Centigrade.
E = coefficient of elasticity in kg./cm.^{2}

$v = \sqrt{\frac{gE}{\gamma}}$ = velocity of sound in cm./sec.

$q = \frac{\psi}{\omega} = \sqrt{\frac{g}{\gamma E}}$ = the proportional factor per unit section.

p = pressure in kg./cm.^{2}

p = 0 to 100

t	$E \times 10^{-4}$	$v \times 10^{-5}$	q
0	1·94	1·38	7·12
10	2·06	1·42	6·90
20	2·13	1·44	6·78
30	2·18	1·46	6·71
40	2·20	1·47	6·69
50	2·22	1·48	6·66
60	2·22	1·48	6·69
70	2·20	1·47	6·71

p = 100 to 200

t	$E \times 10^{-4}$	$v \times 10^{-5}$	q
0	2·00	1·40	7·00
10	2·16	1·45	6·75
20	2·26	1·49	6·60
30	2·30	1·51	6·55
40	2·33	1·52	6·51
50	2·35	1·52	6·50
60	2·35	1·52	6·50
70	2·33	1·52	6·51

TABLE II

1–50 NUMERICAL VALUES

n	n^2	$\sqrt{n}$	$\log_e n$	$\frac{1000}{n}$	πn	$\frac{\pi n^2}{4}$	n
1	1	1,0000	0,00000	1000,000	3,142	0, 78 54	1
2	4	1,4142	0,69315	500,000	6,283	3, 14 16	2
3	9	1,7321	1,09861	333,333	9,425	7, 06 86	3
4	16	2,0000	1,38629	250,000	12,566	12, 56 64	4
5	25	2,2361	1,60944	200,000	15,708	19, 63 50	5
6	36	2,4495	1,79176	166,667	18,850	28, 27 43	6
7	49	2,6458	1,94591	142,857	21,991	38, 48 45	7
8	64	2,8284	2,07944	125,000	25,133	50, 26 55	8
9	81	3,0000	2,19722	111,111	28,274	63, 61 73	9
10	1 00	3,1623	2,30259	100,000	31,416	78, 53 98	**10**
11	1 21	3,3166	2,39790	90,9091	34,558	95, 03 32	11
12	1 44	3,4641	2,48491	83,3333	37,699	1 13, 09 7	12
13	1 69	3,6056	2,56495	76,9231	40,841	1 32, 73 2	13
14	1 96	3,7417	2,63906	71,4286	43,982	1 53, 93 8	14
15	2 25	3,8730	2,70805	66,6667	47,124	1 76, 71 5	15
16	2 56	4,0000	2,77259	62,5000	50,265	2 01, 06 2	16
17	2 89	4,1231	2,83321	58,8235	53,407	2 26, 98 0	17
18	3 24	4,2426	2,89037	55,5556	56,549	2 54, 46 9	18
19	3 61	4,3589	2,94444	52,6316	59,690	2 83, 52 9	19
20	4 00	4,4721	2,99573	50,0000	62,832	3 14, 15 9	**20**
21	4 41	4,5826	3,04452	47,6190	65,973	3 46, 36 1	21
22	4 84	4,6904	3,09104	45,4545	69,115	3 80, 13 3	22
23	5 29	4,7958	3,13549	43,4783	72,257	4 15, 47 6	23
24	5 76	4,8990	3,17805	41,6667	75,398	4 52, 38 9	24
25	6 25	5,0000	3,21888	40,0000	78,540	4 90, 87 4	25
26	6 76	5,0990	3,25810	38,4615	81,681	5 30, 92 9	26
27	7 29	5,1962	3,29584	37,0370	84,823	5 72, 55 5	27
28	7 84	5,2915	3,33220	35,7143	87,965	6 15, 75 2	28
29	8 41	5,3852	3,36730	34,4828	91,106	6 60, 52 0	29
30	9 00	5,4772	3,40120	33,3333	94,248	7 06, 85 8	**30**
31	9 61	5,5678	3,43399	32,2581	97,389	7 54, 76 8	31
32	10 24	5,6569	3,46574	31,2500	100,531	8 04, 24 8	32
33	10 89	5,7446	3,49651	30,3030	103,673	8 55, 29 9	33
34	11 56	5,8310	3,52636	29,4118	106,814	9 07, 92 0	34
35	12 25	5,9161	3,55535	28,5714	109,956	9 62, 11 3	35
36	12 96	6,0000	3,58352	27,7778	113,097	10 17, 88	36
37	13 69	6,0828	3,61092	27,0270	116,239	10 75, 21	37
38	14 44	6,1644	3,63759	26,3158	119,381	11 34, 11	38
39	15 21	6,2450	3,66356	25,6410	122,522	11 94, 59	39
40	16 00	6,3246	3,68888	25,0000	125,66	12 56, 64	**40**
41	16 81	6,4031	3,71357	24,3902	128,81	13 20, 25	41
42	17 64	6,4807	3,73767	23,8095	131,95	13 85, 44	42
43	18 49	6,5574	3,76120	23,2558	135,09	14 52, 20	43
44	19 36	6,6332	3,78419	22,7273	138,23	15 20, 53	44
45	20 25	6,7082	3,80666	22,2222	141,37	15 90, 43	45
46	21 16	6,7823	3,82864	21,7391	144,51	16 61, 90	46
47	22 09	6,8557	3,85015	21,2766	147,65	17 34, 94	47
48	23 04	6,9282	3,87120	20,8333	150,80	18 09, 56	48
49	24 01	7,0000	3,89182	20,4082	153,94	18 85, 74	49
50	25 00	7,0711	3,91202	20,0000	157,08	19 63, 50	**50**

50-100

n	n^2	$\sqrt{n}$	$\log_e n$	$\frac{1000}{n}$	πn	$\frac{\pi n^2}{4}$	n
50	25 00	7,0711	3,91202	20,0000	157,08	19 63, 50	**50**
51	26 01	7,1414	3,93183	19,6078	160,22	20 42, 82	51
52	27 04	7,2111	3,95124	19,2308	163,36	21 23, 72	52
53	28 09	7,2801	3,97029	18,8679	166,50	22 06, 18	53
54	29 16	7,3485	3,98898	18,5185	169,65	22 90, 22	54
55	30 25	7,4162	4,00733	18,1818	172,79	23 75, 83	55
56	31 36	7,4833	4,02535	17,8571	175,93	24 63, 01	56
57	32 49	7,5498	4,04305	17,5439	179,07	25 51, 76	57
58	33 64	7,6158	4,06044	17,2414	182,21	26 42, 08	58
59	34 81	7,6811	4,07754	16,9492	185,35	27 33, 97	59
60	36 00	7,7460	4,09434	16,6667	188,50	28 27, 43	**60**
61	37 21	7,8102	4,11087	16,3934	191,64	29 22, 47	61
62	38 44	7,8740	4,12713	16,1290	194,78	30 19, 07	62
63	39 69	7,9373	4,14313	15,8730	197,92	31 17, 25	63
64	40 96	8,0000	4,15888	15,6250	201,06	32 16, 99	64
65	42 25	8,0623	4,17439	15,3846	204,20	33 18, 31	65
66	43 56	8,1240	4,18965	15,1515	207,35	34 21, 19	66
67	44 89	8,1854	4,20469	14,9254	210,49	35 25, 65	67
68	46 24	8,2462	4,21951	14,7059	213,63	36 31, 68	68
69	47 61	8,3066	4,23411	14,4928	216,77	37 39, 28	69
70	49 00	8,3666	4,24850	14,2857	219,91	38 48, 45	**70**
71	50 41	8,4261	4,26268	14,0845	223,05	39 59, 19	71
72	51 84	8,4853	4,27667	13,8889	226,19	40 71, 50	72
73	53 29	8,5440	4,29046	13,6986	229,34	41 85, 39	73
74	54 76	8,6023	4,30407	13,5135	232,48	43 00, 84	74
75	56 25	8,6603	4,31749	13,3333	235,62	44 17, 86	75
76	57 76	8,7178	4,33073	13,1579	238,76	45 36, 46	76
77	59 29	8,7750	4,34381	12,9870	241,90	46 56, 63	77
78	60 84	8,8318	4,35671	12,8205	245,04	47 78, 36	78
79	62 41	8,8882	4,36945	12,6582	248,19	49 01, 67	79
80	64 00	8,9443	4,38203	12,5000	251,33	50 26, 55	**80**
81	65 61	9,0000	4,39445	12,3457	254,47	51 53, 00	81
82	67 24	9,0554	4,40672	12,1951	257,61	52 81, 02	82
83	68 89	9,1104	4,41884	12,0482	260,75	54 10, 61	83
84	70 56	9,1652	4,43082	11,9048	263,89	55 41, 77	84
85	72 25	9,2195	4,44265	11,7647	267,04	56 74, 50	85
86	73 96	9,2736	4,45435	11,6279	270,18	58 08, 80	86
87	75 69	9,3274	4,46591	11,4943	273,32	59 44, 68	87
88	77 44	9,3808	4,47734	11,3636	276,46	60 82, 12	88
89	79 21	9,4340	4,48864	11,2360	279,60	62 21, 14	89
90	81 00	9,4868	4,49981	11,1111	282,74	63 61, 73	**90**
91	82 81	9,5394	4,51086	10,9890	285,88	65 03, 88	91
92	84 64	9,5917	4,52179	10,8696	289,03	66 47, 61	92
93	86 49	9,6437	4,53260	10,7527	292,17	67 92, 91	93
94	88 36	9,6954	4,54329	10,6383	295,31	69 39, 78	94
95	90 25	9,7468	4,55388	10,5263	298,45	70 88, 22	95
96	92 16	9,7980	4,56435	10,4167	301,59	72 38, 23	96
97	94 09	9,8489	4,57471	10,3093	304,73	73 89, 81	97
98	96 04	9,8995	4,58497	10,2041	307,88	75 42, 96	98
99	98 01	9,9499	4,59512	10,1010	311,02	76 97, 69	99
100	1 00 00	10,0000	4,60517	10,0000	314,16	78 53, 98	**100**

n	n^2	$\sqrt{n}$	$\log_e n$	$\frac{1000}{n}$	πn	$\frac{\pi n^2}{4}$	n
100	1 00 00	10,0000	4,60517	10,0000	314,16	78 53, 98	**100**
101	1 02 01	10,0499	4,61512	9,90099	317,30	80 11, 85	101
102	1 04 04	10,0995	4,62497	9,80392	320,44	81 71, 28	102
103	1 06 09	10,1489	4,63473	9,70874	323,58	83 32, 29	103
104	1 08 16	10,1980	4,64439	9,61538	326,73	84 94, 87	104
105	1 10 25	10,2470	4,65396	9,52381	329,87	86 59, 01	105
106	1 12 36	10,2956	4,66344	9,43396	333,01	88 24, 73	106
107	1 14 49	10,3441	4,67283	9,34579	336,15	89 92, 02	107
108	1 16 64	10,3923	4,68213	9,25926	339,29	91 60, 88	108
109	1 18 81	10,4403	4,69135	9,17431	342,43	93 31, 32	109
110	1 21 00	10,4881	4,70048	9,09091	345,58	95 03, 32	**110**
111	1 23 21	10,5357	4,70953	9,00901	348,72	96 76, 89	111
112	1 25 44	10,5830	4,71850	8,92857	351,86	98 52, 03	112
113	1 27 69	10,6301	4,72739	8,84956	355,00	1 00 28,7	113
114	1 29 96	10,6771	4,73620	8,77193	358,14	1 02 07,0	114
115	1 32 25	10,7238	4,74493	8,69565	361,28	1 03 86,9	115
116	1 34 56	10,7703	4,75359	8,62069	364,42	1 05 68,3	116
117	1 36 89	10,8167	4,76217	8,54701	367,57	1 07 51,3	117
118	1 39 24	10,8628	4,77068	8,47458	370,71	1 09 35,9	118
119	1 41 61	10,9087	4,77912	8,40336	373,85	1 11 22,0	119
120	1 44 00	10,9545	4,78749	8,33333	376,99	1 13 09,7	**120**
121	1 46 41	11,0000	4,79579	8,26446	380,13	1 14 99,0	121
122	1 48 84	11,0454	4,80402	8,19672	383,27	1 16 89,9	122
123	1 51 29	11,0905	4,81218	8,13008	386,42	1 18 82,3	123
124	1 53 76	11,1355	4,82028	8,06452	389,56	1 20 76,3	124
125	1 56 25	11,1803	4,82831	8,00000	392,70	1 22 71,8	125
126	1 58 76	11,2250	4,83628	7,93651	395,84	1 24 69,0	126
127	1 61 29	11,2694	4,84419	7,87402	398,98	1 26 67,7	127
128	1 63 84	11,3137	4,85203	7,81250	402,12	1 28 68,0	128
129	1 66 41	11,3578	4,85981	7,75194	405,27	1 30 69,8	129
130	1 69 00	11,4018	4,86753	7,69231	408,41	1 32 73,2	**130**
131	1 71 61	11,4455	4,87520	7,63359	411,55	1 34 78,2	131
132	1 74 24	11,4891	4,88280	7,57576	414,69	1 36 84,8	132
133	1 76 89	11,5326	4,89035	7,51880	417,83	1 38 92,9	133
134	1 79 56	11,5758	4,89784	7,46269	420,97	1 41 02,6	134
135	1 82 25	11,6190	4,90527	7,40741	424,12	1 43 13,9	135
136	1 84 96	11,6619	4,91265	7,35294	427,26	1 45 26,7	136
137	1 87 69	11,7047	4,91998	7,29927	430,40	1 47 41,1	137
138	1 90 44	11,7473	4,92725	7,24638	433,54	1 49 57,1	138
139	1 93 21	11,7898	4,93447	7,19424	436,68	1 51 74,7	139
140	1 96 00	11,8322	4,94164	7,14286	439,82	1 53 93,8	**140**
141	1 98 81	11,8743	4,94876	7,09220	442,96	1 56 14,5	141
142	2 01 64	11,9164	4,95583	7,04225	446,11	1 58 36,8	142
143	2 04 49	11,9583	4,96284	6,99301	449,25	1 60 60,6	143
144	2 07 36	12,0000	4,96981	6,94444	452,39	1 62 86,0	144
145	2 10 25	12,0416	4,97673	6,89655	455,53	1 65 13,0	145
146	2 13 16	12,0830	4,98361	6,84932	458,67	1 67 41,5	146
147	2 16 09	12,1244	4,99043	6,80272	461,81	1 69 71,7	147
148	2 19 04	12,1655	4,99721	6,75676	464,96	1 72 03,4	148
149	2 22 01	12,2066	5,00395	6,71141	468,10	1 74 36,6	149
150	2 25 00	12,2474	5,01064	6,66667	471,24	1 76 71,5	**150**

150–200

n	n^2	$\sqrt{n}$	$\log_e n$	$\frac{1000}{n}$	πn	$\frac{\pi n^2}{4}$	n
150	2 25 00	12,2474	5,01064	6,66667	471,24	1 76 71,5	**150**
151	2 28 01	12,2882	5,01728	6,62252	474,38	1 79 07,9	151
152	2 31 04	12,3288	5,02388	6,57895	477,52	1 81 45,8	152
153	2 34 09	12,3693	5,03044	6,53595	480,66	1 83 85,4	153
154	2 37 16	12,4097	5,03695	6,49351	483,81	1 86 26,5	154
155	2 40 25	12,4499	5,04343	6,45161	486,95	1 88 69,2	155
156	2 43 36	12,4900	5,04986	6,41026	490,09	1 91 13,4	156
157	2 46 49	12,5300	5,05625	6,36943	493,23	1 93 59,3	157
158	2 49 64	12,5698	5,06260	6,32911	496,37	1 96 06,7	158
159	2 52 81	12,6095	5,06890	6,28931	499,51	1 98 55,7	159
160	2 56 00	12,6491	5,07517	6,25000	502,65	2 01 06,2	**160**
161	2 59 21	12,6886	5,08140	6,21118	505,80	2 03 58,3	161
162	2 62 44	12,7279	5,08760	6,17284	508,94	2 06 12,0	162
163	2 65 69	12,7671	5,09375	6,13497	512,08	2 08 67,2	163
164	2 68 96	12,8062	5,09987	6,09756	515,22	2 11 24,1	164
165	2 72 25	12,8452	5,10595	6,06061	518,36	2 13 82,5	165
166	2 75 56	12,8841	5,11199	6,02410	521,50	2 16 42,4	166
167	2 78 89	12,9228	5,11799	5,98802	524,65	2 19 04,0	167
168	2 82 24	12,9615	5,12396	5,95238	527,79	2 21 67,1	168
169	2 85 61	13,0000	5,12990	5,91716	530,93	2 24 31,8	169
170	2 89 00	13,0384	5,13580	5,88235	534,07	2 26 98,0	**170**
171	2 92 41	13,0767	5,14166	5,84795	537,21	2 29 65,8	171
172	2 95 84	13,1149	5,14749	5,81395	540,35	2 32 35,2	172
173	2 99 29	13,1529	5,15329	5,78035	543,50	2 35 06,2	173
174	3 02 76	13,1909	5,15906	5,74713	546,64	2 37 78,7	174
175	3 06 25	13,2288	5,16479	5,71429	549,78	2 40 52,8	175
176	3 09 76	13,2665	5,17048	5,68182	552,92	2 43 28,5	176
177	3 13 29	13,3041	5,17615	5,64972	556,06	2 46 05,7	177
178	3 16 84	13,3417	5,18178	5,61798	559,20	2 48 84,6	178
179	3 20 41	13,3791	5,18739	5,58659	562,35	2 51 64,9	179
180	3 24 00	13,4164	5,19296	5,55556	565,49	2 54 46,9	**180**
181	3 27 61	13,4536	5,19850	5,52486	568,63	2 57 30,4	181
182	3 31 24	13,4907	5,20401	5,49451	571,77	2 60 15,5	182
183	3 34 89	13,5277	5,20949	5,46448	574,91	2 63 02,2	183
184	3 38 56	13,5647	5,21494	5,43478	578,05	2 65 90,4	184
185	3 42 25	13,6015	5,22036	5,40541	581,19	2 68 80,3	185
186	3 45 96	13,6382	5,22575	5,37634	584,34	2 71 71,6	186
187	3 49 69	13,6748	5,23111	5,34759	587,48	2 74 64,6	187
188	3 53 44	13,7113	5,23644	5,31915	590,62	2 77 59,1	188
189	3 57 21	13,7477	5,24175	5,29101	593,76	2 80 55,2	189
190	3 61 00	13,7840	5,24702	5,26316	596,90	2 83 52,9	**190**
191	3 64 81	13,8203	5,25227	5,23560	600,04	2 86 52,1	191
192	3 68 64	13,8564	5,25750	5,20833	603,19	2 89 52,9	192
193	3 72 49	13,8924	5,26269	5,18135	606,33	2 92 55,3	193
194	3 76 36	13,9284	5,26786	5,15464	609,47	2 95 59,2	194
195	3 80 25	13,9642	5,27300	5,12821	612,61	2 98 64,8	195
196	3 84 16	14,0000	5,27811	5,10204	615,75	3 01 71,9	196
197	3 88 09	14,0357	5,28320	5,07614	618,89	3 04 80,5	197
198	3 92 04	14,0712	5,28827	5,05051	622,04	3 07 90,7	198
199	3 96 01	14,1067	5,29330	5,02513	625,18	3 11 02,6	199
200	4 00 00	14,1421	5,29832	5,00000	628,32	3 14 15,9	**200**

