Beyond the Laffer Curve:
using Kalman ***Observer ID*** & Bass ***Feedback Control*** Technology to Optimize Employment & Stabilize Sustainable Economic Growth!

by

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To provide hard evidence of the credibility of what follows, note that my former protégé Prof. Rudolf E. Kalman received the **National Medal of Science** from President Obama in October 2009, at which time the White House mentioned that his epochal discovery of the Kalman Filter [*aka* **Kalman Observer**] in the field of Guidance, Navigation & Control (**GNC**) had provided what has been called, by the National Academy of Engineering (**NAE**), the **"key enabling technology of the aerospace age,"** and that without Kalman's discoveries the USA would not have been able to beat the Russians to a soft manned lunar landing, nor have given humanity the GPS**!**

Just as many electrical engineers believe that "Wiener Filtering" [and its evolution into "Cybernetics"] was the biggest discovery to result from the World War II era, it has been seriously asserted that Wiener Filtering's generalization from signal-processing & information technology, such as in radar & communication systems, to arbitrary dynamical systems, by Kalman, is the most important technological innovation to have emerged from the Cold War era.

Indeed, Kalman once told me that "the reason that Kalman Filtering turned out to be more important than Wiener Filtering is because Newton is more important than Gauss!"

In truth, Kalman did for Systems Engineering what Euclid did for Geometry and what Newton did for dynamics, and it was the greatest privilege of my lifetime to have learned about Kalman's epochal discovery first-hand when he corrected a naïve assertion by me after I had returned in 1959 from 2 years of active USAF Reserve Service and commented, regarding design of a feedback control system, "if you have *n* state-variables, then you need *n* sensors."

"No Bob, that's not true!" replied Rudy Kalman. "If the system's dynamics is already known, and together with just one or a few sensors satisfies my new criterion for Observability in terms of the system's Markov Parameters, then you can, in real time, estimate adequately, for state-vector feedback control purposes, <u>all</u> of the remaining state-variables from those that <u>are</u> measured! I've been shouting that from the rooftops for the past year! Haven't you been listening?"

"But what if you don't already know the system's dynamics?" I asked. "How do you find that out?"

"That's easy," replied Rudy, who proceeded to tell me something that he had learned at MIT and of which I had never heard. "You just take the cross-correlation matrix of the Output Signal vector with that of the Input Signal vector, and divide it by the auto-correlation matrix of the Input Signal vector, and then use the Inverse Fourier Transform to get the Input/Output (**I/O**) Transfer Matrix, from which the Time-Domain Dynamical & Kinematical Coefficient Matrices can be derived!"

And within 6 years, after the immediately-recognized epochal significance of the Kalman Observer had been rewarded by Kalman's appointment to a faculty position at Stanford University, Rudy and his graduate student B.L. Ho published another epochal contribution to the science of **Empirical System Identification** (**ESID**), in which they showed how to use the matrix ***U*** whose columns are the digital time-histories, in discrete time, of each of the system's measured Inputs, and the matrix ***Y*** whose columns are the discrete-time digital time-histories of each of the system's measured Outputs, to derive, by strictly linear algebra alone, the profound Ho-Kalman Algorithm, from which the coefficient matrices just mentioned are found in one fell swoop!

Specifically, let the positive integers k, ($k = 1, 2, 3, \ldots, N$), label the **epochs** of discrete or digital time at which the dynamical system under consideration may be sampled & modeled, at least near to any given dynamical state, as a Linear Time-Invariant (LTI) system, in terms of a matrix quartet ($\boldsymbol{A},\boldsymbol{B},\boldsymbol{C},\boldsymbol{D}$), as follows, assuming quiescence at the initial time $k = 1$.

Consider an "arbitrary Black Box" into which measured Input signals may be sequentially inserted, and from which measured Output signals may be, subsequently, sequentially extracted.

One desires to model the "unknown dynamics" inside of the Black Box sufficiently precisely that, when new and hitherto untried signals are inserted, then the resultant outputs can be forecasted in advance with sufficient precision as to render the model of significant practical utility.

Let n denote the initially unknown number of "hidden" state-variables, so that what follows may be repeated with various trial values of n, and the "best" value found in pragmatic terms of what choice of n yields the smallest [appropriately weighted] residual prediction-error, as will be explained later.

Of course one already knows the number $\boldsymbol{\ell}$ [lower-case L] of measured outputs, and the number m of measured inputs, and it will turn out that the matrices sought all have their sizes defined in terms of the integer-triad ($\boldsymbol{\ell}$,m,n).

In fact, we begin by defining a column $\boldsymbol{\ell}$-vector $\boldsymbol{y} = \boldsymbol{y}(k)$ at each time-epoch k as the system's output, by stacking each of the $\boldsymbol{\ell}$ measured outputs, and similarly we define a column m-vector $\boldsymbol{u} = \boldsymbol{u}(k)$ as the system's input at time k, by stacking each of the m measured inputs, which taken together are hoped to define a linear system $\mathbb{S}$ such that $\{\, \boldsymbol{y}(k) \mid (\, k = 1, 2, 3, \ldots, N \,)\, \} = \mathbb{S}\{\, \boldsymbol{x}(k) \mid (\, k = 1, 2, 3, \ldots, N \,)\, \}$.

