By substituting for δ'' its value in (56), we obtain expressions for α, β, γ depending on the sums of a sextuple integral and a triple integral, the integrations having to be performed from $-\infty$ to $+\infty$:

$$\alpha = \frac{1}{4\pi n}\iiint \frac{\left\{ m\dfrac{d}{dx}\left[\dfrac{-1}{4\pi(m+n)}\displaystyle\iiint \dfrac{X'(x''-x')+Y'(y''-y')+Z'(z''-z')}{[(x''-x')^2+(y''-y')^2+(z''-z')^2]^{3/2}}\right] dx'\,dy'\,dz' + X''\right\}}{\sqrt{[(x-x')^2+(y-y')^2+(z-z')^2]}}\, dx''\,dy''\,dz''$$

$$\beta = \frac{1}{4\pi n}\iiint \frac{\left\{ m\dfrac{d}{dy}\left[\dfrac{-1}{4\pi(m+n)}\displaystyle\iiint \dfrac{X'(x''-x')+Y'(y''-y')+Z'(z''-z')}{[(x''-x')^2+(y''-y')^2+(z''-z')^2]^{3/2}}\right] dx'\,dy'\,dz' + Y''\right\}}{\sqrt{[(x-x')^2+(y-y')^2+(z-z')^2]}}\, dx''\,dy''\,dz'' \qquad (61)$$

$$\gamma = \frac{1}{4\pi n}\iiint \frac{\left\{ m\dfrac{d}{dz}\left[\dfrac{-1}{4\pi(m+n)}\displaystyle\iiint \dfrac{X'(x''-x')+Y'(y''-y')+Z'(z''-z')}{[(x''-x')^2+(y''-y')^2+(z''-z')^2]^{3/2}}\right] dx'\,dy'\,dz' + Z''\right\}}{\sqrt{[(x-x')^2+(y-y')^2+(z-z')^2]}}\, dx''\,dy''\,dz''.$$

Lord *Kelvin* shows how to simplify these sextuple integrals, and obtains the following general solution for the displacements produced by any distribution of force through an infinite elastic solid filling all space (limits of integration as before $-\infty$ and $+\infty$):

$$\alpha = \frac{1}{24\pi n(m+n)}\iiint \left\{ \frac{2(2m+3n)X'}{\sqrt{[(x-x')^2+(y-y')^2+(z-z')^2]}} - m[(x-x')^2+(y-y')^2+(z-z')^2]\frac{d}{dx}\frac{X'(x-x')+Y'(y-y')+Z'(z-z')}{[(x-x')^2+(y-y')^2+(z-z')^2]^{3/2}}\right\} dx'\,dy'\,dz'$$

$$\beta = \frac{1}{24\pi n(m+n)}\iiint \left\{ \frac{2(2m+3n)Y'}{\sqrt{[(x-x')^2+(y-y')^2+(z-z')^2]}} - m[(x-x')^2+(y-y')^2+(z-z')^2]\frac{d}{dy}\frac{X'(x-x')+Y'(y-y')+Z'(z-z')}{[(x-x')^2+(y-y')^2+(z-z')^2]^{3/2}}\right\} dx'\,dy'\,dz' \qquad (62)$$

$$\gamma = \frac{1}{24\pi n(m+n)}\iiint \left\{ \frac{2(2m+3n)Z'}{\sqrt{[(x-x')^2+(y-y')^2+(z-z')^2]}} - m[(x-x')^2+(y-y')^2+(z-z')^2]\frac{d}{dz}\frac{X'(x-x')+Y'(y-y')+Z'(z-z')}{[(x-x')^2+(y-y')^2+(z-z')^2]^{3/2}}\right\} dx'\,dy'\,dz'.$$

This whole investigation is based upon the integration of the general equations for an infinite isotropic elastic solid: which implies that the density throughout all space shall be equal to ϱ as defined by (46).

Lord *Kelvin*'s definition of X, Y, Z as any arbitrary functions whatever of (x, y, z), either discontinuous and vanishing at all points outside some finite closed surface, or continuous and vanishing at all infinitely distant points with sufficient convergency to make the product of their resultant $R = \sqrt{(X^2+Y^2+Z^2)}$, by the distance

$$r = \sqrt{[(x-x')^2+(y-y')^2+(z-z')^2]}$$

namely Rr, converge to zero as r approaches infinity, implies that the density may vary through changes in the differential elements

$$dX/dx + dY/dy + dZ/dz = \nabla^2 W \qquad (63)$$

as shown below.

But no other changes than those in $\nabla^2 W$, the Laplacian operation on the potential can occur; and even this is chiefly at the boundaries of solid bodies. Accordingly it becomes advisable to investigate these possible changes a little more closely.

12. Geometrical and Physical Conditions which the Forces generated must satisfy.

Suppose X, Y, Z to denote the components of the forces acting on an element of the solid $dm = \varrho\, dx\, dy\, dz$, temporarily imagined to be fluid at (x, y, z), reckoned per unit of the mass. Then the difference of the pressures on the two faces $\delta y\, \delta z$ of the rectangular parallelopiped of the fluid is

$$\delta y\, \delta z\, (d\mathfrak{p}/dx)\, dx \qquad (64)$$

and this fluid element will be in equilibrium when the following equations are satisfied:

$$\begin{aligned} \delta y\, \delta z\, (d\mathfrak{p}/dx)\, \delta x - X\, \delta x\, \delta y\, \delta z &= 0 \\ \delta z\, \delta x\, (d\mathfrak{p}/dy)\, \delta y - Y\, \delta x\, \delta y\, \delta z &= 0 \qquad (65) \\ \delta x\, \delta y\, (d\mathfrak{p}/dz)\, \delta z - Z\, \delta x\, \delta y\, \delta z &= 0\,; \end{aligned}$$

which give the necessary and sufficient condition for the equilibrium of any fluid mass:

$$d\mathfrak{p}/dx = X \qquad d\mathfrak{p}/dy = Y \qquad d\mathfrak{p}/dz = Z. \qquad (66)$$

From these equations we obtain immediately

$$\begin{aligned} d\mathfrak{p} &= d\mathfrak{p}/dx \cdot dx + d\mathfrak{p}/dy \cdot dy + d\mathfrak{p}/dz \cdot dz \\ &= \varrho\,(X\,dx + Y\,dy + Z\,dz). \qquad (67) \end{aligned}$$

This equation shows that $X\,dx + Y\,dy + Z\,dz$ is the completed differential of a function $\mathfrak{p}\,(x, y, z)$ of three independent variables, or may be made so by a factor. Physically this is equivalent to concluding that the pressure in the fluid is along the lines of force, and thus a series of surfaces exists which cuts the lines of force at right angles. If the forces belong to a conservative system, say when a gravitational mass has attained a state of internal equilibrium, as in the theory of the figures of the heavenly bodies, no factor is required to render the differential complete, and we may put

$$X\,dx + Y\,dy + Z\,dz = -dV \qquad (68)$$

or by (67)

$$d\mathfrak{p} = -\varrho\, dV. \qquad (69)$$

This expression shows that the pressure $\mathfrak{p}$ is constant over the equipotential surfaces,

$$\varrho = -d\mathfrak{p}/dV \qquad (70)$$

and the density also is a function of the potential V. This condition arises when the density of the body is uniform, over the equipotential surfaces, for the distribution of force