200–250

n	n^2	$\sqrt{n}$	$\log_e n$	$\frac{1000}{n}$	πn	$\frac{\pi n^2}{4}$	n
200	4 00 00	14,1421	5,29832	5,00000	628,32	3 14 15,9	**200**
201	4 04 01	14,1774	5,30330	4,97512	631,46	3 17 30,9	201
202	4 08 04	14,2127	5,30827	4,95050	634,60	3 20 47,4	202
203	4 12 09	14,2478	5,31321	4,92611	637,74	3 23 65,5	203
204	4 16 16	14,2829	5,31812	4,90196	640,88	3 26 85,1	204
205	4 20 25	14,3178	5,32301	4,87805	644,03	3 30 06,4	205
206	4 24 36	14,3527	5,32788	4,85437	647,17	3 33 29,2	206
207	4 28 49	14,3875	5,33272	4,83092	650,31	3 36 53,5	207
208	4 32 64	14,4222	5,33754	4,80769	653,45	3 39 79,5	208
209	4 36 81	14,4568	5,34233	4,78469	656,59	3 43 07,0	209
210	4 41 00	14,4914	5,34711	4,76190	659,73	3 46 36,1	**210**
211	4 45 21	14,5258	5,35186	4,73934	662,88	3 49 66,7	211
212	4 49 44	14,5602	5,35659	4,71698	666,02	3 52 98,9	212
213	4 53 69	14,5945	5,36129	4,69484	669,16	3 56 32,7	213
214	4 57 96	14,6287	5,36598	4,67290	672,30	3 59 68,1	214
215	4 62 25	14,6629	5,37064	4,65116	675,44	3 63 05,0	215
216	4 66 56	14,6969	5,37528	4,62963	678,58	3 66 43,5	216
217	4 70 89	14,7309	5,37990	4,60829	681,73	3 69 83,6	217
218	4 75 24	14,7648	5,38450	4,58716	684,87	3 73 25,3	218
219	4 79 61	14,7986	5,38907	4,56621	688,01	3 76 68,5	219
220	4 84 00	14,8324	5,39363	4,54545	691,15	3 80 13,3	**220**
221	4 88 41	14,8661	5,39816	4,52489	694,29	3 83 59,6	221
222	4 92 84	14,8997	5,40268	4,50450	697,43	3 87 07,6	222
223	4 97 29	14,9332	5,40717	4,48430	700,58	3 90 57,1	223
224	5 01 76	14,9666	5,41165	4,46429	703,72	3 94 08,1	224
225	5 06 25	15,0000	5,41610	4,44444	706,86	3 97 60,8	225
226	5 10 76	15,0333	5,42053	4,42478	710,00	4 01 15,0	226
227	5 15 29	15,0665	5,42495	4,40529	713,14	4 04 70,8	227
228	5 19 84	15,0997	5,42935	4,38596	716,28	4 08 28,1	228
229	5 24 41	15,1327	5,43372	4,36681	719,42	4 11 87,1	229
230	5 29 00	15,1658	5,43808	4,34783	722,57	4 15 47,6	**230**
231	5 33 61	15,1987	5,44242	4,32900	725,71	4 19 09,6	231
232	5 38 24	15,2315	5,44674	4,31034	728,85	4 22 73,3	232
233	5 42 89	15,2643	5,45104	4,29185	731,99	4 26 38,5	233
234	5 47 56	15,2971	5,45532	4,27350	735,13	4 30 05,3	234
235	5 52 25	15,3297	5,45959	4,25532	738,27	4 33 73,6	235
236	5 56 96	15,3623	5,46383	4,23729	741,42	4 37 43,5	236
237	5 61 69	15,3948	5,46806	4,21941	744,56	4 41 15,0	237
238	5 66 44	15,4272	5,47227	4,20168	747,70	4 44 88,1	238
239	5 71 21	15,4596	5,47646	4,18410	750,84	4 48 62,7	239
240	5 76 00	15,4919	5,48064	4,16667	753,98	4 52 38,9	**240**
241	5 80 81	15,5242	5,48480	4,14938	757,12	4 56 16,7	241
242	5 85 64	15,5563	5,48894	4,13223	760,27	4 59 96,1	242
243	5 90 49	15,5885	5,49306	4,11523	763,41	4 63 77,0	243
244	5 95 36	15,6205	5,49717	4,09836	766,55	4 67 59,5	244
245	6 00 25	15,6525	5,50126	4,08163	769,69	4 71 43,5	245
246	6 05 16	15,6844	5,50533	4,06504	772,83	4 75 29,2	246
247	6 10 09	15,7162	5,50939	4,04858	775,97	4 79 16,4	247
248	6 15 04	15,7480	5,51343	4,03226	779,11	4 83 05,1	248
249	6 20 01	15,7797	5,51745	4,01606	782,26	4 86 95,5	249
250	6 25 00	15,8114	5,52146	4,00000	785,40	4 90 87,4	**250**

250–300

n	n^2	$\sqrt{n}$	$\log_e n$	$\frac{1000}{n}$	πn	$\frac{\pi n^2}{4}$	n
250	6 25 00	15,8114	5,52146	4,00000	785,40	4 90 87,4	**250**
251	6 30 01	15,8430	5,52545	3,98406	788,54	4 94 80,9	251
252	6 35 04	15,8745	5,52943	3,96825	791,68	4 98 75,9	252
253	6 40 09	15,9060	5,53339	3,95257	794,82	5 02 72,6	253
254	6 45 16	15,9374	5,53733	3,93701	797,96	5 06 70,7	254
255	6 50 25	15,9687	5,54126	3,92157	801,11	5 10 70,5	255
256	6 55 36	16,0000	5,54518	3,90625	804,25	5 14 71,9	256
257	6 60 49	16,0312	5,54908	3,89105	807,39	5 18 74,8	257
258	6 65 64	16,0624	5,55296	3,87597	810,53	5 22 79,2	258
259	6 70 81	16,0935	5,55683	3,86100	813,67	5 26 85,3	259
260	6 76 00	16,1245	5,56068	3,84615	816,81	5 30 92,9	**260**
261	6 81 21	16,1555	5,56452	3,83142	819,96	5 35 02,1	261
262	6 86 44	16,1864	5,56834	3,81679	823,10	5 39 12,9	262
263	6 91 69	16,2173	5,57215	3,80228	826,24	5 43 25,2	263
264	6 96 96	16,2481	5,57595	3,78788	829,38	5 47 39,1	264
265	7 02 25	16,2788	5,57973	3,77358	832,52	5 51 54,6	265
266	7 07 56	16,3095	5,58350	3,75940	835,66	5 55 71,6	266
267	7 12 89	16,3401	5,58725	3,74532	838,81	5 59 90,2	267
268	7 18 24	16,3707	5,59099	3,73134	841,95	5 64 10,4	268
269	7 23 61	16,4012	5,59471	3,71747	845,09	5 68 32,2	269
270	7 29 00	16,4317	5,59842	3,70370	848,23	5 72 55,5	**270**
271	7 34 41	16,4621	5,60212	3,69004	851,37	5 76 80,4	271
272	7 39 84	16,4924	5,60580	3,67647	854,51	5 81 06,9	272
273	7 45 29	16,5227	5,60947	3,66300	857,65	5 85 34,9	273
274	7 50 76	16,5529	5,61313	3,64964	860,80	5 89 64,6	274
275	7 56 25	16,5831	5,61677	6,63636	863,94	5 93 95,7	275
276	7 61 76	16,6132	5,62040	3,62319	867,08	5 98 28,5	276
277	7 67 29	16,6433	5,62402	3,61011	870,22	6 02 62,8	277
278	7 72 84	16,6733	5,62762	3,59712	873,36	6 06 98,7	278
279	7 78 41	16,7033	5,63121	3,58423	876,50	6 11 36,2	279
280	7 84 00	16,7332	5,63479	3,57143	879,65	6 15 75,2	**280**
281	7 89 61	16,7631	5,63835	3,55872	882,79	6 20 15,8	281
282	7 95 24	16,7929	5,64191	3,54610	885,93	6 24 58,0	282
283	8 00 89	16,8226	5,64545	3,53357	889,07	6 29 01,8	283
284	8 06 56	16,8523	5,64897	3,52113	892,21	6 33 47,1	284
285	8 12 25	16,8819	5,65249	3,50877	895,35	6 37 94,0	285
286	8 17 96	16,9115	5,65599	3,49650	898,50	6 42 42,4	286
287	8 23 69	16,9411	5,65948	3,48432	901,64	6 46 92,5	287
288	8 29 44	16,9706	5,66296	3,47222	904,78	6 51 44,1	288
289	8 35 21	17,0000	5,66643	3,46021	907,92	6 55 97,2	289
290	8 41 00	17,0294	5,66988	3,44828	911,06	6 60 52,0	**290**
291	8 46 81	17,0587	5,67332	3,43643	914,20	6 65 08,3	291
292	8 52 64	17,0880	5,67675	3,42466	917,35	6 69 66,2	292
293	8 58 49	17,1172	5,68017	3,41297	920,49	6 74 25,6	293
294	8 64 36	17,1464	5,68358	3,40136	923,63	6 78 86,7	294
295	8 70 25	17,1756	5,68698	3,38983	926,77	6 83 49,3	295
296	8 76 16	17,2047	5,69036	3,37838	929,91	6 88 13,4	296
297	8 82 09	17,2337	5,69373	3,36700	933,05	6 92 79,2	297
298	8 88 04	17,2627	5,69709	3,35570	936,19	6 97 46,5	298
299	8 94 01	17,2916	5,70044	3,34448	939,34	7 02 15,4	299
300	9 00 00	17,3205	5,70378	3,33333	942,48	7 06 85,8	**300**

n	n^2	$\sqrt{n}$	$\log_e n$	$\frac{1000}{n}$	πn	$\frac{\pi n^2}{4}$	n
300	9 00 00	17,3205	5,70378	3,33333	942,48	7 06 85,8	**300**
301	9 06 01	17,3494	5,70711	3,32226	945,62	7 11 57,9	301
302	9 12 04	17,3781	5,71043	3,31126	948,76	7 16 31,5	302
303	9 18 09	17,4069	5,71373	3,30033	951,90	7 21 06,6	303
304	9 24 16	17,4356	5,71703	3,28947	955,04	7 25 83,4	304
305	9 30 25	17,4642	5,72031	3,27869	958,19	7 30 61,7	305
306	9 36 36	17,4929	5,72359	3,26797	961,33	7 35 41,5	306
307	9 42 49	17,5214	5,72685	3,25733	964,47	7 40 23,0	307
308	9 48 64	17,5499	5,73010	3,24675	967,61	7 45 06,0	308
309	9 54 81	17,5784	5,73334	3,23625	970,75	7 49 90,6	309
310	9 61 00	17,6068	5,73657	3,22581	973,89	7 54 76,8	**310**
311	9 67 21	17,6352	5,73979	3,21543	977,04	7 59 64,5	311
312	9 73 44	17,6635	5,74300	3,20513	980,18	7 64 53,8	312
313	9 79 69	17,6918	5,74620	3,19489	983,32	7 69 44,7	313
314	9 85 96	17,7200	5,74939	3,18471	986,46	7 74 37,1	314
315	9 92 25	17,7482	5,75257	3,17460	989,60	7 79 31,1	315
316	9 98 56	17,7764	5,75574	3,16456	992,74	7 84 26,7	316
317	10 04 89	17,8045	5,75890	3,15457	995,88	7 89 23,9	317
318	10 11 24	17,8326	5,76205	3,14465	999,03	7 94 22,6	318
319	10 17 61	17,8606	5,76519	3,13480	1002,2	7 99 22,9	319
320	10 24 00	17,8885	5,76832	3,12500	1005,3	8 04 24,8	**320**
321	10 30 41	17,9165	5,77144	3,11526	1008,5	8 09 28,2	321
322	10 36 84	17,9444	5,77455	3,10559	1011,6	8 14 33,2	322
323	10 43 29	17,9722	5,77765	3,09598	1014,7	8 19 39,8	323
324	10 49 76	18,0000	5,78074	3,08642	1017,9	8 24 48,0	324
325	10 56 25	18,0278	5,78383	3,07692	1021,0	8 29 57,7	325
326	10 62 76	18,0555	5,78690	3,06748	1024,2	8 34 69,0	326
327	10 69 29	18,0831	5,78996	3,05810	1027,3	8 39 81,8	327
328	10 75 84	18,1108	5,79301	3,04878	1030,4	8 44 96,3	328
329	10 82 41	18,1384	5,79606	3,03951	1033,6	8 50 12,3	329
330	10 89 00	18,1659	5,79909	3,03030	1036,7	8 55 29,9	**330**
331	10 95 61	18,1934	5,80212	3,02115	1039,9	8 60 49,0	331
332	11 02 24	18,2209	5,80513	3,01205	1043,0	8 65 69,7	332
333	11 08 89	18,2483	5,80814	3,00300	1046,2	8 70 92,0	333
334	11 15 56	18,2757	5,81114	2,99401	1049,3	8 76 15,9	334
335	11 22 25	18,3030	5,81413	2,98507	1052,4	8 81 41,3	335
336	11 28 96	18,3303	5,81711	2,97619	1055,6	8 86 68,3	336
337	11 35 69	18,3576	5,82008	2,96736	1058,7	8 91 96,9	337
338	11 42 44	18,3848	5,82305	2,95858	1061,9	8 97 27,0	338
339	11 49 21	18,4120	5,82600	2,94985	1065,0	9 02 58,7	339
340	11 56 00	18,4391	5,82895	2,94118	1068,1	9 07 92,0	**340**
341	11 62 81	18,4662	5,83188	2,93255	1071,3	9 13 26,9	341
342	11 69 64	18,4932	5,83481	2,92398	1074,4	9 18 63,3	342
343	11 76 49	18,5203	5,83773	2,91545	1077,6	9 24 01,3	343
344	11 83 36	18,5472	5,84064	2,90698	1080,7	9 29 40,9	344
345	11 90 25	18,5742	5,84354	2,89855	1083,8	9 34 82,0	345
346	11 97 16	18,6011	5,84644	2,89017	1087,0	9 40 24,7	346
347	12 04 09	18,6279	5,84932	2,88184	1090,1	9 45 69,0	347
348	12 11 04	18,6548	5,85220	2,87356	1093,3	9 51 14,9	348
349	12 18 01	18,6815	5,85507	2,86533	1096,4	9 56 62,3	349
350	12 25 00	18,7083	5,85793	2,85714	1099,6	9 62 11,3	**350**

350-400

n	n^2	$\sqrt{n}$	$\log_e n$	$\frac{1000}{n}$	πn	$\frac{\pi n^2}{4}$	n
350	12 25 00	18,7083	5,85793	2,85714	1099,6	9 62 11,3	**350**
351	12 32 01	18,7350	5,86079	2,84900	1102,7	9 67 61,8	351
352	12 39 04	18,7617	5,86363	2,84091	1105,8	9 73 14,0	352
353	12 46 09	18,7883	5,86647	2,83286	1109,0	9 78 67,7	353
354	12 53 16	18,8149	5,86930	2,82486	1112,1	9 84 23,0	354
355	12 60 25	18,8414	5,87212	2,81690	1115,3	9 89 79,8	355
356	12 67 36	18,8680	5,87493	2,80899	1118,4	9 95 38,2	356
357	12 74 49	18,8944	5,87774	2,80112	1121,5	10 00 98	357
358	12 81 64	18,9209	5,88053	2,79330	1124,7	10 06 60	358
359	12 88 81	18,9473	5,88332	2,78552	1127,8	10 12 23	359
360	12 96 00	18,9737	5,88610	2,77778	1131,0	10 17 88	**360**
361	13 03 21	19,0000	5,88888	2,77008	1134,1	10 23 54	361
362	13 10 44	19,0263	5,89164	2,76243	1137,3	10 29 22	362
363	13 17 69	19,0526	5,89440	2,75482	1140,4	10 34 91	363
364	13 24 96	19,0788	5,89715	2,74725	1143,5	10 40 62	364
365	13 32 25	19,1050	5,89990	2,73973	1146,7	10 46 35	365
366	13 39 56	19,1311	5,90263	2,73224	1149,8	10 52 09	366
367	13 46 89	19,1572	5,90536	2,72480	1153,0	10 57 85	367
368	13 54 24	19,1833	5,90808	2,71739	1156,1	10 63 62	368
369	13 61 61	19,2094	5,91080	2,71003	1159,2	10 69 41	369
370	13 69 00	19,2354	5,91350	2,70270	1162,4	10 75 21	**370**
371	13 76 41	19,2614	5,91620	2,69542	1165,5	10 81 03	371
372	13 83 84	19,2873	5,91889	2,68817	1168,7	10 86 87	372
373	13 91 29	19,3132	5,92158	2,68097	1171,8	10 92 72	373
374	13 98 76	19,3391	5,92426	2,67380	1175,0	10 98 58	374
375	14 06 25	19,3649	5,92693	2,66667	1178,1	11 04 47	375
376	14 13 76	19,3907	5,92959	2,65957	1181,2	11 10 36	376
377	14 21 29	19,4165	5,93225	2,65252	1184,4	11 16 28	377
378	14 28 84	19,4422	5,93489	2,64550	1187,5	11 22 21	378
379	14 36 41	19,4679	5,93754	2,63852	1190,7	11 28 15	379
380	14 44 00	19,4936	5,94017	2,63158	1193,8	11 34 11	**380**
381	14 51 61	19,5192	5,94280	2,62467	1196,9	11 40 09	381
382	14 59 24	19,5448	5,94542	2,61780	1200,1	11 46 08	382
383	14 66 89	19,5704	5,94803	2,61097	1203,2	11 52 09	383
384	14 74 56	19,5959	5,95064	2,60417	1206,4	11 58 12	384
385	14 82 25	19,6214	5,95324	2,59740	1209,5	11 64 16	385
386	14 89 96	19,6469	5,95584	2,59067	1212,7	11 70 21	386
387	14 97 69	19,6723	5,95842	2,58398	1215,8	11 76 28	387
388	15 05 44	19,6977	5,96101	2,57732	1218,9	11 82 37	388
389	15 13 21	19,7231	5,96358	2,57069	1222,1	11 88 47	389
390	15 21 00	19,7484	5,96615	2,56410	1225,2	11 94 59	**390**
391	15 28 81	19,7737	5,96871	2,55754	1228,4	12 00 72	391
392	15 36 64	19,7990	5,97126	2,55102	1231,5	12 06 87	392
393	15 44 49	19,8242	5,97381	2,54453	1234,6	12 13 04	393
394	15 52 36	19,8494	5,97635	2,53807	1237,8	12 19 22	394
395	15 60 25	19,8746	5,97889	2,53165	1240,9	12 25 42	395
396	15 68 16	19,8997	5,98141	2,52525	1244,1	12 31 63	396
397	15 76 09	19,9249	5,98394	2,51889	1247,2	12 37 86	397
398	15 84 04	19,9499	5,98645	2,51256	1250,4	12 44 10	398
399	15 92 01	19,9750	5,98896	2,50627	1253,5	12 50 36	399
400	16 00 00	20,0000	5,99146	2,50000	1256,6	12 56 64	**400**

400–450

n	n^2	$\sqrt{n}$	$\log_e n$	$\frac{1000}{n}$	πn	$\frac{\pi n^2}{4}$	n
400	16 00 00	20,0000	5,99146	2,50000	1256,6	12 56 64	**400**
401	16 08 01	20,0250	5,99396	2,49377	1259,8	12 62 93	401
402	16 16 04	20,0499	5,99645	2,48756	1262,9	12 69 23	402
403	16 24 09	20,0749	5,99894	2,48139	1266,1	12 75 56	403
404	16 32 16	20,0998	6,00141	2,47525	1269,2	12 81 90	404
405	16 40 25	20,1246	6,00389	2,46914	1272,3	12 88 25	405
406	16 48 36	20,1494	6,00635	2,46305	1275,5	12 94 62	406
407	16 56 49	20,1742	6,00881	2,45700	1278,6	13 01 00	407
408	16 64 64	20,1990	6,01127	2,45098	1281,8	13 07 41	408
409	16 72 81	20,2237	6,01372	2,44499	1284,9	13 13 82	409
410	16 81 00	20,2485	6,01616	2,43902	1288,1	13 20 25	**410**
411	16 89 21	20,2731	6,01859	2,43309	1291,2	13 26 70	411
412	16 97 44	20,2978	6,02102	2,42718	1294,3	13 33 17	412
413	17 05 69	20,3224	6,02345	2,42131	1297,5	13 39 65	413
414	17 13 96	20,3470	6,02587	2,41546	1300,6	13 46 14	414
415	17 22 25	20,3715	6,02828	2,40964	1303,8	13 52 65	415
416	17 30 56	20,3961	6,03069	2,40385	1306,9	13 59 18	416
417	17 38 89	20,4206	6,03309	2,39808	1310,0	13 65 72	417
418	17 47 24	20,4450	6,03548	2,39234	1313,2	13 72 28	418
419	17 55 61	20,4695	6,03787	2,38663	1316,3	13 78 85	419
420	17 64 00	20,4939	6,04025	2,38095	1319,5	13 85 44	**420**
421	17 72 41	20,5183	6,04263	2,37530	1322,6	13 92 05	421
422	17 80 84	20,5426	6,04501	2,36967	1325,8	13 98 67	422
423	17 89 29	20,5670	6,04737	2,36407	1328,9	14 05 31	423
424	17 97 76	20,5913	6,04973	2,35849	1332,0	14 11 96	424
425	18 06 25	20,6155	6,05209	2,35294	1335,2	14 18 63	425
426	18 14 76	20,6398	6,05444	2,34742	1338,3	14 25 31	426
427	18 23 29	20,6640	6,05678	2,34192	1341,5	14 32 01	427
428	18 31 84	20,6882	6,05912	2,33645	1344,6	14 38 72	428
429	18 40 41	20,7123	6,06146	2,33100	1347,7	14 45 45	429
430	18 49 00	20,7364	6,06379	2,32558	1350,9	14 52 20	**430**
431	18 57 61	20,7605	6,06611	2,32019	1354,0	14 58 96	431
432	18 66 24	20,7846	6,06843	2,31481	1357,2	14 65 74	432
433	18 74 89	20,8087	6,07074	2,30947	1360,3	14 72 54	433
434	18 83 56	20,8327	6,07304	2,30415	1363,5	14 79 34	434
435	18 92 25	20,8567	6,07535	2,29885	1366,6	14 86 17	435
436	19 00 96	20,8806	6,07764	2,29358	1369,7	14 93 01	436
437	19 09 69	20,9045	6,07993	2,28833	1372,9	14 99 87	437
438	19 18 44	20,9284	6,08222	2,28311	1376,0	15 06 74	438
439	19 27 21	20,9523	6,08450	2,27790	1379,2	15 13 63	439
440	19 36 00	20,9762	6,08677	2,27273	1382,3	15 20 53	**440**
441	19 44 81	21,0000	6,08904	2,26757	1385,4	15 27 45	441
442	19 53 64	21,0238	6,09131	2,26244	1388,6	15 34 39	442
443	19 62 49	21,0476	6,09357	2,25734	1391,7	15 41 34	443
444	19 71 36	21,0713	6,09582	2,25225	1394,9	15 48 30	444
445	19 80 25	21,0950	6,09807	2,24719	1398,0	15 55 28	445
446	19 89 16	21,1187	6,10032	2,24215	1401,2	15 62 28	446
447	19 98 09	21,1424	6,10256	2,23714	1404,3	15 69 30	447
448	20 07 04	21,1660	6,10479	2,23214	1407,4	15 76 33	448
449	20 16 01	21,1896	6,10702	2,22717	1410,6	15 83 37	449
450	20 25 00	21,2132	6,10925	2,22222	1413,7	15 90 43	**450**

450–500

n	n^2	$\sqrt{n}$	$\log_e n$	$\frac{1000}{n}$	πn	$\frac{\pi n^2}{4}$	n
450	20 25 00	21,2132	6,10925	2,22222	1413,7	15 90 43	**450**
451	20 34 01	21,2368	6,11147	2,21729	1416,9	15 97 51	451
452	20 43 04	21,2603	6,11368	2,21239	1420,0	16 04 60	452
453	20 52 09	21,2838	6,11589	2,20751	1423,1	16 11 71	453
454	20 61 16	21,3073	6,11810	2,20264	1426,3	16 18 83	454
455	20 70 25	21,3307	6,12030	2,19780	1429,4	16 25 97	455
456	20 79 36	21,3542	6,12249	2,19298	1432,6	16 33 13	456
457	20 88 49	21,3776	6,12468	2,18818	1435,7	16 40 30	457
458	20 97 64	21,4009	6,12687	2,18341	1438,8	16 47 48	458
459	21 06 81	21,4243	6,12905	2,17865	1442,0	16 54 68	459
460	21 16 00	21,4476	6,13123	2,17391	1445,1	16 61 90	**460**
461	21 25 21	21,4709	6,13340	2,16920	1448,3	16 69 14	461
462	21 34 44	21,4942	6,13556	2,16450	1451,4	16 76 39	462
463	21 43 69	21,5174	6,13773	2,15983	1454,6	16 83 65	463
464	21 52 96	21,5407	6,13988	2,15517	1457,7	16 90 93	464
465	21 62 25	21,5639	6,14204	2,15054	1460,8	16 98 23	465
466	21 71 56	21,5870	6,14419	2,14592	1464,0	17 05 54	466
467	21 80 89	21,6102	6,14633	2,14133	1467,1	17 12 87	467
468	21 90 24	21,6333	6,14847	2,13675	1470,3	17 20 21	468
469	21 99 61	21,6564	6,15060	2,13220	1473,4	17 27 57	469
470	22 09 00	21,6795	6,15273	2,12766	1476,5	17 34 94	**470**
471	22 18 41	21,7025	6,15486	2,12314	1479,7	17 42 34	471
472	22 27 84	21,7256	6,15698	2,11864	1482,8	17 49 74	472
473	22 37 29	21,7486	6,15910	2,11416	1486,0	17 57 16	473
474	22 46 76	21,7715	6,16121	2,10970	1489,1	17 64 60	474
475	22 56 25	21,7945	6,16331	2,10526	1492,3	17 72 05	475
476	22 65 76	21,8174	6,16542	2,10084	1495,4	17 79 52	476
477	22 75 29	21,8403	6,16752	2,09644	1498,5	17 87 01	477
478	22 84 84	21,8632	6,16961	2,09205	1501,7	17 94 51	478
479	22 94 41	21,8861	6,17170	2,08768	1504,8	18 02 03	479
480	23 04 00	21,9089	6,17379	2,08333	1508,0	18 09 56	**480**
481	23 13 61	21,9317	6,17587	2,07900	1511,1	18 17 11	481
482	23 23 24	21,9545	6,17794	2,07469	1514,2	18 24 67	482
483	23 32 89	21,9773	6,18002	2,07039	1517,4	18 32 25	483
484	23 42 56	22,0000	6,18208	2,06612	1520,5	18 39 84	484
485	23 52 25	22,0227	6,18415	2,06186	1523,7	18 47 45	485
486	23 61 96	22,0454	6,18621	2,05761	1526,8	18 55 08	486
487	23 71 69	22,0681	6,18826	2,05339	1530,0	18 62 72	487
488	23 81 44	22,0907	6,19032	2,04918	1533,1	18 70 38	488
489	23 91 21	22,1133	6,19236	2,04499	1536,2	18 78 05	489
490	24 01 00	22,1359	6,19441	2,04082	1539,4	18 85 74	**490**
491	24 10 81	22,1585	6,19644	2,03666	1542,5	18 93 45	491
492	24 20 64	22,1811	6,19848	2,03252	1545,7	19 01 17	492
493	24 30 49	22,2036	6,20051	2,02840	1548,8	19 08 90	493
494	24 40 36	22,2261	6,20254	2,02429	1551,9	19 16 65	494
495	24 50 25	22,2486	6,20456	2,02020	1555,1	19 24 42	495
496	24 60 16	22,2711	6,20658	2,01613	1558,2	19 32 21	496
497	24 70 09	22,2935	6,20859	2,01207	1561,4	19 40 00	497
498	24 80 04	22,3159	6,21060	2,00803	1564,5	19 47 82	498
499	24 90 01	22,3383	6,21261	2,00401	1567,7	19 55 65	499
500	25 00 00	22,3607	6,21461	2,00000	1570,8	19 63 50	**500**

500–550

n	n^2	$\sqrt{n}$	$\log_e n$	$\frac{1000}{n}$	πn	$\frac{\pi n^2}{4}$	n
500	25 00 00	22,3607	6,21461	2,00000	1570,8	19 63 50	**500**
501	25 10 01	22,3830	6,21661	1,99601	1573,9	19 71 36	501
502	25 20 04	22,4054	6,21860	1,99203	1577,1	19 79 23	502
503	25 30 09	22,4277	6,22059	1,98807	1580,2	19 87 13	503
504	25 40 16	22,4499	6,22258	1,98413	1583,4	19 95 04	504
505	25 50 25	22,4722	6,22456	1,98020	1586,5	20 02 96	505
506	25 60 36	22,4944	6,22654	1,97628	1589,6	20 10 90	506
507	25 70 49	22,5167	6,22851	1,97239	1592,8	20 18 86	507
508	25 80 64	22,5389	6,23048	1,96850	1595,9	20 26 83	508
509	25 90 81	22,5610	6,23245	1,96464	1599,1	20 34 82	509
510	26 01 00	22,5832	6,23441	1,96078	1602,2	20 42 82	**510**
511	26 11 21	22,6053	6,23637	1,95695	1605,4	20 50 84	511
512	26 21 44	22,6274	6,23832	1,95312	1608,5	20 58 87	512
513	26 31 69	22,6495	6,24028	1,94932	1611,6	20 66 92	513
514	26 41 96	22,6716	6,24222	1,94553	1614,8	20 74 99	514
515	26 52 25	22,6936	6,24417	1,94175	1617,9	20 83 07	515
516	26 62 56	22,7156	6,24611	1,93798	1621,1	20 91 17	516
517	26 72 89	22,7376	6,24804	1,93424	1624,2	20 99 28	517
518	26 83 24	22,7596	6,24998	1,93050	1627,3	21 07 41	518
519	26 93 61	22,7816	6,25190	1,92678	1630,5	21 15 56	519
520	27 04 00	22,8035	6,25383	1,92308	1633,6	21 23 72	**520**
521	27 14 41	22,8254	6,25575	1,91939	1636,8	21 31 89	521
522	27 24 84	22,8473	6,25767	1,91571	1639,9	21 40 08	522
523	27 35 29	22,8692	6,25958	1,91205	1643,1	21 48 29	523
524	27 45 76	22,8910	6,26149	1,90840	1646,2	21 56 51	524
525	27 56 25	22,9129	6,26340	1,90476	1649,3	21 64 75	525
526	27 66 76	22,9347	6,26530	1,90114	1652,5	21 73 01	526
527	27 77 29	22,9565	6,26720	1,89753	1655,6	21 81 28	527
528	27 87 84	22,9783	6,26910	1,89394	1658,8	21 89 56	528
529	27 98 41	23,0000	6,27099	1,89036	1661,9	21 97 87	529
530	28 09 00	23,0217	6,27288	1,88679	1665,0	22 06 18	**530**
531	28 19 61	23,0434	6,27476	1,88324	1668,2	22 14 52	531
532	28 30 24	23,0651	6,27664	1,87970	1671,3	22 22 87	532
533	28 40 89	23,0868	6,27852	1,87617	1674,5	22 31 23	533
534	28 51 56	23,1084	6,28040	1,87266	1677,6	22 39 61	534
535	28 62 25	23,1301	6,28227	1,86916	1680,8	22 48 01	535
536	28 72 96	23,1517	6,28413	1,86567	1683,9	22 56 42	536
537	28 83 69	23,1733	6,28600	1,86220	1687,0	22 64 84	537
538	28 94 44	23,1948	6,28786	1,85874	1690,2	22 73 29	538
539	29 05 21	23,2164	6,28972	1,85529	1693,3	22 81 75	539
540	29 16 00	23,2379	6,29157	1,85185	1696,5	22 90 22	**540**
541	29 26 81	23,2594	6,29342	1,84843	1699,6	22 98 71	541
542	29 37 64	23,2809	6,29527	1,84502	1702,7	23 07 22	542
543	29 48 49	23,3024	6,29711	1,84162	1705,9	23 15 74	543
544	29 59 36	23,3238	6,29895	1,83824	1709,0	23 24 28	544
545	29 70 25	23,3452	6,30079	1,83486	1712,2	23 32 83	545
546	29 81 16	23,3666	6,30262	1,83150	1715,3	23 41 40	546
547	29 92 09	23,3880	6,30445	1,82815	1718,5	23 49 98	547
548	30 03 04	23,4094	6,30628	1,82482	1721,6	23 58 58	548
549	30 14 01	23,4307	6,30810	1,82149	1724,7	23 67 20	549
550	30 25 00	23,4521	6,30992	1,81818	1727,9	23 75 83	**550**

550–600

n	n^2	$\sqrt{n}$	$\log_e n$	$\frac{1000}{n}$	πn	$\frac{\pi n^2}{4}$	n
550	30 25 00	23,4521	6,30992	1,81818	1727,9	23 75 83	**550**
551	30 36 01	23,4734	6,31173	1,81488	1731,0	23 84 48	551
552	30 47 04	23,4947	6,31355	1,81159	1734,2	23 93 14	552
553	30 58 09	23,5160	6,31536	1,80832	1737,3	24 01 82	553
554	30 69 16	23,5372	6,31716	1,80505	1740,4	24 10 51	554
555	30 80 25	23,5584	6,31897	1,80180	1743,6	24 19 22	555
556	30 91 36	23,5797	6,32077	1,79856	1746,7	24 27 95	556
557	31 02 49	23,6008	6,32257	1,79533	1749,9	24 36 69	557
558	31 13 64	23,6220	6,32436	1,79211	1753,0	24 45 45	558
559	31 24 81	23,6432	6,32615	1,78891	1756,2	24 54 22	559
560	31 36 00	23,6643	6,32794	1,78571	1759,3	24 63 01	**560**
561	31 47 21	23,6854	6,32972	1,78253	1762,4	24 71 81	561
562	31 58 44	23,7065	6,33150	1,77936	1765,6	24 80 63	562
563	31 69 69	23,7276	6,33328	1,77620	1768,7	24 89 47	563
564	31 80 96	23,7487	6,33505	1,77305	1771,9	24 98 32	564
565	31 92 25	23,7697	6,33683	1,76991	1775,0	25 07 19	565
566	32 03 56	23,7908	6,33859	1,76678	1778,1	25 16 07	566
567	32 14 89	23,8118	6,34036	1,76367	1781,3	25 24 97	567
568	32 26 24	23,8328	6,34212	1,76056	1784,4	25 33 88	568
569	32 37 61	23,8537	6,34388	1,75747	1787,6	25 42 81	569
570	32 49 00	23,8747	6,34564	1,75439	1790,7	25 51 76	**570**
571	32 60 41	23,8956	6,34739	1,75131	1793,8	25 60 72	571
572	32 71 84	23,9165	6,34914	1,74825	1797,0	25 69 70	572
573	32 83 29	23,9374	6,35089	1,74520	1800,1	25 78 69	573
574	32 94 76	23,9583	6,35263	1,74216	1803,3	25 87 70	574
575	33 06 25	23,9792	6,35437	1,73913	1806,4	25 96 72	575
576	33 17 76	24,0000	6,35611	1,73611	1809,6	26 05 76	576
577	33 29 29	24,0208	6,35784	1,73310	1812,7	26 14 82	577
578	33 40 84	24,0416	6,35957	1,73010	1815,8	26 23 89	578
579	33 52 41	24,0624	6,36130	1,72712	1819,0	26 32 98	579
580	33 64 00	24,0832	6,36303	1,72414	1822,1	26 42 08	**580**
581	33 75 61	24,1039	6,36475	1,72117	1825,3	26 51 20	581
582	33 87 24	24,1247	6,36647	1,71821	1828,4	26 60 33	582
583	33 98 89	24,1454	6,36819	1,71527	1831,6	26 69 48	583
584	34 10 56	24,1661	6,36990	1,71233	1834,7	26 78 65	584
585	34 22 25	24,1868	6,37161	1,70940	1837,8	26 87 83	585
586	34 33 96	24,2074	6,37332	1,70648	1841,0	26 97 03	586
587	34 45 69	24,2281	6,37502	1,70358	1844,1	27 06 24	587
588	34 57 44	24,2487	6,37673	1,70068	1847,3	27 15 47	588
589	34 69 21	24,2693	6,37843	1,69779	1850,4	27 24 71	589
590	34 81 00	24,2899	6,38012	1,69492	1853,5	27 33 97	**590**
591	34 92 81	24,3105	6,38182	1,69205	1856,7	27 43 25	591
592	35 04 64	24,3311	6,38351	1,68919	1859,8	27 52 54	592
593	35 16 49	24,3516	6,38519	1,68634	1863,0	27 61 84	593
594	35 28 36	24,3721	6,38688	1,68350	1866,1	27 71 17	594
595	35 40 25	24,3926	6,38856	1,68067	1869,2	27 80 51	595
596	35 52 16	24,4131	6,39024	1,67785	1872,4	27 89 86	596
597	35 64 09	24,4336	6,39192	1,67504	1875,5	27 99 23	597
598	35 76 04	24,4540	6,39359	1,67224	1878,7	28 08 62	598
599	35 88 01	24,4745	6,39526	1,66945	1881,8	28 18 02	599
600	36 00 00	24,4949	6,39693	1,66667	1885,0	28 27 43	**600**

n	n^2	$\sqrt{n}$	$\log_e n$	$\frac{1000}{n}$	πn	$\frac{\pi n^2}{4}$	n
600	36 00 00	24,4949	6,39693	1,66667	1885,0	28 27 43	**600**
601	36 12 01	24,5153	6,39859	1,66389	1888,1	28 36 87	601
602	36 24 04	24,5357	6,40026	1,66113	1891,2	28 46 31	602
603	36 36 09	24,5561	6,40192	1,65837	1894,4	28 55 78	603
604	36 48 16	24,5764	6,40357	1,65563	1897,5	28 65 26	604
605	36 60 25	24,5967	6,40523	1,65289	1900,7	28 74 75	605
606	36 72 36	24,6171	6,40688	1,65017	1903,8	28 84 26	606
607	36 84 49	24,6374	6,40853	1,64745	1906,9	28 93 79	607
608	36 96 64	24,6577	6,41017	1,64474	1910,1	29 03 33	608
609	37 08 81	24,6779	6,41182	1,64204	1913,2	29 12 89	609
610	37 21 00	24,6982	6,41346	1,63934	1916,4	29 22 47	**610**
611	37 33 21	24,7184	6,41510	1,63666	1919,5	29 32 06	611
612	37 45 44	24,7386	6,41673	1,63399	1922,7	29 41 66	612
613	37 57 69	24,7588	6,41836	1,63132	1925,8	29 51 28	613
614	37 69 96	24,7790	6,41999	1,62866	1928,9	29 60 92	614
615	37 82 25	24,7992	6,42162	1,62602	1932,1	29 70 57	615
616	37 94 56	24,8193	6,42325	1,62338	1935,2	29 80 24	616
617	38 06 89	24,8395	6,42487	1,62075	1938,4	29 89 92	617
618	38 19 24	24,8596	6,42649	1,61812	1941,5	29 99 62	618
619	38 31 61	24,8797	6,42811	1,61551	1944,6	30 09 34	619
620	38 44 00	24,8998	6,42972	1 61290	1947,8	30 19 07	**620**
621	38 56 41	24,9199	6,43133	1,61031	1950,9	30 28 82	621
622	38 68 84	24,9399	6,43294	1,60772	1954,1	30 38 58	622
623	38 81 29	24,9600	6,43455	1,60514	1957,2	30 48 36	623
624	38 93 76	24,9800	6,43615	1,60256	1960,4	30 58 15	624
625	39 06 25	25,0000	6,43775	1,60000	1963,5	30 67 96	625
626	39 18 76	25,0200	6,43935	1,59744	1966,6	30 77 79	626
627	39 31 29	25,0400	6,44095	1,59490	1969,8	30 87 63	627
628	39 43 84	25,0599	6,44254	1,59236	1972,9	30 97 48	628
629	39 56 41	25,0799	6,44413	1,58983	1976,1	31 07 36	629
630	39 69 00	25,0998	6,44572	1,58730	1979,2	31 17 25	**630**
631	39 81 61	25,1197	6,44731	1,58479	1982,3	31 27 15	631
632	39 94 24	25,1396	6,44889	1,58228	1985,5	31 37 07	632
633	40 06 89	25,1595	6,45047	1,57978	1988,6	31 47 00	633
634	40 19 56	25,1794	6,45205	1,57729	1991,8	31 56 96	634
635	40 32 25	25,1992	6,45362	1,57480	1994,9	31 66 92	635
636	40 44 96	25,2190	6,45520	1,57233	1998,1	31 76 90	636
637	40 57 69	25,2389	6,45677	1,56986	2001,2	31 86 90	637
638	40 70 44	25,2587	6,45834	1,56740	2004,3	31 96 92	638
639	40 83 21	25,2784	6,45990	1,56495	2007,5	32 06 95	639
640	40 96 00	25,2982	6,46147	1,56250	2010,6	32 16 99	**640**
641	41 08 81	25,3180	6,46303	1,56006	2013,8	32 27 05	641
642	41 21 64	25,3377	6,46459	1,55763	2016,9	32 37 13	642
643	41 34 49	25,3574	6,46614	1,55521	2020,0	32 47 22	643
644	41 47 36	25,3772	6,46770	1,55280	2023,2	32 57 33	644
645	41 60 25	25,3969	6,46925	1,55039	2026,3	32 67 45	645
646	41 73 16	25,4165	6,47080	1,54799	2029,5	32 77 59	646
647	41 86 09	25,4362	6,47235	1,54560	2032,6	32 87 75	647
648	41 99 04	25,4558	6,47389	1,54321	2035,8	32 97 92	648
649	42 12 01	25,4755	6,47543	1,54083	2038,9	33 08 10	649
650	42 25 00	25,4951	6,47697	1,53846	2042,0	33 18 31	**650**

650 700

n	n^2	$\sqrt{n}$	$\log_e n$	$\frac{1000}{n}$	πn	$\frac{\pi n^2}{4}$	n
650	42 25 00	25,4951	6,47697	1,53846	2042,0	33 18 31	**650**
651	42 38 01	25,5147	6,47851	1,53610	2045,2	33 28 53	651
652	42 51 04	25,5343	6,48004	1,53374	2048,3	33 38 76	652
653	42 64 09	25,5539	6,48158	1,53139	2051,5	33 49 01	653
654	42 77 16	25,5734	6,48311	1,52905	2054,6	33 59 27	654
655	42 90 25	25,5930	6,48464	1,52672	2057,7	33 69 55	655
656	43 03 36	25,6125	6,48616	1,52439	2060,9	33 79 85	656
657	43 16 49	25,6320	6,48768	1,52207	2064,0	33 90 16	657
658	43 29 64	25,6515	6,48920	1,51976	2067,2	34 00 49	658
659	43 42 81	25,6710	6,49072	1,51745	2070,3	34 10 83	659
660	43 56 00	25,6905	6,49224	1,51515	2073,5	34 21 19	**660**
661	43 69 21	25,7099	6,49375	1,51286	2076,6	34 31 57	661
662	43 82 44	25,7294	6,49527	1,51057	2079,7	34 41 96	662
663	43 95 69	25,7488	6,49677	1,50830	2082,9	34 52 37	663
664	44 08 96	25,7682	6,49828	1,50602	2086,0	34 62 79	664
665	44 22 25	25,7876	6,49979	1,50376	2089,2	34 73 23	665
666	44 35 56	25,8070	6,50129	1,50150	2092,3	34 83 68	666
667	44 48 89	25,8263	6,50279	1,49925	2095,4	34 94 15	667
668	44 62 24	25,8457	6,50429	1,49701	2098,6	35 04 64	668
669	44 75 61	25,8650	6,50578	1,49477	2101,7	35 15 14	669
670	44 89 00	25,8844	6,50728	1,49254	2104,9	35 25 65	**670**
671	45 02 41	25,9037	6,50877	1,49031	2108,0	35 36 18	671
672	45 15 84	25,9230	6,51026	1,48810	2111,2	35 46 73	672
673	45 29 29	25,9422	6,51175	1,48588	2114,3	35 57 30	673
674	45 42 76	25,9615	6,51323	1,48368	2117,4	35 67 88	674
675	45 56 25	25,9808	6,51471	1,48148	2120,6	35 78 47	675
676	45 69 76	26,0000	6,51619	1,47929	2123,7	35 89 08	676
677	45 83 29	26,0192	6,51767	1,47710	2126,9	35 99 71	677
678	45 96 84	26,0384	6,51915	1,47493	2130,0	36 10 35	678
679	46 10 41	26,0576	6,52062	1,47275	2133,1	36 21 01	679
680	46 24 00	26,0768	6,52209	1,47059	2136,3	36 31 68	**680**
681	46 37 61	26,0960	6,52356	1,46843	2139,4	36 42 37	681
682	46 51 24	26,1151	6,52503	1,46628	2142,6	36 53 08	682
683	46 64 89	26,1343	6,52649	1,46413	2145,7	36 63 80	683
684	46 78 56	26,1534	6,52796	1,46199	2148,8	36 74 53	684
685	46 92 25	26,1725	6,52942	1,45985	2152,0	36 85 28	685
686	47 05 96	26,1916	6,53088	1,45773	2155,1	36 96 05	686
687	47 19 69	26,2107	6,53233	1,45560	2158,3	37 06 84	687
688	47 33 44	26,2298	6,53379	1,45349	2161,4	37 17 64	688
689	47 47 21	26,2488	6,53524	1,45138	2164,6	37 28 45	689
690	47 61 00	26,2679	6,53669	1,44928	2167,7	37 39 28	**690**
691	47 74 81	26,2869	6,53814	1,44718	2170,8	37 50 13	691
692	47 88 64	26,3059	6,53959	1,44509	2174,0	37 60 99	692
693	48 02 49	26,3249	6,54103	1,44300	2177,1	37 71 87	693
694	48 16 36	26,3439	6,54247	1,44092	2180,3	37 82 76	694
695	48 30 25	26,3629	6,54391	1,43885	2183,4	37 93 67	695
696	48 44 16	26,3818	6,54535	1,43678	2186,5	38 04 59	696
697	48 58 09	26,4008	6,54679	1,43472	2189,7	38 15 53	697
698	48 72 04	26,4197	6,54822	1,43266	2192,8	38 26 49	698
699	48 86 01	26,4386	6,54965	1,43062	2196,0	38 37 46	699
700	49 00 00	26,4575	6,55108	1,42857	2199,1	38 48 45	**700**

700–750

n	n^2	$\sqrt{n}$	$\log_e n$	$\frac{1000}{n}$	πn	$\frac{\pi n^2}{4}$	n
700	49 00 00	26,4575	6,55108	1,42857	2199,1	38 48 45	**700**
701	49 14 01	26,4764	6,55251	1,42653	2202,3	38 59 45	701
702	49 28 04	26,4953	6,55393	1,42450	2205,4	38 70 47	702
703	49 42 09	26,5141	6,55536	1,42248	2208,5	38 81 51	703
704	49 56 16	26,5330	6,55678	1,42045	2211,7	38 92 56	704
705	49 70 25	26,5518	6,55820	1,41844	2214,8	39 03 63	705
706	49 84 36	26,5707	6,55962	1,41643	2218,0	39 14 71	706
707	49 98 49	26,5895	6,56103	1,41443	2221,1	39 25 80	707
708	50 12 64	26,6083	6,56244	1,41243	2224,2	39 36 92	708
709	50 26 81	26,6271	6,56386	1,41044	2227,4	39 48 05	709
710	50 41 00	26,6458	6,56526	1,40845	2230,5	39 59 19	**710**
711	50 55 21	26,6646	6,56667	1,40647	2233,7	39 70 35	711
712	50 69 44	26,6833	6,56808	1,40449	2236,8	39 81 53	712
713	50 83 69	26,7021	6,56948	1,40252	2240,0	39 92 72	713
714	50 97 96	26,7208	6,57088	1,40056	2243,1	40 03 93	714
715	51 12 25	26,7395	6,57228	1,39860	2246,2	40 15 15	715
716	51 26 56	26,7582	6,57368	1,39665	2249,4	40 26 39	716
717	51 40 89	26,7769	6,57508	1,39470	2252,5	40 37 65	717
718	51 55 24	26,7955	6,57647	1,39276	2255,7	40 48 92	718
719	51 69 61	26,8142	6,57786	1,39082	2258,8	40 60 20	719
720	51 84 00	26,8328	6,57925	1,38889	2261,9	40 71 50	**720**
721	51 98 41	26,8514	6,58064	1,38696	2265,1	40 82 82	721
722	52 12 84	26,8701	6,58203	1,38504	2268,2	40 94 15	722
723	52 27 29	26,8887	6,58341	1,38313	2271,4	41 05 50	723
724	52 41 76	26,9072	6,58479	1,38122	2274,5	41 16 87	724
725	52 56 25	26,9258	6,58617	1,37931	2277,7	41 28 25	725
726	52 70 76	26,9444	6,58755	1,37741	2280,8	41 39 65	726
727	52 85 29	26,9629	6,58893	1,37552	2283,9	41 51 06	727
728	52 99 84	26,9815	6,59030	1,37363	2287,1	41 62 48	728
729	53 14 41	27,0000	6,59167	1,37174	2290,2	41 73 93	729
730	53 29 00	27,0185	6,59304	1,36986	2293,4	41 85 39	**730**
731	53 43 61	27,0370	6,59441	1,36799	2296,5	41 96 86	731
732	53 58 24	27,0555	6,59578	1,36612	2299,6	42 08 35	732
733	53 72 89	27,0740	6,59715	1,36426	2302,8	42 19 86	733
734	53 87 56	27,0924	6,59851	1,36240	2305,9	42 31 38	734
735	54 02 25	27,1109	6,59987	1,36054	2309,1	42 42 93	735
736	54 16 96	27,1293	6,60123	1,35870	2312,2	42 54 47	736
737	54 31 69	27,1477	6,60259	1,35685	2315,4	42 66 04	737
738	54 46 44	27,1662	6,60394	1,35501	2318,5	42 77 62	738
739	54 61 21	27,1846	6,60530	1,35318	2321,6	42 89 22	739
740	54 76 00	27,2029	6,60665	1,35135	2324,8	43 00 84	**740**
741	54 90 81	27,2213	6,60800	1,34953	2327,9	43 12 47	741
742	55 05 64	27,2397	6,60935	1,34771	2331,1	43 24 12	742
743	55 20 49	27,2580	6,61070	1,34590	2334,2	43 35 78	743
744	55 35 36	27,2764	6,61204	1,34409	2337,3	43 47 46	744
745	55 50 25	27,2947	6,61338	1,34228	2340,5	43 59 16	745
746	55 65 16	27,3130	6,61473	1,34048	2343,6	43 70 87	746
747	55 80 09	27,3313	6,61607	1,33869	2346,8	43 82 59	747
748	55 95 04	27,3496	6,61740	1,33690	2349,9	43 94 33	748
749	56 10 01	27,3679	6,61874	1,33511	2353,1	44 06 09	749
750	56 25 00	27,3861	6,62007	1,33333	2356,2	44 17 86	**750**

750–800

n	n^2	$\sqrt{n}$	$\log_e n$	$\frac{1000}{n}$	πn	$\frac{\pi n^2}{4}$	n
750	56 25 00	27,3861	6,62007	1,33333	2356,2	44 17 86	**750**
751	56 40 01	27,4044	6,62141	1,33156	2359,3	44 29 65	751
752	56 55 04	27,4226	6,62274	1,32979	2362,5	44 41 46	752
753	56 70 09	27,4408	6,62407	1,32802	2365,6	44 53 28	753
754	56 85 16	27,4591	6,62539	1,32626	2368,8	44 65 11	754
755	57 00 25	27,4773	6,62672	1,32450	2371,9	44 76 97	755
756	57 15 36	27,4955	6,62804	1,32275	2375,0	44 88 83	756
757	57 30 49	27,5136	6,62936	1,32100	2378,2	45 00 72	757
758	57 45 64	27,5318	6,63068	1,31926	2381,3	45 12 62	758
759	57 60 81	27,5500	6,63200	1,31752	2384,5	45 24 53	759
760	57 76 00	27,5681	6,63332	1,31579	2387,6	45 36 46	**760**
761	57 91 21	27,5862	6,63463	1,31406	2390,8	45 48 41	761
762	58 06 44	27,6043	6,63595	1,31234	2393,9	45 60 37	762
763	58 21 69	27,6225	6,63726	1,31062	2397,0	45 72 34	763
764	58 36 96	27,6405	6,63857	1,30890	2400,2	45 84 34	764
765	58 52 25	27,6586	6,63988	1,30719	2403,3	45 96 35	765
766	58 67 56	27,6767	6,64118	1,30548	2406,5	46 08 37	766
767	58 82 89	27,6948	6,64249	1,30378	2409,6	46 20 41	767
768	58 98 24	27,7128	6,64379	1,30208	2412,7	46 32 47	768
769	59 13 61	27,7308	6,64509	1,30039	2415,9	46 44 54	769
770	59 29 00	27,7489	6,64639	1,29870	2419,0	46 56 63	**770**
771	59 44 41	27,7669	6,64769	1,29702	2422,2	46 68 73	771
772	59 59 84	27,7849	6,64898	1,29534	2425,3	46 80 85	772
773	59 75 29	27,8029	6,65028	1,29366	2428,5	46 92 98	773
774	59 90 76	27,8209	6,65157	1,29199	2431,6	47 05 13	774
775	60 06 25	27,8388	6,65286	1,29032	2434,7	47 17 30	775
776	60 21 76	27,8568	6,65415	1,28866	2437,9	47 29 48	776
777	60 37 29	27,8747	6,65544	1,28700	2441,0	47 41 68	777
778	60 52 84	27,8927	6,65673	1,28535	2444,2	47 53 89	778
779	60 68 41	27,9106	6,65801	1,28370	2447,3	47 66 12	779
780	60 84 00	27,9285	6,65929	1,28205	2450,4	47 78 36	**780**
781	60 99 61	27,9464	6,66058	1,28041	2453,6	47 90 62	781
782	61 15 24	27,9643	6,66185	1,27877	2456,7	48 02 90	782
783	61 30 89	27,9821	6,66313	1,27714	2459,9	48 15 19	783
784	61 46 56	28,0000	6,66441	1,27551	2463,0	48 27 50	784
785	61 62 25	28,0179	6,66568	1,27389	2466,2	48 39 82	785
786	61 77 96	28,0357	6,66696	1,27226	2469,3	48 52 16	786
787	61 93 69	28,0535	6,66823	1,27065	2472,4	48 64 51	787
788	62 09 44	28,0713	6,66950	1,26904	2475,6	48 76 88	788
789	62 25 21	28,0891	6,67077	1,26743	2478,7	48 89 27	789
790	62 41 00	28,1069	6,67203	1,26582	2481,9	49 01 67	**790**
791	62 56 81	28,1247	6,67330	1,26422	2485,0	49 14 09	791
792	62 72 64	28,1425	6,67456	1,26263	2488,1	49 26 52	792
793	62 88 49	28,1603	6,67582	1,26103	2491,3	49 38 97	793
794	63 04 36	28,1780	6,67708	1,25945	2494,4	49 51 43	794
795	63 20 25	28,1957	6,67834	1,25786	2497,6	49 63 91	795
796	63 36 16	28,2135	6,67960	1,25628	2500,7	49 76 41	796
797	63 52 09	28,2312	6,68085	1,25471	2503,8	49 88 92	797
798	63 68 04	28,2489	6,68211	1,25313	2507,0	50 01 45	798
799	63 84 01	28,2666	6,68336	1,25156	2510,1	50 13 99	799
800	64 00 00	28,2843	6,68461	1,25000	2513,3	50 26 55	**800**

n	n^2	$\sqrt{n}$	$\log_e n$	$\frac{1000}{n}$	πn	$\frac{\pi n^2}{4}$	n
800	64 00 00	28,2843	6,68461	1,25000	2513,3	50 26 55	**800**
801	64 16 01	28,3019	6,68586	1,24844	2516,4	50 39 12	801
802	64 32 04	28,3196	6,68711	1,24688	2519,6	50 51 71	802
803	64 48 09	28,3373	6,68835	1,24533	2522,7	50 64 32	803
804	64 64 16	28,3549	6,68960	1,24378	2525,8	50 76 94	804
805	64 80 25	28,3725	6,69084	1,24224	2529,0	50 89 58	805
806	64 96 36	28,3901	6,69208	1,24069	2532,1	51 02 23	806
807	65 12 49	28,4077	6,69332	1,23916	2535,3	51 14 90	807
808	65 28 64	28,4253	6,69456	1,23762	2538,4	51 27 58	808
809	65 44 81	28,4429	6,69580	1,23609	2541,5	51 40 28	809
810	65 61 00	28,4605	6,69703	1,23457	2544,7	51 53 00	**810**
811	65 77 21	28,4781	6,69827	1,23305	2547,8	51 65 73	811
812	65 93 44	28,4956	6,69950	1,23153	2551,0	51 78 48	812
813	66 09 69	28,5132	6,70073	1,23001	2554,1	51 91 24	813
814	66 25 96	28,5307	6,70196	1,22850	2557,3	52 04 02	814
815	66 42 25	28,5482	6,70319	1,22699	2560,4	52 16 81	815
816	66 58 56	28,5657	6,70441	1,22549	2563,5	52 29 62	816
817	66 74 89	28,5832	6,70564	1,22399	2566,7	52 42 45	817
818	66 91 24	28,6007	6,70686	1,22249	2569,8	52 55 29	818
819	67 07 61	28,6182	6,70808	1,22100	2573,0	52 68 14	819
820	67 24 00	28,6356	6,70930	1,21951	2576,1	52 81 02	**820**
821	67 40 41	28,6531	6,71052	1,21803	2579,2	52 93 91	821
822	67 56 84	28,6705	6,71174	1,21655	2582,4	53 06 81	822
823	67 73 29	28,6880	6,71296	1,21507	2585,5	53 19 73	823
824	67 89 76	28,7054	6,71417	1,21359	2588,7	53 32 67	824
825	68 06 25	28,7228	6,71538	1,21212	2591,8	53 45 62	825
826	68 22 76	28,7402	6,71659	1,21065	2595,0	53 58 58	826
827	68 39 29	28,7576	6,71780	1,20919	2598,1	53 71 57	827
828	68 55 84	28,7750	6,71901	1,20773	2601,2	53 84 56	828
829	68 72 41	28,7924	6,72022	1,20627	2604,4	53 97 58	829
830	68 89 00	28,8097	6,72143	1,20482	2607,5	54 10 61	**830**
831	69 05 61	28,8271	6,72263	1,20337	2610,7	54 23 65	831
832	69 22 24	28,8444	6,72383	1,20192	2613,8	54 36 71	832
833	69 38 89	28,8617	6,72503	1,20048	2616,9	54 49 79	833
834	69 55 56	28,8791	6,72623	1,19904	2620,1	54 62 88	834
835	69 72 25	28,8964	6,72743	1,19760	2623,2	54 75 99	835
836	69 88 96	28,9137	6,72863	1,19617	2626,4	54 89 12	836
837	70 05 69	28,9310	6,72982	1,19474	2629,5	55 02 26	837
838	70 22 44	28,9482	6,73102	1,19332	2632,7	55 15 41	838
839	70 39 21	28,9655	6,73221	1,19190	2635,8	55 28 58	839
840	70 56 00	28,9828	6,73340	1,19048	2638,9	55 41 77	**840**
841	70 72 81	29,0000	6,73459	1,18906	2642,1	55 54 97	841
842	70 89 64	29,0172	6,73578	1,18765	2645,2	55 68 19	842
843	71 06 49	29,0345	6,73697	1,18624	2648,4	55 81 42	843
844	71 23 36	29,0517	6,73815	1,18483	2651,5	55 94 67	844
845	71 40 25	29,0689	6,73934	1,18343	2654,6	56 07 94	845
846	71 57 16	29,0861	6,74052	1,18203	2657,8	56 21 22	846
847	71 74 09	29,1033	6,74170	1,18064	2660,9	56 34 52	847
848	71 91 04	29,1204	6,74288	1,17925	2664,1	56 47 83	848
849	72 08 01	29,1376	6,74406	1,17786	2667,2	56 61 16	849
850	72 25 00	29,1548	6,74524	1,17647	2670,4	56 74 50	**850**

850-900

n	n²	√n	logₑ n	1000/n	π n	π n²/4	n
850	.72 25 00	29,1548	6,74524	1,17647	2670,4	56 74 50	**850**
851	72 42 01	29,1719	6,74641	1,17509	2673,5	56 87 86	851
852	72 59 04	29,1890	6,74759	1,17371	2676,6	57 01 24	852
853	72 76 09	29,2062	6,74876	1,17233	2679,8	57 14 63	853
854	72 93 16	29,2233	6,74993	1,17096	2682,9	57 28 03	854
855	73 10 25	29,2404	6,75110	1,16959	2686,1	57 41 46	855
856	73 27 36	29,2575	6,75227	1,16822	2689,2	57 54 90	856
857	73 44 49	29,2746	6,75344	1,16686	2692,3	57 68 35	857
858	73 61 64	29,2916	6,75460	1,16550	2695,5	57 81 82	858
859	73 78 81	29,3087	6,75577	1,16414	2698,6	57 95 30	859
860	73 96 00	29,3258	6,75693	1,16279	2701,8	58 08 80	**860**
861	74 13 21	29,3428	6,75809	1,16144	2704,9	58 22 32	861
862	74 30 44	29,3598	6,75926	1,16009	2708,1	58 35 85	862
863	74 47 69	29,3769	6,76041	1,15875	2711,2	58 49 40	863
864	74 64 96	29,3939	6,76157	1,15741	2714,3	58 62 97	864
865	74 82 25	29,4109	6,76273	1,15607	2717,5	58 76 55	865
866	74 99 56	29,4279	6,76388	1,15473	2720,6	58 90 14	866
867	75 16 89	29,4449	6,76504	1,15340	2723,8	59 03 75	867
868	75 34 24	29,4618	6,76619	1,15207	2726,9	59 17 38	868
869	75 51 61	29,4788	6,76734	1,15075	2730,0	59 31 02	869
870	75 69 00	29,4958	6,76849	1,14943	2733,2	59 44 68	**870**
871	75 86 41	29,5127	6,76964	1,14811	2736,3	59 58 35	871
872	76 03 84	29,5296	6,77079	1,14679	2739,5	59 72 04	872
873	76 21 29	29,5466	6,77194	1,14548	2742,6	59 85 75	873
874	76 38 76	29,5635	6,77308	1,14416	2745,8	59 99 47	874
875	76 56 25	29,5804	6,77422	1,14286	2748,9	60 13 20	875
876	76 73 76	29,5973	6,77537	1,14155	2752,0	60 26 96	876
877	76 91 29	29,6142	6,77651	1,14025	2755,2	60 40 73	877
878	77 08 84	29,6311	6,77765	1,13895	2758,3	60 54 51	878
879	77 26 41	29,6479	6,77878	1,13766	2761,5	60 68 31	879
880	77 44 00	29,6648	6,77992	1,13636	2764,6	60 82 12	**880**
881	77 61 61	29,6816	6,78106	1,13507	2767,7	60 95 95	881
882	77 79 24	29,6985	6,78219	1,13379	2770,9	61 09 80	882
883	77 96 89	29,7153	6,78333	1,13250	2774,0	61 23 66	883
884	78 14 56	29,7321	6,78446	1,13122	2777,2	61 37 54	884
885	78 32 25	29,7489	6,78559	1,12994	2780,3	61 51 43	885
886	78 49 96	29,7658	6,78672	1,12867	2783,5	61 65 34	886
887	78 67 69	29,7825	6,78784	1,12740	2786,6	61 79 27	887
888	78 85 44	29,7993	6,78897	1,12613	2789,7	61 93 21	888
889	79 03 21	29,8161	6,79010	1,12486	2792,9	62 07 17	889
890	79 21 00	29,8329	6,79122	1,12360	2796,0	62 21 14	**890**
891	79 38 81	29,8496	6,79234	1,12233	2799,2	62 35 13	891
892	79 56 64	29,8664	6,79347	1,12108	2802,3	62 49 13	892
893	79 74 49	29,8831	6,79459	1,11982	2805,4	62 63 15	893
894	79 92 36	29,8998	6,79571	1,11857	2808,6	62 77 18	894
895	80 10 25	29,9166	6,79682	1,11732	2811,7	62 91 24	895
896	80 28 16	29,9333	6,79794	1,11607	2814,9	63 05 30	896
897	80 46 09	29,9500	6,79906	1,11483	2818,0	63 19 38	897
898	80 64 04	29,9666	6,80017	1,11359	2821,2	63 33 48	898
899	80 82 01	29,9833	6,80128	1,11235	2824,3	63 47 60	899
900	81 00 00	30,0000	6,80239	1,11111	2827,4	63 61 73	**900**

n	n^2	$\sqrt{n}$	$\log_e n$	$\frac{1000}{n}$	πn	$\frac{\pi n^2}{4}$	n
900	81 00 00	30,0000	6,80239	1,11111	2827,4	63 61 73	**900**
901	81 18 01	30,0167	6,80351	1,10988	2830,6	63 75 87	901
902	81 36 04	30,0333	6,80461	1,10865	2833,7	63 90 03	902
903	81 54 09	30,0500	6,80572	1,10742	2836,9	64 04 21	903
904	81 72 16	30,0666	6,80683	1,10619	2840,0	64 18 40	904
905	81 90 25	30,0832	6,80793	1,10497	2843,1	64 32 61	905
906	82 08 36	30,0998	6,80904	1,10375	2846,3	64 46 83	906
907	82 26 49	30,1164	6,81014	1,10254	2849,4	64 61 07	907
908	82 44 64	30,1330	6,81124	1,10132	2852,6	64 75 33	908
909	82 62 81	30,1496	6,81235	1,10011	2855,7	64 89 60	909
910	82 81 00	30,1662	6,81344	1,09890	2858,8	65 03 88	**910**
911	82 99 21	30,1828	6,81454	1,09769	2862,0	65 18 18	911
912	83 17 44	30,1993	6,81564	1,09649	2865,1	65 32 50	912
913	83 35 69	30,2159	6,81674	1,09529	2868,3	65 46 84	913
914	83 53 96	30,2324	6,81783	1,09409	2871,4	65 61 18	914
915	83 72 25	30,2490	6,81892	1,09290	2874,6	65 75 55	915
916	83 90 56	30,2655	6,82002	1,09170	2877,7	65 89 93	916
917	84 08 89	30,2820	6,82111	1,09051	2880,8	66 04 33	917
918	84 27 24	30,2985	6,82220	1,08932	2884,0	66 18 74	918
919	84 45 61	30,3150	6,82329	1,08814	2887,1	66 33 17	919
920	84 64 00	30,3315	6,82437	1,08696	2890,3	66 47 61	**920**
921	84 82 41	30,3480	6,82546	1,08578	2893,4	66 62 07	921
922	85 00 84	30,3645	6,82655	1,08460	2896,5	66 76 54	922
923	85 19 29	30,3809	6,82763	1,08342	2899,7	66 91 03	923
924	85 37 76	30,3974	6,82871	1,08225	2902,8	67 05 54	924
925	85 56 25	30,4138	6,82979	1,08108	2906,0	67 20 06	925
926	85 74 76	30,4302	6,83087	1,07991	2909,1	67 34 60	926
927	85 93 29	30,4467	6,83195	1,07875	2912,3	67 49 15	927
928	86 11 84	30,4631	6,83303	1,07759	2915,4	67 63 72	928
929	86 30 41	30,4795	6,83411	1,07643	2918,5	67 78 31	929
930	86 49 00	30,4959	6,83518	1,07527	2921,7	67 92 91	**930**
931	86 67 61	30,5123	6,83626	1,07411	2924,8	68 07 52	931
932	86 86 24	30,5287	6,83733	1,07296	2928,0	68 22 16	932
933	87 04 89	30,5450	6,83841	1,07181	2931,1	68 36 80	933
934	87 23 56	30,5614	6,83948	1,07066	2934,2	68 51 47	934
935	87 42 25	30,5778	6,84055	1,06952	2937,4	68 66 15	935
936	87 60 96	30,5941	6,84162	1,06838	2940,5	68 80 84	936
937	87 79 69	30,6105	6,84268	1,06724	2943,7	68 95 55	937
938	87 98 44	30,6268	6,84375	1,06610	2946,8	69 10 28	938
939	88 17 21	30,6431	6,84482	1,06496	2950,0	69 25 02	939
940	88 36 00	30,6594	6,84588	1,06383	2953,1	69 39 78	**940**
941	88 54 81	30,6757	6,84694	1,06270	2956,2	69 54 55	941
942	88 73 64	30,6920	6,84801	1,06157	2959,4	69 69 34	942
943	88 92 49	30,7083	6,84907	1,06045	2962,5	69 84 15	943
944	89 11 36	30,7246	6,85013	1,05932	2965,7	69 98 97	944
945	89 30 25	30,7409	6,85118	1,05820	2968,8	70 13 80	945
946	89 49 16	30,7571	6,85224	1,05708	2971,9	70 28 65	946
947	89 68 09	30,7734	6,85330	1,05597	2975,1	70 43 52	947
948	89 87 04	30,7896	6,85435	1,05485	2978,2	70 58 40	948
949	90 06 01	30,8058	6,85541	1,05374	2981,4	70 73 30	949
950	90 25 00	30,8221	6,85646	1,05263	2984,5	70 88 22	**950**

950–1000

n	n^2	$\sqrt{n}$	$\log_e n$	$\frac{1000}{n}$	πn	$\frac{\pi n^2}{4}$	n
950	90 25 00	30,8221	6,85646	1,05263	2984,5	70 88 22	**950**
951	90 44 01	30,8383	6,85751	1,05152	2987,7	71 03 15	951
952	90 63 04	30,8545	6,85857	1,05042	2990,8	71 18 09	952
953	90 82 09	30,8707	6,85961	1,04932	2993,9	71 33 06	953
954	91 01 16	30,8869	6,86066	1,04822	2997,1	71 48 03	954
955	91 20 25	30,9031	6,86171	1,04712	3000,2	71 63 03	955
956	91 39 36	30,9192	6,86276	1,04603	3003,4	71 78 04	956
957	91 58 49	30,9354	6,86380	1,04493	3006,5	71 93 06	957
958	91 77 64	30,9516	6,86485	1,04384	3009,6	72 08 10	958
959	91 96 81	30,9677	6,86589	1,04275	3012,8	72 23 16	959
960	92 16 00	30,9839	6,86693	1,04167	3015,9	72 38 23	**960**
961	92 35 21	31,0000	6,86797	1,04058	3019,1	72 53 32	961
962	92 54 44	31,0161	6,86901	1,03950	3022,2	72 68 42	962
963	92 73 69	31,0322	6,87005	1,03842	3025,4	72 83 54	963
964	92 92 96	31,0483	6,87109	1,03734	3028,5	72 98 67	964
965	93 12 25	31,0644	6,87213	1,03627	3031,6	73 13 82	965
966	93 31 56	31,0805	6,87316	1,03520	3034,8	73 28 99	966
967	93 50 89	31,0966	6,87420	1,03413	3037,9	73 44 17	967
968	93 70 24	31,1127	6,87523	1,03306	3041,1	73 59 37	968
969	93 89 61	31,1288	6,87626	1,03199	3044,2	73 74 58	969
970	94 09 00	31,1448	6,87730	1,03093	3047,3	73 89 81	**970**
971	94 28 41	31,1609	6,87833	1,02987	3050,5	74 05 06	971
972	94 47 84	31,1769	6,87936	1,02881	3053,6	74 20 32	972
973	94 67 29	31,1929	6,88038	1,02775	3056,8	74 35 59	973
974	94 86 76	31,2090	6,88141	1,02669	3059,9	74 50 88	974
975	95 06 25	31,2250	6,88244	1,02564	3063,1	74 66 19	975
976	95 25 76	31,2410	6,88346	1,02459	3066,2	74 81 51	976
977	95 45 29	31,2570	6,88449	1,02354	3069,3	74 96 85	977
978	95 64 84	31,2730	6,88551	1,02249	3072,5	75 12 21	978
979	95 84 41	31,2890	6,88653	1,02145	3075,6	75 27 58	979
980	96 04 00	31,3050	6,88755	1,02041	3078,8	75 42 96	**980**
981	96 23 61	31,3209	6,88857	1,01937	3081,9	75 58 37	981
982	96 43 24	31,3369	6,88959	1,01833	3085,0	75 73 78	982
983	96 62 89	31,3528	6,89061	1,01729	3088,2	75 89 22	983
984	96 82 56	31,3688	6,89163	1,01626	3091,3	76 04 66	984
985	97 02 25	31,3847	6,89264	1,01523	3094,5	76 20 13	985
986	97 21 96	31,4006	6,89366	1,01420	3097,6	76 35 61	986
987	97 41 69	31,4166	6,89467	1,01317	3100,8	76 51 11	987
988	97 61 44	31,4325	6,89568	1,01215	3103,9	76 66 62	988
989	97 81 21	31,4484	6,89669	1,01112	3107,0	76 82 14	989
990	98 01 00	31,4643	6,89770	1,01010	3110,2	76 97 69	**990**
991	98 20 81	31,4802	6,89871	1,00908	3113,3	77 13 25	991
992	98 40 64	31,4960	6,89972	1,00806	3116,5	77 28 82	992
993	98 60 49	31,5119	6,90073	1,00705	3119,6	77 44 41	993
994	98 80 36	31,5278	6,90174	1,00604	3122,7	77 60 02	994
995	99 00 25	31,5436	6,90274	1,00503	3125,9	77 75 64	995
996	99 20 16	31,5595	6,90375	1,00402	3129,0	77 91 28	996
997	99 40 09	31,5753	6,90475	1,00301	3132,2	78 06 93	997
998	99 60 04	31,5911	6,90575	1,00200	3135,3	78 22 60	998
999	99 80 01	31,6070	6,90675	1,00100	3138,5	78 38 28	999
1000	100 00 00	31,6228	6,90776	1,00000	3141,6	78 53 98	**1000**

n	n^2	$\sqrt{n}$	$\log_e n$	$\frac{1000}{n}$	πn	$\frac{\pi n^2}{4}$	n
1000	1 000 000	31,6228	6,90776	1,00000	3141,6	78 53 98	**1000**
1001	1 002 001	31,6386	6,90875	0,99900	3144,7	78 69 70	1001
1002	1 004 004	31,6544	6,90975	0,99800	3147,9	78 85 43	1002
1003	1 006 009	31,6702	6,91075	0,99701	3151,0	79 01 18	1003
1004	1 008 016	31,6860	6,91175	0,99602	3154,2	79 16 94	1004
1005	1 010 025	31,7017	6,91275	0,99502	3157,3	79 32 72	1005
1006	1 012 036	31,7175	6,91374	0,99404	3160,4	79 48 51	1006
1007	1 014 049	31,7333	6,91474	0,99305	3163,6	79 64 32	1007
1008	1 016 064	31,7490	6,91573	0,99206	3166,7	79 80 15	1008
1009	1 018 081	31,7648	6,91672	0,99108	3169,9	79 95 99	1009
1010	1 020 100	31,7805	6,91771	0,99010	3173,0	80 11 84	**1010**
1011	1 022 121	31,7962	6,91869	0,98912	3176,2	80 27 72	1011
1012	1 024 144	31,8119	6,91968	0,98814	3179,3	80 43 61	1012
1013	1 026 169	31,8277	6,92067	0,98717	3182,4	80 59 51	1013
1014	1 028 196	31,8434	6,92166	0,98619	3185,6	80 75 43	1014
1015	1 030 225	31,8591	6,92264	0,98522	3188,7	80 91 37	1015
1016	1 032 256	31,8748	6,92363	0,98425	3191,9	81 07 32	1016
1017	1 034 289	31,8904	6,92461	0,98328	3195,0	81 23 29	1017
1018	1 036 324	31,9061	6,92559	0,98232	3198,1	81 39 27	1018
1019	1 038 361	31,9218	6,92658	0,98135	3201,3	81 55 27	1019
1020	1 040 400	31,9374	6,92756	0,98039	3204,4	81 71 28	**1020**
1021	1 042 441	31,9531	6,92854	0,97943	3207,6	81 87 31	1021
1022	1 044 484	31,9687	6,92952	0,97847	3210,7	82 03 36	1022
1023	1 046 529	31,9844	6,93049	0,97752	3213,9	82 19 42	1023
1024	1 048 576	32,0000	6,93147	0,97656	3217,0	82 35 49	1024
1025	1 050 625	32,0156	6,93245	0,97561	3220,1	82 51 59	1025
1026	1 052 676	32,0312	6,93342	0,97466	3223,3	82 67 70	1026
1027	1 054 729	32,0468	6,93440	0,97371	3226,4	82 83 82	1027
1028	1 056 784	32,0624	6,93537	0,97276	3229,6	82 99 96	1028
1029	1 058 841	32,0780	6,93634	0,97182	3232,7	83 16 12	1029
1030	1 060 900	32,0936	6,93731	0,97087	3235,8	83 32 29	**1030**
1031	1 062 961	32,1092	6,93828	0,96993	3239,0	83 48 47	1031
1032	1 065 024	32,1248	6,93925	0,96899	3242,1	83 64 68	1032
1033	1 067 089	32,1403	6,94022	0,96805	3245,3	83 80 90	1033
1034	1 069 156	32,1559	6,94119	0,96712	3248,4	83 97 13	1034
1035	1 071 225	32,1714	6,94216	0,96618	3251,5	84 13 38	1035
1036	1 073 296	32,1870	6,94312	0,96525	3254,7	84 29 65	1036
1037	1 075 369	32,2025	6,94409	0,96432	3257,8	84 45 93	1037
1038	1 077 444	32,2180	6,94505	0,96339	3261,0	84 62 22	1038
1039	1 079 521	32,2335	6,94601	0,96246	3264,1	84 78 54	1039
1040	1 081 600	32,2490	6,94698	0,96154	3267,3	84 94 87	**1040**
1041	1 083 681	32,2645	6,94794	0,96061	3270,4	85 11 21	1041
1042	1 085 764	32,2800	6,94890	0,95969	3273,5	85 27 57	1042
1043	1 087 849	32,2955	6,94986	0,95877	3276,7	85 43 94	1043
1044	1 089 936	32,3110	6,95081	0,95785	3279,8	85 60 34	1044
1045	1 092 025	32,3265	6,95177	0,95694	3283,0	85 76 74	1045
1046	1 094 116	32,3419	6,95273	0,95602	3286,1	85 93 17	1046
1047	1 096 209	32,3574	6,95368	0,95511	3289,2	86 09 60	1047
1048	1 098 304	32,3728	6,95464	0,95420	3292,4	86 26 06	1048
1049	1 100 401	32,3883	6,95559	0,95329	3295,5	86 42 53	1049
1050	1 102 500	32,4037	6,95655	0,95238	3298,7	86 59 01	**1050**

1050–1100

n	n^2	$\sqrt{n}$	$\log_e n$	$\frac{1000}{n}$	πn	$\frac{\pi n^2}{4}$	n
1050	1 102 500	32,4037	6,95655	0,95238	3298,7	86 59 01	**1050**
1051	1 104 601	32,4191	6,95750	0,95147	3301,8	86 75 51	1051
1052	1 106 704	32,4345	6,95845	0,95057	3305,0	86 92 03	1052
1053	1 108 809	32,4500	6,95940	0,94967	3308,1	87 08 56	1053
1054	1 110 916	32,4654	6,96035	0,94877	3311,2	87 25 11	1054
1055	1 113 025	32,4808	6,96130	0,94787	3314,4	87 41 68	1055
1056	1 115 136	32,4962	6,96224	0,94697	3317,5	87 58 26	1056
1057	1 117 249	32,5115	6,96319	0,94607	3320,7	87 74 85	1057
1058	1 119 364	32,5269	6,96414	0,94518	3323,8	87 91 46	1058
1059	1 121 481	32,5423	6,96508	0,94429	3326,9	88 08 09	1059
1060	1 123 600	32,5576	6,96602	0,94340	3330,1	88 24 73	**1060**
1061	1 125 721	32,5730	6,96697	0,94251	3333,2	88 41 39	1061
1062	1 127 844	32,5883	6,96791	0,94162	3336,4	88 58 06	1062
1063	1 129 969	32,6037	6,96885	0,94073	3339,5	88 74 75	1063
1064	1 132 096	32,6190	6,96979	0,93985	3342,7	88 91 46	1064
1065	1 134 225	32,6343	6,97073	0,93897	3345,8	89 08 18	1065
1066	1 136 356	32,6497	6,97167	0,93809	3348,9	89 24 92	1066
1067	1 138 489	32,6650	6,97261	0,93721	3352,1	89 41 67	1067
1068	1 140 624	32,6803	6,97354	0,93633	3355,2	89 58 43	1068
1069	1 142 761	32,6956	6,97448	0,93545	3358,4	89 75 22	1069
1070	1 144 900	32,7109	6,97541	0,93458	3361,5	89 92 02	**1070**
1071	1 147 041	32,7261	6,97635	0,93371	3364,6	90 08 84	1071
1072	1 149 184	32,7414	6,97728	0,93284	3367,8	90 25 67	1072
1073	1 151 329	32,7567	6,97821	0,93197	3370,9	90 42 51	1073
1074	1 153 476	32,7719	6,97914	0,93110	3374,1	90 59 38	1074
1075	1 155 625	32,7872	6,98008	0,93023	3377,2	90 76 26	1075
1076	1 157 776	32,8024	6,98101	0,92937	3380,4	90 93 15	1076
1077	1 159 929	32,8177	6,98193	0,92851	3383,5	91 10 06	1077
1078	1 162 084	32,8329	6,98286	0,92764	3386,6	91 26 98	1078
1079	1 164 241	32,8481	6,98379	0,92678	3389,8	91 43 92	1079
1080	1 166 400	32,8634	6,98472	0,92593	3392,9	91 60 88	**1080**
1081	1 168 561	32,8786	6,98564	0,92507	3396,1	91 77 85	1081
1082	1 170 724	32,8938	6,98657	0,92421	3399,2	91 94 84	1082
1083	1 172 889	32,9090	6,98749	0,92336	3402,3	92 11 85	1083
1084	1 175 056	32,9242	6,98841	0,92251	3405,5	92 28 87	1084
1085	1 177 225	32,9393	6,98933	0,92166	3408,6	92 45 90	1085
1086	1 179 396	32,9545	6,99026	0,92081	3411,8	92 62 95	1086
1087	1 181 569	32,9697	6,99118	0,91996	3414,9	92 80 02	1087
1088	1 183 744	32,9848	6,99210	0,91912	3418,1	92 97 10	1088
1089	1 185 921	33,0000	6,99301	0,91827	3421,2	93 14 20	1089
1090	1 188 100	33,0151	6,99393	0,91743	3424,3	93 31 31	**1090**
1091	1 190 281	33,0303	6,99485	0,91659	3427,5	93 48 44	1091
1092	1 192 464	33,0454	6,99577	0,91575	3430,6	93 65 59	1092
1093	1 194 649	33,0606	6,99668	0,91491	3433,8	93 82 75	1093
1094	1 196 836	33,0757	6,99760	0,91408	3436,9	93 99 93	1094
1095	1 199 025	33,0908	6,99851	0,91324	3440,0	94 17 12	1095
1096	1 201 216	33,1059	6,99942	0,91240	3443,2	94 34 33	1096
1097	1 203 409	33,1210	7,00033	0,91158	3446,3	94 51 55	1097
1098	1 205 604	33,1361	7,00125	0,91075	3449,5	94 68 79	1098
1099	1 207 801	33,1512	7,00216	0,90992	3452,6	94 86 04	1099
1100	1 210 000	33,1662	7,00307	0,90909	3455,8	95 03 32	**1100**

TABLE III

CIRCULAR (TRIGONOMETRIC) FUNCTIONS

(Taken from B. O. Peirce's "Short Table of Integrals," Ginn & Co.)

Radians.	Degrees.	Sines.	Cosines.	Tangents.	Cotangents.		
0·0000	0°00′	·0000	1·0000	·0000	∞	90°00′	1·5708
0·0029	10	·0029	1·0000	·0029	343·77	50	1·5679
0·0058	20	·0058	1·0000	·0058	171·89	40	1·5650
0·0087	30	·0087	1·0000	·0087	114·59	30	1·5621
0·0116	40	·0116	·9999	·0116	85·940	20	1·5592
0·0145	50	·0145	·9999	·0145	68·750	10	1·5563
0·0175	1°00′	·0175	·9998	·0175	57·290	89°00′	1·5533
0·0204	10	·0204	·9998	·0204	49·104	50	1·5504
0·0233	20	·0233	·9997	·0233	42·964	40	1·5475
0·0262	30	·0262	·9997	·0262	38·188	30	1·5446
0·0291	40	·0291	·9996	·0291	34·368	20	1·5417
0·0320	50	·0320	·9995	·0320	31·242	10	1·5388
0·0349	2°00′	·0349	·9994	·0349	28·636	88°00′	1·5359
0·0378	10	·0378	·9993	·0378	26·432	50	1·5330
0·0407	20	·0407	·9992	·0407	24·542	40	1·5301
0·0436	30	·0436	·9990	·0437	22·904	30	1·5272
0·0465	40	·0465	·9989	·0466	21·470	20	1·5243
0·0495	50	·0494	·9988	·0495	20·206	10	1·5213
0·0524	3°00′	·0523	·9986	·0524	19·081	87°00′	1·5184
0·0553	10	·0552	·9985	·0553	18·075	50	1·5155
0·0582	20	·0581	·9983	·0582	17·169	40	1·5126
0·0611	30	·0610	·9981	·0612	16·350	30	1·5097
0·0640	40	·0640	·9980	·0641	15·605	20	1·5068
0·0669	50	·0669	·9978	·0670	14·924	10	1·5039
0·0698	4°00′	·0698	·9976	·0699	14·301	86°00′	1·5010
0·0727	10	·0727	·9974	·0729	13·727	50	1·4981
0·0756	20	·0756	·9971	·0758	13·197	40	1·4952
0·0785	30	·0785	·9969	·0787	12·706	30	1·4923
0·0814	40	·0814	·9967	·0816	12·251	20	1·4893
0.0844	50	·0843	·9964	·0846	11·826	10	1·4864
0·0873	5°00′	·0872	·9962	·0875	11·430	85°00′	1·4835
0·0902	10	·0901	·9959	·0904	11·059	50	1·4806
0·0931	20	·0929	·9957	·0934	10·712	40	1·4777
0·0960	30	·0958	·9954	·0963	10·385	30	1·4748
0·0989	40	·0987	·9951	·0992	10·078	20	1·4719
0·1018	50	·1016	·9948	·1022	9·7882	10	1·4690
0·1047	6°00′	·1045	·9945	·1051	9·5144	84°00′	1·4661
0·1076	10	·1074	·9942	·1080	9·2553	50	1·4632
0·1105	20	·1103	·9939	·1110	9·0098	40	1·4603
0·1134	30	·1132	·9936	·1139	8·7769	30	1·4574
0·1164	40	·1161	·9932	·1169	8·5555	20	1·4544
0·1193	50	·1190	·9929	·1198	8·3450	10	1·4515
0·1222	7°00′	·1219	·9925	·1228	8·1443	83°00′	1·4486
0·1251	10	·1248	·9922	·1257	7·9530	50	1·4457
0·1280	20	·1276	·9918	·1287	7·7704	40	1·4428
0·1309	30	·1305	·9914	·1317	7·5958	30	1·4399
0·1338	40	·1334	·9911	·1346	7·4287	20	1·4370
0·1367	50	·1363	·9907	·1376	7·2687	10	1·4341
0·1396	8°00′	·1392	·9903	·1405	7·1154	82°00′	1·4312
0·1425	10	·1421	·9899	·1435	6·9682	50	1·4283
0·1454	20	·1449	·9894	·1465	6·8269	40	1·4254
0·1484	30	·1478	·9890	·1495	6·6912	30	1·4224
0·1513	40	·1507	·9886	·1524	6·5606	20	1·4195
0·1542	50	·1536	·9881	·1554	6·4348	10	1·4166
0·1571	9°00′	·1564	·9877	·1584	6·3138	81°00′	1·4137
		Cosines.	Sines.	Cotangents.	Tangents.	Degrees.	Radians.

CIRCULAR (TRIGONOMETRIC) FUNCTIONS

Radians.	Degrees.	Sines.	Cosines.	Tangents.	Cotangents.		
0·1571	9°00′	·1564	·9877	·1584	6·3138	81°00′	1·4137
0·1600	10	·1593	·9872	·1614	6·1970	50	1·4108
0·1629	20	·1622	·9868	·1644	6·0844	40	1·4079
0·1658	30	·1650	·9863	·1673	5·9758	30	1·4050
0·1687	40	·1679	·9858	·1703	5·8708	20	1·4021
0·1716	50	·1708	·9853	·1733	5·7694	10	1·3992
0·1745	10°00′	·1736	·9848	·1763	5·6713	80°00′	1·3963
0·1774	10	·1765	·9843	·1793	5·5764	50	1·3934
0·1804	20	·1794	·9838	·1823	5·4845	40	1·3904
0·1833	30	·1822	·9833	·1853	5·3955	30	1·3875
0·1862	40	·1851	·9827	·1883	5·3093	20	1·3846
0·1891	50	·1880	·9822	·1914	5·2257	10	1·3817
0·1920	11°00′	·1908	·9816	·1944	5·1446	79°00′	1·3788
0·1949	10	·1937	·9811	·1974	5·0658	50	1·3759
0·1978	20	·1965	·9805	·2004	4·9894	40	1·3730
0·2007	30	·1994	·9799	·2035	4·9152	30	1·3701
0·2036	40	·2022	·9793	·2065	4·8430	20	1·3672
0·2065	50	·2051	·9787	·2095	4·7729	10	1·3643
0·2094	12°00′	·2079	·9781	·2126	4·7046	78°00′	1·3614
0·2123	10	·2108	·9775	·2156	4·6382	50	1·3584
0·2153	20	·2136	·9769	·2186	4·5736	40	1·3555
0·2182	30	·2164	·9763	·2217	4·5107	30	1·3526
0·2211	40	·2193	·9757	·2247	4·4494	20	1·3497
0·2240	50	·2221	·9750	·2278	4·3897	10	1·3468
0·2269	13°00′	·2250	·9744	·2309	4·3315	77°00′	1·3439
0·2298	10	·2278	·9737	·2339	4·2747	50	1·3410
0·2327	20	·2306	·9730	·2370	4·2193	40	1·3381
0·2356	30	·2334	·9724	·2401	4·1653	30	1·3352
0·2385	40	·2363	·9717	·2432	4·1126	20	1·3323
0·2414	50	·2391	·9710	·2462	4·0611	10	1·3294
0·2443	14°00′	·2419	·9703	·2493	4·0108	76°00′	1·3265
0·2473	10	·2447	·9696	·2524	3·9617	50	1·3235
0·2502	20	·2476	·9689	·2555	3·9136	40	1·3206
0·2531	30	·2504	·9681	·2586	3·8667	30	1·3177
0·2560	40	·2532	·9674	·2617	3·8208	20	1·3148
0·2589	50	·2560	·9667	·2648	3·7760	10	1·3119
0·2618	15°00′	·2588	·9659	·2679	3·7321	75°00′	1·3090
0·2647	10	·2616	·9652	·2711	3·6891	50	1·3061
0·2676	20	·2644	·9644	·2742	3·6470	40	1·3032
0·2705	30	·2672	·9636	·2773	3·6059	30	1·3003
0·2734	40	·2700	·9628	·2805	3·5656	20	1·2974
0·2763	50	·2728	·9621	·2836	3·5261	10	1·2945
0·2793	16°00′	·2756	·9613	·2867	3·4874	74°00′	1·2915
0·2822	10	·2784	·9605	·2899	3·4495	50	1·2886
0·2851	20	·2812	·9596	·2931	3·4124	40	1·2857
0·2880	30	·2840	·9588	·2962	3·3759	30	1·2828
0·2909	40	·2868	·9580	·2994	3·3402	20	1·2799
0·2938	50	·2896	·9572	·3026	3·3052	10	1·2770
0·2967	17°00′	·2924	·9563	·3057	3·2709	73°00′	1·2741
0·2996	10	·2952	·9555	·3089	3·2371	50	1·2712
0·3025	20	·2979	·9546	·3121	3·2041	40	1·2683
0·3054	30	·3007	·9537	·3153	3·1716	30	1·2654
0·3083	40	·3035	·9528	·3185	3·1397	20	1·2625
0·3113	50	·3062	·9520	·3217	3·1084	10	1·2595
0·3142	18°00′	·3090	·9511	·3249	3·0777	72°00′	1·2566
		Cosines.	Sines.	Cotangents.	Tangents.	Degrees.	Radians.

CIRCULAR (TRIGONOMETRIC) FUNCTIONS

Radians.	Degrees.	Sines.	Cosines.	Tangents.	Cotangents.		
0·3142	18°00′	·3090	·9511	·3249	3·0777	72°00′	1·2566
0·3171	10	·3118	·9502	·3281	3·0475	50	1·2537
0·3200	20	·3145	·9492	·3314	3·0178	40	1·2508
0·3229	30	·3173	·9483	·3346	2·9887	30	1·2479
0·3258	40	·3201	·9474	·3378	2·9600	20	1·2450
0·3287	50	·3228	·9465	·3411	2·9319	10	1·2421
0·3316	19°00′	·3256	·9455	·3443	2·9042	71°00′	1·2392
0·3345	10	·3283	·9446	·3476	2·8770	50	1·2363
0·3374	20	·3311	·9436	·3508	2·8502	40	1·2334
0·3403	30	·3338	·9426	·3541	2·8239	30	1·2305
0·3432	40	·3365	·9417	·3574	2·7980	20	1·2275
0·3462	50	·3393	·9407	·3607	2·7725	10	1·2246
0·3491	20°00′	·3420	·9397	·3640	2·7475	70°00′	1·2217
0·3520	10	·3448	·9387	·3673	2·7228	50	1·2188
0·3549	20	·3475	·9377	·3706	2·6985	40	1·2159
0·3578	30	·3502	·9367	·3739	2·6746	30	1·2130
0·3607	40	·3529	·9356	·3772	2·6511	20	1·2101
0·3636	50	·3557	·9346	·3805	2·6279	10	1·2072
0·3665	21°00′	·3584	·9336	·3839	2·6051	69°00′	1·2043
0·3694	10	·3611	·9325	·3872	2·5826	50	1·2014
0·3723	20	·3638	·9315	·3906	2·5605	40	1·1985
0·3752	30	·3665	·9304	·3939	2·5386	30	1·1956
0·3782	40	·3692	·9293	·3973	2·5172	20	1·1926
0·3811	50	·3719	·9283	·4006	2·4960	10	1·1897
0·3840	22°00′	·3746	·9272	·4040	2·4751	68°00′	1·1868
0·3869	10	·3773	·9261	·4074	2·4545	50	1·1839
0·3898	20	·3800	·9250	·4108	2·4342	40	1·1810
0·3927	30	·3827	·9239	·4142	2·4142	30	1·1781
0·3956	40	·3854	·9228	·4176	2·3945	20	1·1752
0·3985	50	·3881	·9216	·4210	2·3750	10	1·1723
0·4014	23°00′	·3907	·9205	·4245	2·3559	67°00′	1·1694
0·4043	10	·3934	·9194	·4279	2·3369	50	1·1665
0·4072	20	·3961	·9182	·4314	2·3183	40	1·1636
0·4102	30	·3987	·9171	·4348	2·2998	30	1·1606
0·4131	40	·4014	·9159	·4383	2·2817	20	1·1577
0·4160	50	·4041	·9147	·4417	2·2637	10	1·1548
0·4189	24°00′	·4067	·9135	·4452	2·2460	66°00′	1·1519
0·4218	10	·4094	·9124	·4487	2·2286	50	1·1490
0·4247	20	·4120	·9112	·4522	2·2113	40	1·1461
0·4276	30	·4147	·9100	·4557	2·1943	30	1·1432
0·4305	40	·4173	·9088	·4592	2·1775	20	1·1403
0·4334	50	·4200	·9075	·4628	2·1609	10	1·1374
0·4363	25°00′	·4226	·9063	·4663	2·1445	65°00′	1·1345
0·4392	10	·4253	·9051	·4699	2·1283	50	1·1316
0·4422	20	·4279	·9038	·4734	2·1123	40	1·1286
0·4451	30	·4305	·9026	·4770	2·0965	30	1·1257
0·4480	40	·4331	·9013	·4806	2·0809	20	1·1228
0·4509	50	·4358	·9001	·4841	2·0655	10	1·1199
0·4538	26°00′	·4384	·8988	·4877	2·0503	64°00′	1·1170
0·4567	10	·4410	·8975	·4913	2·0353	50	1·1141
0·4596	20	·4436	·8962	·4950	2·0204	40	1·1112
0·4625	30	·4462	·8949	·4986	2·0057	30	1·1083
0·4654	40	·4488	·8936	·5022	1·9912	20	1·1054
0·4683	50	·4514	·8923	·5059	1·9768	10	1·1025
0·4712	27°00′	·4540	·8910	·5095	1·9626	63°00′	1·0996
		Cosines.	Sines.	Cotangents.	Tangents.	Degrees.	Radians.

CIRCULAR (TRIGONOMETRIC) FUNCTIONS

Radians.	Degrees.	Sines.	Cosines.	Tangents.	Cotangents.		
0·4712	27°00′	·4540	·8910	·5095	1·9626	63°00′	1·0996
0·4741	10	·4566	·8897	·5132	1·9486	50	1·0966
0·4771	20	·4592	·8884	·5169	1·9347	40	1·0937
0·4800	30	·4617	·8870	·5206	1·9210	30	1·0908
0·4829	40	·4643	·8857	·5243	1·9074	20	1·0879
0·4858	50	·4669	·8843	·5280	1·8940	10	1·0850
0·4887	28°00′	·4695	·8829	·5317	1·8807	62°00′	1·0821
0·4916	10	·4720	·8816	·5354	1·8676	50	1·0792
0·4945	20	·4746	·8802	·5392	1·8546	40	1·0763
0·4974	30	·4772	·8788	·5430	1·8418	30	1·0734
0·5003	40	·4797	·8774	·5467	1·8291	20	1·0705
0·5032	50	·4823	·8760	·5505	1·8165	10	1·0676
0·5061	29°00′	·4848	·8746	·5543	1·8040	61°00′	1·0647
0·5091	10	·4874	·8732	·5581	1·7917	50	1·0617
0·5120	20	·4899	·8718	·5619	1·7796	40	1·0588
0·5149	30	·4924	·8704	·5658	1·7675	30	1·0559
0·5178	40	·4950	·8689	·5696	1·7556	20	1·0530
0·5207	50	·4975	·8675	·5735	1·7437	10	1·0501
0·5236	30°00′	·5000	·8660	·5774	1·7321	60°00′	1·0472
0·5265	10	·5025	·8646	·5812	1·7205	50	1·0443
0·5294	20	·5050	·8631	·5851	1·7090	40	1·0414
0·5323	30	·5075	·8616	·5890	1·6977	30	1·0385
0·5352	40	·5100	·8601	·5930	1·6864	20	1·0356
0·5381	50	·5125	·8587	·5969	1·6753	10	1·0327
0·5411	31°00′	·5150	·8572	·6009	1·6643	59°00′	1·0297
0·5440	10	·5175	·8557	·6048	1·6534	50	1·0268
0·5469	20	·5200	·8542	·6088	1·6426	40	1·0239
0·5498	30	·5225	·8526	·6128	1·6319	30	1·0210
0·5527	40	·5250	·8511	·6168	1·6212	20	1·0181
0·5556	50	·5275	·8496	·6208	1·6107	10	1·0152
0·5585	32°00′	·5299	·8480	·6249	1·6003	58°00′	1·0123
0·5614	10	·5324	·8465	·6289	1·5900	50	1·0094
0·5643	20	·5348	·8450	·6330	1·5798	40	1·0065
0·5672	30	·5373	·8434	·6371	1·5697	30	1·0036
0·5701	40	·5398	·8418	·6412	1·5597	20	1·0007
0·5730	50	·5422	·8403	·6453	1·5497	10	0·9977
0·5760	33°00′	·5446	·8387	·6494	1·5399	57°00′	0·9948
0·5789	10	·5471	·8371	·6536	1·5301	50	0·9919
0·5818	20	·5495	·8355	·6577	1·5204	40	0·9890
0·5847	30	·5519	·8339	·6619	1·5108	30	0·9861
0·5876	40	·5544	·8323	·6661	1·5013	20	0·9832
0·5905	50	·5568	·8307	·6703	1·4919	10	0·9803
0·5[illegible]34	34°00′	·5592	·8290	·6745	1·4826	56°00′	0·9774
0·5963	10	·5616	·8274	·6787	1·4733	50	0·9745
0·5992	20	·5640	·8258	·6830	1·4641	40	0·9716
0·6021	30	·5664	·8241	·6873	1·4550	30	0·9687
0·6050	40	·5688	·8225	·6916	1·4460	20	0·9657
0·6080	50	·5712	·8208	·6959	1·4370	10	0·9628
0·6109	55°00′	·5736	·8192	·7002	1·4281	55°00′	0·9599
0·6138	10	·5760	·8175	·7046	1·4193	50	0·9570
0·6167	20	·5783	·8158	·7089	1·4106	40	0·9541
0·6196	30	·5807	·8141	·7133	1·4019	30	0·9512
0·6225	40	·5831	·8124	·7177	1·3934	20	0·9483
0·6254	50	·5854	·8107	·7221	1·3848	10	0·9454
0·6283	36°00	·5878	·8090	·7265	1·3764	54°00′	0·9425
		Cosines.	Sines.	Cotangents.	Tangents.	Degrees.	Radians.

CIRCULAR (TRIGONOMETRIC) FUNCTIONS

Radians.	Degrees.	Sines.	Cosines.	Tangents.	Cotangents.		
0·6283	36°00′	·5878	·8090	·7265	1·3764	54°00′	0·9425
0·6312	10	·5901	·8073	·7310	1·3680	50	0·9396
0·6341	20	·5925	·8056	·7355	1·3597	40	0·9367
0·6370	30	·5948	·8039	·7400	1·3514	30	0·9338
0·6400	40	·5972	·8021	·7445	1·3432	20	0·9308
0·6429	50	·5995	·8004	·7490	1·3351	10	0·9279
0·6458	37°00′	·6018	·7986	·7536	1·3270	53°00′	0·9250
0·6487	10	·6041	·7969	·7581	1·3190	50	0·9221
0·6516	20	·6065	·7951	·7627	1·3111	40	0·9192
0·6545	30	·6088	·7934	·7673	1·3032	30	0·9163
0·6574	40	·6111	·7916	·7720	1·2954	20	0·9134
0·6603	50	·6134	·7898	·7766	1·2876	10	0·9105
0·6632	38°00′	·6157	·7880	·7813	1·2799	52°00′	0·9076
0·6661	10	·6180	·7862	·7860	1·2723	50	0·9047
0·6690	20	·6202	·7844	·7907	1·2647	40	0·9018
0·6720	30	·6225	·7826	·7954	1·2572	30	0·8988
0·6749	40	·6248	·7808	·8002	1·2497	20	0·8959
0·6778	50	·6271	·7790	·8050	1·2423	10	0·8930
0·6807	39°00′	·6293	·7771	·8098	1·2349	51°00′	0·8901
0·6836	10	·6316	·7753	·8146	1·2276	50	0·8872
0·6865	20	·6338	·7735	·8195	1·2203	40	0·8843
0·6894	30	·6361	·7716	·8243	1·2131	30	0·8814
0·6923	40	·6383	·7698	·8292	1·2059	20	0·8785
0·6952	50	·6406	·7679	·8342	1·1988	10	0·8756
0·6981	40°00′	·6428	·7660	·8391	1·1918	50°00′	0·8727
0·7010	10	·6450	·7642	·8441	1·1847	50	0·8698
0·7039	20	·6472	·7623	·8491	1·1778	40	0·8668
0·7069	30	·6494	·7604	·8541	1·1708	30	0·8639
0·7098	40	·6517	·7585	·8591	1·1640	20	0·8610
0·7127	50	·6539	·7566	·8642	1·1571	10	0·8581
0·7156	41°00′	·6561	·7547	·8693	1·1504	49°00′	0·8552
0·7185	10	·6583	·7528	·8744	1·1436	50	0·8523
0·7214	20	·6604	·7509	·8796	1·1369	40	0·8494
0·7243	30	·6626	·7490	·8847	1·1303	30	0·8465
0·7272	40	·6648	·7470	·8899	1·1237	20	0·8436
0·7301	50	·6670	·7451	·8952	1·1171	10	0·8407
0·7330	42°00′	·6691	·7431	·9004	1·1106	48°00′	0·8378
0·7359	10	·6713	·7412	·9057	1·1041	50	0·8348
0·7389	20	·6734	·7392	·9110	1·0977	40	0·8319
0·7418	30	·6756	·7373	·9163	1·0913	30	0·8290
0·7447	40	·6777	·7353	·9217	1·0850	20	0·8261
0·7476	50	·6799	·7333	·9271	1·0786	10	0·8232
0·7505	43°00′	·6820	·7314	·9325	1·0724	47°00′	0·82[illegible]3
0·7534	10	·6841	·7294	·9380	1·0661	50	0·8174
0·7563	20	·6862	·7274	·9435	1·0599	40	0·8145
0·7592	30	·6884	·7254	·94[illegible]0	1·0538	30	0·8116
0·7621	40	·6905	·7234	·9545	1·0477	20	0·8[illegible]87
0·7650	50	·6926	·7214	·9601	1·0416	10	0·8058
0·7679	44°00′	·6947	·7193	·9657	1·0355	46°00′	0·80[illegible]9
0·7709	10	·6967	·7173	·9713	1·0295	50	0·7999
0·7738	20	·6988	·7153	·9770	1·0235	40	0·7970
0·7767	30	·7009	·7133	·9827	1·0176	30	0·7941
0·7796	40	·7030	·7112	·9884	1·0117	20	0·7912
0·7825	50	·7050	·7092	·9942	1·0058	10	0·7883
0·7854	45°00′	·7071	·7071	1·0000	1·0000	45°00′	0·7854
		Cosines.	Sines.	Cotangents.	Tangents.	Degrees.	Radians.

TABLE IV

HYPERBOLIC FUNCTIONS

a.	sinh a.	cosh a.	tanh a.	coth a.	a.	sinh a.	cosh a.	tanh a.	coth a.
0·00	0·00000	1·00000	0·00000	∞	0·50	0·52110	1·12763	0·46212	2·1640
·01	·01000	·00005	·01000	100·003	·51	·53240	·13289	·46995	·1279
·02	·02000	·00020	·02000	50·007	·52	·54375	·13827	·47770	·0934
·03	·03000	·00045	·02999	33·343	·53	·55516	·14377	·48538	·0602
·04	·04001	·00080	·03998	25·013	·54	·56663	·14938	·49299	·0284
0·05	0·05002	1·00125	0·04996	20·017	0·55	0·57815	1·15510	0·50052	1·9979
·06	·06004	·00180	·05993	16·687	·56	·58973	·16094	·50798	·9686
·07	·07006	·00245	·06989	14·309	·57	·60137	·16690	·51536	·9404
·08	·08009	·00320	·07983	12·527	·58	·61307	·17297	·52267	·9133
·09	·09012	·00405	·08976	11·141	·59	·62483	·17916	·52990	·8872
0·10	0·10017	1·00500	0·09967	10·0333	0·60	0·63665	1·18547	0·53705	1·8620
·11	·11022	·00606	·10956	9·1275	·61	·64854	·19189	·54413	·8378
·12	·12029	·00721	·11943	8·3733	·62	·66049	·19844	·55113	·8145
·13	·13037	·00846	·12927	7·7356	·63	·67251	·20510	·55805	·7919
·14	·14046	·00982	·13909	7·1895	·64	·68459	·21189	·56490	·7702
0·15	0·15056	1·01127	0·14889	6·7166	0·65	0·69675	1·21879	0·57167	1·7493
·16	·16068	·01283	·15865	6·3032	·66	·70897	·22582	·57836	·7290
·17	·17082	·01448	·16838	5·9389	·67	·72126	·23297	·58498	·7095
·18	·18097	·01624	·17808	5·6154	·68	·73363	·24025	·59152	·6906
·19	·19115	·01810	·18775	5·3263	·69	·74607	·24765	·59798	·6723
0·20	0·20134	1·02007	0·19738	5·0665	0·70	0·75858	1·25517	0·60437	1·6546
·21	·21155	·02213	·20697	4·8317	·71	·77117	·26282	·61068	·6375
·22	·22178	·02430	·21652	4·6186	·72	·78384	·27059	·61691	·6210
·23	·23203	·02657	·22603	4·4242	·73	·79659	·27849	·62307	·6050
·24	·24231	·02894	·23550	4·2464	·74	·80941	·28652	·62915	·5895
0·25	0·25261	1·03141	0·24492	4·0830	0·75	0·82232	1·29468	0·63515	1·5744
·26	·26294	·03399	·25430	3·9324	·76	·83530	·30297	·64108	·5599
·27	·27329	·03667	·26362	3·7933	·77	·84838	·31139	·64693	·5458
·28	·28367	·03946	·27291	3·6643	·78	·86153	·31994	·65271	·5321
·29	·29408	·04235	·28213	3·5444	·79	·87478	·32862	·65841	·5188
0·30	0·30452	1·04534	0·29131	3·4327	0·80	0·88811	1·33743	0·66404	1·5059
·31	·31499	·04844	·30044	·3285	·81	·90152	·34638	·66959	·4935
·32	·32549	·05164	·30951	·2309	·82	·91503	·35547	·67507	·4813
·33	·33602	·05495	·31852	·1395	·83	·92863	·36468	·68048	·4696
·34	·34659	·05836	·32748	·0536	·84	·94233	·37404	·68581	·4581
0·35	0·35719	1·06188	0·33638	2·9729	0·85	0·95612	1·38353	0·69107	1·4470
·36	·36783	·06550	·34521	·8968	·86	·97000	·39316	·69626	·4362
·37	·37850	·06923	·35399	·8249	·87	·98398	·40293	·70137	·4258
·38	·38921	·07307	·36271	·7570	·88	·99806	·41284	·70642	·4156
·39	·39996	·07702	·37136	·6928	·89	1·01224	·42289	·71139	·4057
0·40	0·41075	1·08107	0·37995	2·6319	0·90	1·02652	1·43309	0·71630	1·3961
·41	·42158	·08523	·38847	·5742	·91	·04090	·44342	·72113	·3867
·42	·43246	·08950	·39693	·5193	·92	·05539	·45390	·72590	·3776
·43	·44337	·09388	·40532	·4672	·93	·06998	·46453	·73059	·3687
·44	·45434	·09837	·41364	·4175	·94	·08468	47530	·73522	·3601
0·45	0·46534	1·102970	0·42190	2·3702	0·95	1·09948	1·48623	0·73978	1·3517
·46	·47640	·10768	·43008	·3251	·96	·11440	·49729	·74428	·3436
·47	·48750	·11250	·43820	·2821	·97	·12943	·50851	·74870	·3356
·48	·49865	·11743	·44624	·2409	·98	·14457	·51988	·75307	·3279
·49	·50984	·12247	·45422	·2016	·99	·15983	·53141	·75736	·3204
0·50	0·52110	1·12763	0·46212	2·1640	1·00	1·17520	1·54308	0·76159	1·3130

HYPERBOLIC FUNCTIONS

a.	sinh a.	cosh a.	tanh a.	coth a.	a.	sinh a.	cosh a.	tanh a.	coth a.
1·00	1·17520	1·54308	0·76159	1·3130	1·50	2·12928	2·35241	0·90515	1·1048
·01	·19069	·55491	·76576	·3059	·51	·15291	·37382	·90694	·1026
·02	·20630	·56689	·76987	·2989	·52	·17676	·39547	·90870	·1005
·03	·22203	·57904	·77391	·2921	·53	·20082	·41736	·91042	·0984
·04	·23788	·59134	·77789	·2855	·54	·22510	·43949	·91212	·0963
1·05	1·25386	1·60379	0·78181	1·2791	1·55	2·24961	2·46186	0·91379	1·0943
·06	·26996	·61641	·78566	·2728	·56	·27434	·48448	·91542	·0924
·07	·28619	·62919	·78946	·2667	·57	·29930	·50735	·91703	·0905
·08	·30254	·64214	·79320	·2607	·58	·32449	·53047	·91860	·0886
·09	·31903	·65525	·79688	·2549	·59	·34991	·55384	·92015	·0868
1·10	1·33565	1·66852	0·80050	1·2492	1·60	2·37557	2·57746	0·92167	1·0850
·11	·35240	·68196	·80406	·2437	·61	·40146	·60135	·92316	·0832
·12	·36929	·69557	·80757	·2383	·62	·42760	·62549	·92462	·0815
·13	·38631	·70934	·81102	·2330	·63	·45397	·64990	·92606	·0798
·14	·40347	·72329	·81441	·2279	·64	·48059	·67457	·92747	·0782
1·15	1·42078	1·73741	0·81775	1·2229	1·65	2·50746	2·69951	0·92886	1·0766
·16	·43822	·75171	·82104	·2180	·66	·53459	·72472	·93022	·0750
·17	·45581	·76618	·82427	·2132	·67	·56196	·75021	·93155	·0735
·18	·47355	·78083	·82745	·2085	·68	·58959	·77596	·93286	·0720
·19	·49143	·79565	·83058	·2040	·69	·61748	·80200	·93415	·0705
1·20	1·50946	1·81066	0·83365	1·1995	1·70	2·64563	2·82832	0·93541	1·0691
·21	·52764	·82584	·83668	·1952	·71	·67405	·85491	·93665	·0676
·22	·54598	·84121	·83965	·1910	·72	·70273	·88180	·93786	·0663
·23	·56447	·85676	·84258	·1868	·73	·73168	·90897	·93906	·0649
·24	·58311	·87250	·84546	·1828	·74	·76091	·93643	·94023	·0636
1·25	1·60192	1·88842	0·84828	1·1789	1·75	2·79041	2·96419	0·94138	1·0623
·26	·62088	·90454	·85106	·1750	·76	·82020	·99224	·94250	·0610
·27	·64001	·92084	·85380	·1712	·77	·85026	3·02059	·94361	·0598
·28	·65930	·93734	·85648	·1676	·78	·88061	·04925	·94470	·0585
·29	·67876	·95403	·85913	·1640	·79	·91125	·07821	·94576	·0574
1·30	1·69838	1·97091	0·86172	1·1605	1·80	2·94217	3·10747	0·94681	1·0562
·31	·71818	·98800	·86428	·1570	·81	·97340	·13705	·94783	·0550
·32	·73814	2·00528	·86678	·1537	·82	3·00492	·16694	·94884	·0539
·33	·75828	·02276	·86925	·1504	·83	·03674	·19715	·94983	·0528
·34	·77860	·04044	·87167	·1472	·84	·06886	·22768	·95080	·0518
1·35	1·79909	2·05833	0·87405	1·1441	1·85	3·10129	3·25853	0·95175	1·0507
·36	·81977	·07643	·87639	·1410	·86	·13403	·28970	·95268	·0497
·37	·84062	·09473	·87869	·1381	·87	·16709	·32121	·95359	·0487
·38	·86166	·11324	·88095	·1351	·88	·20046	·35305	·95449	·0477
·39	·88289	·13196	·88317	·1323	·89	·23415	·38522	·95537	·0467
1·40	·190430	2·15090	0·88535	1·1295	1·90	3·26816	3·41773	0·95624	1·0458
·41	·92591	·17005	·88749	·1268	·91	·30250	·45058	·95709	·0448
·42	·94770	·18942	·88960	·1241	·92	·33718	·48378	·95792	·0439
·43	·96970	·20900	·89167	·1215	·93	·37218	·51733	·95873	·043[illegible]
·44	·99188	·22881	·89370	·1189	·94	·40752	·55123	·9[illegible]953	·0422
1·45	2·01427	2·24884	0·89569	1·1165	1·95	3·44321	3·58548	0·96032	1·0413
·46	·03686	·26910	·89765	·1140	·96	·47923	·62009	·961[illegible]9	·04[illegible]5
·47	·05965	·28958	·89958	·1116	·97	·51561	·65507	·96185	·03[illegible]7
·48	·08265	·31029	·90147	·1093	·98	·55234	·69041	·96259	·0389
·49	·10586	·33123	·90332	·1070	·99	·58942	·72611	·96331	·0381
1·50	2·12928	2·35241	0·90515	1·1048	2·00	3·62686	3·76220	0·964[illegible]3	1·0373

HYPERBOLIC FUNCTIONS

a.	sinh *a*.	cosh *a*.	tanh *a*.	coth *a*.	*a*.	sinh *a*.	cosh *a*.	tanh *a*.	coth *a*.
2·00	3·62686	3·76220	0·96403	1·0373	2·50	6·05020	6·13229	0·98661	1·0136
·01	·66466	·79865	·96473	·0366	·51	·11183	·19310	·98688	0133
·02	·70283	·83549	·96541	·0358	·52	·17407	·25453	·98714	·0130
·03	·74138	·87271	·96609	·0351	·53	·23692	·31658	·98739	·0128
·04	·78029	·91032	·96675	·0344	·54	·30040	·37927	·98764	·0125
2·05	3·81958	3·94832	0·96740	1·0337	2·55	6·36451	6·44259	0·98788	1·0123
·06	·85926	·98671	·96803	·0330	·56	·42926	·50656	·98812	·0120
·07	·89932	4·02550	·96865	·0324	·57	·49464	·57118	·98835	·0118
·08	·93977	·06470	·96926	·0317	·58	·56068	·63646	·98858	·0115
·09	·98061	·10430	·96986	·0311	·59	·62738	·70240	·98881	·0113
2·10	4·02186	4·14431	0·97045	1·0304	2·60	6·69473	6·76901	0·98903	1·0111
·11	·06350	·18474	·97103	·0298	·61	·76276	·83629	·98924	·0109
·12	·10555	·22558	·97159	·0292	·62	·83146	·90426	·98946	·0107
·13	·14801	·26685	·97215	·0286	·63	·90085	·97292	·98966	·0104
·14	·19089	·30855	·97269	·0281	·64	·97092	7·04228	·98987	·0102
2·15	4·23419	4·35067	·097323	1·0275	2·65	7·04169	7·11234	0·99007	1·0100
·16	·27791	·39323	·97375	·0270	·66	·11317	·18312	·99026	·0098
·17	·32205	·43623	·97426	·0264	·67	·18536	·25461	·99045	·0096
·18	·36663	·47967	·97477	·0259	·68	·25827	·32683	·99064	·0094
·19	·41165	·52356	·97526	·0254	·69	·33190	·39978	·99083	·0093
2·20	4·45711	4·56791	0·97574	1·0249	2·70	7·40626	7·47347	0·99101	1·0091
·21	·50301	·61271	·97622	·0244	·71	·48137	·54791	·99118	·0089
·22	·54936	·65797	·97668	·0239	·72	·55722	·62310	·99136	·0087
·23	·59617	·70370	·97714	·0234	·73	·63383	·69905	·99153	·0085
·24	·64344	·74989	·97759	·0229	·74	·71121	·77578	·99170	·0084
2·25	4·69117	4·79657	0·97803	1·0225	2·75	7·78935	7·85328	0·99186	1·0082
·26	·73937	·84372	·97846	·0220	·76	·86828	·93157	·99202	·0080
·27	·78804	·89136	·97888	·0216	·77	·94799	8·01065	·99218	·0079
·28	·83720	·93948	·97929	·0211	·78	8·02849	·09053	·99233	·0077
·29	·88684	·98810	·97970	·0207	·79	·10980	·17122	·99248	·0076
2·30	4·93696	5·03722	0·98010	1·0203	2·80	8·19192	8·25273	0·99263	1·0074
·31	·98758	·08684	·98049	·0199	·81	·27486	·33506	·99278	·0073
·32	5·03870	·13697	·98087	·0195	·82	·35862	·41823	·99292	·0071
·33	·09032	·18762	·98124	·0191	·83	·44322	·50224	·99306	·0070
·34	·14245	·23878	·98161	·0187	·84	·52867	·58710	·99320	·0069
2·35	5·19510	5·29047	0·98197	1·0184	2·85	8·61497	8·67281	0·99333	1·0067
·36	·24827	·34269	·98233	·0180	·86	·70213	·75940	·99346	·0066
·37	·30196	·39544	·98267	·0176	·87	·79016	·84686	·99359	·0065
·38	·35618	·44873	·98301	·0173	·88	·87907	·93520	·99372	·0063
·39	·41093	·50256	·98335	·0169	·89	·96887	9·02444	·99384	·0062
2·40	5·46623	5·55695	0·98367	1·0166	2·90	9·05956	9·11458	0·99396	1·0061
·41	·52207	·61189	·98400	·0163	·91	·15116	·20564	·99408	·0060
·42	·57847	·66739	·98431	·0159	·92	·24368	·29761	·99420	·0058
·43	·63542	·72346	·98462	·0156	·93	·33712	·39051	·99431	·0057
·44	·69294	·78010	·98492	·0153	·94	·43149	·48436	·99443	·0056
2·45	5·75103	5·83732	0·98522	1·0150	2·95	9·52681	9·57915	0·99454	1·0055
·46	·80969	·89512	·98551	·0147	·96	·62308	·67490	·99464	·0054
·47	·86893	·95352	·98579	·0144	·97	·72031	·77161	·99475	·0053
·48	·92876	6·01250	·98607	·0141	·98	·81851	·86930	·99485	·0052
·49	·98918	·07209	·98635	·0138	·99	·91770	·96798	·99496	·0051
2·50	6·05020	6·13229	0·98661	1·0136	3·00	10·01787	10·06766	0·99505	1·0050

HYPERBOLIC FUNCTIONS

a.	sinh *a*.	cosh *a*.	tanh *a*.	coth *a*.	*a*.	sinh *a*.	cosh *a*.	tanh *a*.	coth *a*.
3·0	10·0179	10·0677	0·99505	1·0050	4·0	27·2899	27·3082	0·99933	1·0007
·1	11·0765	11·1215	·99595	·0041	·1	30·1619	30·1784	·99945	·0005
·2	12·2459	12·2866	·99668	·0033	·2	33·3357	33·3507	·99955	·0004
3	13·5379	13·5748	·99728	·0027	·3	36·8431	36·8567	·99963	·0004
·4	14·9654	14·9987	·99777	·0022	·4	40·7193	40·7316	·99970	·0003
3·5	16·5426	16·5728	0·99818	1·0018	4·5	45·0030	45·0141	0·99975	1·0002
·6	18·2855	18·3128	·99851	·0015	·6	49·7371	49·7472	·99980	·0002
·7	20·2113	20·2360	·99878	·0012	·7	54·9690	54·9781	·99983	·0002
·8	22·3394	22·3618	·99900	·0010	·8	60·7511	60·7593	·99986	·0001
·9	24·6911	24·7113	·99918	·0008	·9	67·1412	67·1486	·99989	·0001
4·0	27·2899	27·3082	0·99933	1·0007	5·0	74·2032	74·2099	0·99991	1·0001

TABLE V

EXPONENTIAL FUNCTION

x	e^x	e^{-x}	x	e^x	e^{-x}
0·00	1·0000	1·000000	0·50	1·6487	0·606531
·01	·0101	0·990050	·51	·6653	·600496
·02	·0202	·980199	·52	·6820	·594521
·03	·0305	·970446	·53	·6989	·588605
·04	·0408	·960789	·54	·7160	·582748
0·05	1·0513	0·951229	0·55	1·7333	0·576950
·06	·0618	·941765	·56	·7507	·571209
·07	·0725	·932394	·57	·7683	·565525
·08	·0833	·923116	·58	·7860	·559898
·09	·0942	·913931	·59	·8040	·554327
0·10	1·1052	0·904837	0·60	1·8221	0·548812
·11	·1163	·895834	·61	·8404	·543351
·12	·1275	·886920	·62	·8589	·537944
·13	·1388	·878095	·63	·8776	·532592
·14	·1503	·869358	·64	·8965	·527292
0·15	1·1618	0·860708	0·65	1·9155	0·522046
·16	·1735	·852144	·66	·9348	·516851
·17	·1853	·843665	·67	·9542	·511709
·18	·1972	·835270	·68	·9739	·506617
·19	·2092	·826959	·69	·9937	·501576
0·20	1·2214	0·818731	0·70	2·0138	0·496585
·21	·2337	·810584	·71	·0340	·491644
·22	·2461	·802519	·72	·0544	·486752
·23	·2586	·794534	·73	·0751	·481909
·24	·2712	·786628	·74	·0959	·477114
0·25	1·2840	0·778801	0·75	2·1170	0·472367
·26	·2969	·771052	·76	·1383	·467666
·27	·3100	·763379	·77	·1598	·463013
·28	·3231	·755784	·78	·1815	·458406
·29	·3364	·748264	·79	·2034	·453845
0·30	1·3499	0·740818	0·80	2·2255	0·449329
·31	·3634	·733447	·81	·2479	·444858
·32	·3771	·726149	·82	·2705	·440432
·33	·3910	·718924	·83	·2933	·436049
·34	·4049	·711770	·84	·3164	·431711
0·35	1·4191	0·704688	0·85	2·3396	0·427415
·36	·4333	·697676	·86	·3632	·423162
·37	·4477	·690734	·87	·3869	·418952
·38	·4623	·683861	·88	·4109	·414783
·39	·4770	·677057	·89	·4351	·410656
0·40	1·4918	0·670320	0·90	2·4596	0·406570
·41	·5068	·663650	·91	·4843	·402524
·42	·5220	·657047	·92	·5093	·398519
·43	·5373	·650509	·93	·5345	·394554
·44	·5527	·644036	·94	·5600	·390628
0·45	1·5683	0·637628	0·95	2·5857	0·386741
·46	·5841	·631284	·96	·6117	·382893
·47	·6000	·625002	·97	·6379	·379083
·48	·6161	·618783	·98	·6645	·375311
·49	·6323	·612626	·99	·6912	·371577
0·50	1·6487	0·606531	1·00	2·7183	0·367879

EXPONENTIAL FUNCTION

x	e^x	e^{-x}	x	e^x	e^{-x}
1·00	2·7183	0·367879	1·50	4·4817	0·223130
·01	·7456	·364219	·51	·5267	·220910
·02	·7732	·360595	·52	·5722	·218712
·03	·8011	·357007	·53	·6182	·216536
·04	·8292	·353455	·54	·6646	·214381
1·05	2·8577	0·349938	1·55	4·7115	0·212248
·06	·8864	·346456	·56	·7588	·210136
·07	·9154	·343009	·57	·8066	·208045
·08	·9447	·339596	·58	·8550	·205975
·09	·9743	·336216	·59	·9037	·203926
1·10	3·0042	0·332871	1·60	4·9530	0·201897
·11	·0344	·329559	·61	5·0028	·199888
·12	·0649	·326280	·62	·0531	·197899
·13	·0957	·323033	·63	·1039	·195930
·14	·1268	·319819	·64	·1552	·193980
1·15	3·1582	0·316637	1·65	5·2070	0·192050
·16	·1899	·313486	·66	·2593	·190139
·17	·2220	·310367	·67	·3122	·188247
·18	·2544	·307279	·68	·3656	·186374
·19	·2871	·304221	·69	·4195	·184520
1·20	3·3201	0·301194	1·70	5·4739	0·182684
·21	·3535	·298197	·71	·5290	·180866
·22	·3872	·295230	·72	·5845	·179066
·23	·4212	·292293	·73	·6407	·177284
·24	·4556	·289384	·74	·6973	·175520
1·25	3·4903	0·286505	1·75	5·7546	0·173774
·26	·5254	·283654	·76	·8124	·172045
·27	·5609	·280832	·77	·8709	·170333
·28	·5966	·278037	·78	·9299	·168638
·29	·6328	·275271	·79	·9895	·166960
1·30	3·6693	0·272532	1·80	6·0496	0·165299
·31	·7062	·269820	·81	·1104	·163654
·32	·7434	·267135	·82	·1719	·162026
·33	·7810	·264477	·83	·2339	·160414
·34	·8190	·261846	·84	·2965	·158817
1·35	3·8574	0·259240	1·85	6·3598	0·157237
·36	·8962	·256661	·86	·4237	·155673
·37	·9354	·254107	·87	·4883	·154124
·38	·9749	·251579	·88	·5535	·152590
·39	4·0149	·249075	·89	·6194	·151072
1·40	4·0552	0·246597	1·90	6·6859	0·149569
·41	·0960	·244143	·91	·7531	·148080
·42	·1371	·241714	·92	·8210	·146607
·43	·1787	·239309	·93	·8895	·145148
·44	·2207	·236928	·94	·9588	·1437[illegible]4
1·45	4·2631	0·234570	1·95	7·0287	0·142274
·46	·3060	·232236	·96	·0993	·140858
·47	·3492	·229925	·97	·1707	·139457
·48	·3929	·227638	·98	·2427	·138069
·49	·4371	·225373	·99	·31[illegible]5	·13[illegible]695
1·50	4·4817	0·223130	2·00	7·3891	0·13[illegible]335

EXPONENTIAL FUNCTION

x	e^x	e^{-x}	x	e^x	e^{-x}
2·00	7·3891	0·135335	2·50	12·182	0·082085
·01	·4633	·133989	·51	·305	·081268
·02	·5383	·132655	·52	·429	·080460
·03	·6141	·131336	·53	·554	·079659
·04	·6906	·130029	·54	·680	·078866
2·05	7·7679	0·128735	2·55	12·807	0·078082
·06	·8460	·127454	·56	·936	·077305
·07	·9248	·126186	·57	13·066	·076536
·08	8·0045	·124930	·58	·197	·075774
·09	·0849	·123687	·59	·330	·075020
2·10	8·1662	0·122456	2·60	13·464	0·074274
·11	·2482	·121238	·61	·599	·073535
·12	·3311	·120032	·62	·736	·072803
·13	·4149	·118837	·63	·874	·072078
·14	·4994	·117655	·64	14·013	·071361
2·15	8·5849	0·116484	2·65	14·154	0·070651
·16	·6711	·115325	·66	·296	·069948
·17	·7583	·114178	·67	·440	·069252
·18	·8463	·113042	·68	·585	·068563
·19	·9352	·111917	·69	·732	·067881
2·20	9·0250	0·110803	2·70	14·880	0·067206
·21	·1157	·109701	·71	15·029	·066537
·22	·2073	·108609	·72	·180	·065875
·23	·2999	·107528	·73	·333	·065219
·24	·3933	·106459	·74	·487	·064570
2·25	9·4877	0·105399	2·75	15·643	0·063928
·26	·5831	·104350	·76	·800	·063292
·27	·6794	·103312	·77	·959	·062662
·28	·7767	·102284	·78	16·119	·062039
·29	·8749	·101266	·79	·281	·061421
2·30	9·9742	0·100259	2·80	16·445	0·060810
·31	10·074	·099261	·81	·610	·060205
·32	·176	·098274	·82	·777	·059606
·33	·278	·097296	·83	·945	·059013
·34	·381	·096328	·84	17·116	·058426
2·35	10·486	0·095369	2·85	17·288	0·057844
·36	·591	·094420	·86	·462	·057269
·37	·697	·093481	·87	·637	·056699
·38	·805	·092551	·88	·814	·056135
·39	·913	·091630	·89	·993	·055576
2·40	11·023	0·090718	2·90	18·174	0·055023
·41	·134	·089815	·91	·357	·054476
·42	·246	·088922	·92	·541	·053934
·43	·359	·088037	·93	·728	·053397
·44	·473	·087161	·94	·916	·052866
2·45	11·588	0·086294	2·95	19·106	0·052340
·46	·705	·085435	·96	·298	·051819
·47	·822	·084585	·97	·492	·051303
·48	·941	·083743	·98	·688	·050793
·49	12·061	·082910	·99	·886	·050287
2·50	12·182	0·082085	3·00	20·086	0·049787

EXPONENTIAL FUNCTION

x	e^x	e^{-x}	x	e^x	e^{-x}
3·00	20·086	0·049787	3·50	33·115	0·030197
·01	·287	·049292	·51	·448	·029897
·02	·491	·048801	·52	·784	·029599
·03	·697	·048316	·53	34·124	·029305
·04	·905	·047835	·54	·467	·029013
3·05	21·115	0·047359	3·55	34·813	0·028725
·06	·328	·046888	·56	35·163	·028439
·07	·542	·046421	·57	·517	·028156
·08	·758	·045959	·58	·874	·027876
·09	·977	·045502	·59	36·234	·027598
3·10	22·198	0·045049	3·60	36·598	0·027324
·11	·421	·044601	·61	·966	·027052
·12	·646	·044157	·62	37·338	·026783
·13	·874	·043718	·63	·713	·026516
·14	23·104	·043283	·64	38·092	·026252
3·15	23·336	0·042852	3·65	38·475	0·025991
·16	·571	·042426	·66	·861	·025733
·17	·807	·042004	·67	39·252	·025476
·18	24·047	·041586	·68	·646	·025223
·19	·288	·041172	·69	40·045	·024972
3·20	24·533	0·040762	3·70	40·447	0·024724
·21	·779	·040357	·71	·854	·024478
·22	25·028	·039955	·72	41·264	·024234
·23	·280	·039557	·73	·679	·023993
·24	·534	·039164	·74	42·098	·023754
3·25	25·790	0·038774	3·75	42·521	0·023518
·26	26·050	·038388	·76	·948	·023284
·27	·311	·038006	·77	43·380	·023052
·28	·576	·037628	·78	·816	·022823
·29	·843	·037254	·79	44·256	·022596
3·30	27·113	0·036883	3·80	44·701	0·022371
·31	·385	·036516	·81	45·150	·022148
·32	·660	·036153	·82	·604	·021928
·33	·938	·035793	·83	46·063	·021710
·34	28·219	·035437	·84	·525	·021494
3·35	28·503	0·035084	3·85	46·993	0·021280
·36	·789	·034735	·86	47·465	·021068
·37	29·079	·034390	·87	·942	·020858
·38	·371	·034047	·88	48·424	·020651
·39	·666	·033709	·89	·911	·020445
3·40	29·964	0·033373	3·90	49·402	0·020242
·41	30·265	·033041	·91	·899	·020041
·42	·569	·032712	·92	50·400	·019841
·43	·877	·032387	·93	·907	·019644
·44	31·187	0·32065	·94	51·419	·019448
3·45	31·500	0·031746	3·95	51·935	0·019255
·46	·817	·031430	·96	52·457	·019063
·47	32·137	·031117	·97	·985	·018873
·48	·460	·030807	·98	53·517	·018686
·49	·786	·030501	·99	54·055	·018500
3·50	35·115	0·030197	4·00	54·598	0·018316

EXPONENTIAL FUNCTION

x	e^x	e^{-x}	x	e^x	e^{-x}
4·00	54·598	0·018316	4·50	90·017	0·011109
·01	55·147	·018133	·51	·922	·010998
·02	·701	·017953	·52	91·836	·010889
·03	56·261	·017774	·53	92·759	·010781
·04	·826	·017597	·54	93·691	·010673
4·05	57·397	0·017422	4·55	94·632	0·010567
·06	·974	·017249	·56	95·583	·010462
·07	58·557	·017077	·57	96·544	·010358
·08	59·145	·016907	·58	97·514	·010255
·09	·740	·016739	·59	98·494	·010153
4·10	60·340	0·016573	4·60	99·484	0·010052
·11	·947	·016408	·61	100·48	·009952
·12	61·559	·016245	·62	101·49	·009853
·13	62·178	·016083	·63	102·51	·009755
·14	·803	·015923	·64	103·54	·009658
4·15	63·434	0·015764	4·65	104·58	0·009562
·16	64·072	·015608	·66	105·64	·009466
·17	·715	·015452	·67	106·70	·009372
·18	65·366	·015299	·68	107·77	·009279
·19	66·023	·015146	·69	108·85	·009187
4·20	66·686	0·014996	4·70	109·95	0·009095
·21	67·357	·014846	·71	111·05	·009005
·22	68·033	·014699	·72	112·17	·008915
·23	·717	·014552	·73	113·30	·008826
·24	69·408	·014408	·74	114·43	·008739
4·25	70·105	0·014264	4·75	115·58	0·008652
·26	·810	·014122	·76	116·75	·008566
·27	71·522	·013982	·77	117·92	·008480
·28	72·240	·013843	·78	119·10	·008396
·29	·966	·013705	·79	120·30	·008312
4·30	73·700	0·013569	4·80	121·51	0·008230
·31	74·440	·013434	·81	122·73	·008148
·32	75·189	·013300	·82	123·97	·008067
·33	·944	·013168	·83	125·21	·007987
·34	76·708	·013037	·84	126·47	·007907
4·35	77·478	0·012907	4·85	127·74	0·007828
·36	78·257	·012778	·86	129·02	·007750
·37	79·044	·012651	·87	130·32	·007673
·38	79·838	·012525	·88	131·63	·007597
·39	80·640	·012401	·89	132·95	·007521
4·40	81·451	0·012277	4·90	134·29	0·007447
·41	82·269	·012155	·91	135·64	·007372
·42	83·096	·012034	·92	137·00	·007299
·43	·931	·011914	·93	138·38	·007227
·44	84·775	·011796	·94	139·77	·007155
4·45	85·627	0·011679	4·95	141·17	0·007083
·46	86·488	·011562	·96	142·59	·007013
·47	87·357	·011447	·97	144·03	·006943
·48	88·235	·011333	·98	145·47	·006874
·49	89·121	·011221	·99	146·94	·006806
4·[illegible]	90·017	0·011109	5·00	148·41	0·006738

EXPONENTIAL FUNCTION

x	e^x	e^{-x}	x	e^x	e^{-x}
5·00	148·41	0·006738	5·0	148·41	0·006738
·01	149·90	·006671	·1	164·02	·006097
·02	151·41	·006605	·2	181·27	·005517
·03	152·93	·006539	·3	200·34	·004992
·04	154·47	·006474	·4	221·41	·004517
5·05	156·02	0·006409	5·5	244·69	0·004087
·06	157·59	·006346	·6	270·43	·003698
·07	159·17	·006282	·7	298·87	·003346
·08	160·77	·006220	·8	330·30	·003028
·09	162·39	·006158	·9	365·04	·002739
5·10	164·02	0·006097	6·0	403·43	0·002479
·11	165·67	·006036	·1	445·86	·002243
·12	167·34	·005976	·2	492·75	·002029
·13	169·02	·005917	·3	544·57	·001836
·14	170·72	·005858	·4	601·85	·001662
5·15	172·43	0·005799	6·5	665·14	0·001503
·16	174·16	·005742	·6	735·10	·001360
·17	175·91	·005685	·7	812·41	·001231
·18	177·68	·005628	·8	897·85	·001114
·19	179·47	·005572	·9	992·27	·001008
5·20	181·27	0·005517	7·0	1096·6	0·000912
·21	183·09	·005462	·1	1212·0	·000825
·22	184·93	·005407	·2	1339·4	·000747
·23	186·79	·005354	·3	1480·3	·000676
·24	188·67	·005300	·4	1636·0	·000611
5·25	190·57	0·005248	7·5	1808·0	0·000553
·26	192·48	·005195	·6	1998·2	·000500
·27	194·42	·005144	·7	2208·3	·000453
·28	196·37	·005092	·8	2440·6	·000410
·29	198·34	·005042	·9	2697·3	·000371
5·30	200·34	0·004992	8·0	2981·0	0·000335
·31	202·35	·004942	·1	3294·5	·000304
·32	204·38	·004893	·2	3641·0	·000275
·33	206·44	·004844	·3	4023·9	·000249
·34	208·51	·004796	·4	4447·1	·000225
5·35	210·61	0·004748	8·5	4914·8	0·000203
·36	212·72	·004701	·6	5431·7	·000184
·37	214·86	·004654	·7	6002·9	·000167
·38	217·02	·004608	·8	6634·2	·000151
·39	219·20	·004562	·9	7332·0	·000136
5·40	221·41	0·004517	9·0	8103·1	0·000123
·41	223·63	·004472	·1	8955·3	·000112
·42	225·88	·004427	·2	9897·1	·000101
·43	228·15	·004383	·3	10938.	·000091
·44	230·44	·004339	·4	12088·	·000083
5·45	232·76	0·004296	9·5	13360·	0·000075
·46	235·10	·004254	·6	14765·	·000068
·47	237·46	·004211	·7	16318·	·000061
·48	239·85	·004169	·8	18034·	·000055
·49	242·26	·004128	·9	19930·	·000050
5·50	244·69	0·004087	10·0	22026·	0·000045

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