Indeed, in terms of the initially unknown n-dimensional column-vectors $\boldsymbol{x} = \boldsymbol{x}(k)$, or **hidden states** $\{\boldsymbol{x}(k) \mid k = 1, 2, 3, \ldots, N\}$, the LTI system $\mathbb{S}$ is postulated to **evolve in discrete time** according to

$$\boldsymbol{x}(k{+}1) \;=\; \boldsymbol{A}\cdot\boldsymbol{x}(k) \;+\; \boldsymbol{B}\cdot\boldsymbol{u}(k), \quad \boldsymbol{x}(1) = 0, \quad (\, k = 1, 2, 3, \ldots, N \,),$$

$$\boldsymbol{y}(k) \;=\; \boldsymbol{C}\cdot\boldsymbol{x}(k) \;+\; \boldsymbol{D}\cdot\boldsymbol{u}(k),$$

where the real-valued constant matrices ($\boldsymbol{A},\boldsymbol{B},\boldsymbol{C},\boldsymbol{D}$) are respectively of dimensions $n\times n$, $n\times m$, $\boldsymbol{\ell}\times n$, $\boldsymbol{\ell}\times m$.

The **ESID** problem is**:** given the $N\times m$ Input-data matrix $\boldsymbol{U} = [\, \boldsymbol{u}(1)^{\mathrm{T}}, \boldsymbol{u}(2)^{\mathrm{T}}, \ldots, \boldsymbol{u}(N)^{\mathrm{T}} \,]^{\mathrm{T}}$, and the $N\times\boldsymbol{\ell}$ Output-data matrix $\boldsymbol{Y} = [\, \boldsymbol{y}(1)^{\mathrm{T}}, \boldsymbol{y}(2)^{\mathrm{T}}, \ldots, \boldsymbol{y}(N)^{\mathrm{T}} \,]^{\mathrm{T}}$, where $^{\mathrm{T}}$ denotes the operation of vector-matrix row-column transposition, or, more briefly, given the I/O-data ***pair*** ($\boldsymbol{U}$,$\boldsymbol{Y}$), find the LTI dynamical-coefficient ***quartet*** ($\boldsymbol{A}$,$\boldsymbol{B}$,$\boldsymbol{C}$,$\boldsymbol{D}$).

The profound Ho-Kalman Algorithm defines, by purely Linear Algebra procedures, a System ID function $\boldsymbol{\mathcal{L}}$ such that ($\boldsymbol{A},\boldsymbol{B},\boldsymbol{C},\boldsymbol{D}$) = $\boldsymbol{\mathcal{L}}(\boldsymbol{U},\boldsymbol{Y})$.

And I flatter myself that I have derived the most numerically robust MATLAB implementation of $\boldsymbol{\mathcal{L}}$ as the result of a 2005 Purchase Order from DARPA, whose Final Report may be found on my website www.innoventek.com and whose .*m*-file program-suite I will happily forward *gratis*, via a zip-file attachment, to any Licensed MATLAB user, upon request.

I will now show how to perfect the concept of the Laffer Curve in three steps, the second & third of which use other profound discoveries of Kalman which have not been mentioned so far, but will be explained later, in contest.

The first step is to use the Ho-Kalman algorithm to make an Empirical Identification of the macro-economic LTI model which best predicts the National Economy, using at least about 8 years of weekly economic data.

The second step is to use the Bass Formula to find an Optimal Observer Gain matrix for implementation of a Kalman Observer which will optimally estimate all hidden state-variables.

The third step is to use the Bass Formula to find an Optimal Controller Feedback Gain matrix for implementing weekly or monthly changes in those variables which the government can control, such as Federal Reserve Interest Rates, and various kinds of [Sales & (graduated) Income & Capital Gains] Tax Rates and Tariff Rates, etc.

The result will be a **scientific macro-economic approach** to minimizing inflation, maximizing employment, and stabilizing growth at a sustainable rate!

Firstly one defines the m components of the m-dimensional column-vector $\boldsymbol{u}_{\mathbf{i}}$, ($i = 1, 2, \ldots, m$), in terms of economic time variables which are **exogenous** inputs in the following sense. Either they are

(i) ***geophysical*** variables well-defined for many epochs in advance, such as the gravitational/electromagnetic [1] phases of the moon, which Robert D. Taylor [2] has proved beyond dispute, via 36 consecutive months of bi-weekly "up"/"down" predictions of the DJIA (or S&P 500) direction with better than 90% accuracy[!!], affect mass-psychology optimism/pessimism via stimulation of the human hormones melatonin & serotonin [3] and have an observable effect upon public-auction-market prices, or

(ii) ***demographic*** variables such as the birth-rate 25 years ago & the birth-rate 57 years ago (whose appropriately-weighted combination Harry S. Dent, Jr. [4] has demonstrated can be added together to produce a predictive curve whose maxima & maxima coincide EXACTLY with the DJIA maxima & maxima for the past century!), or else

(iii) they can be affected by **governmental** actions, such as the Federal Reserve Interest Rate, or various kinds of Internal Tax Rates, External Tariff Rates, or what percentage of the *GNP* is government spending for procurement, such as by DoD or NASA, etc..

Secondly, one defines the ℓ components of the ℓ–dimensional column-vector $\boldsymbol{y}_{\mathbf{i}}$, ($i = 1, 2, \ldots, \ell$), in terms of economic time-variables which are to be regarded as outputs in the sense that it is conjectured that changes in the exogenous inputs may affect the evolution in time of the chosen outputs, such as (*i*) the domestic gross national product (*GNP*), and (*ii*) the [normalized] percentage *GNP* **rate** of growth or recession (*GR*), $-1 < GR < 1$, and (*iii*) the percentage Inflation Rate (*IR*), and (*iv*) the percentage employment rate E, where $0 < E <$!, etc.

One must also choose the sampling-epoch duration, i.e. whether the data is sampled daily, weekly, monthly, or annually, and the total length N of the I/O data-sample sets.

In order to speak more concretely of a possible numerical example, suppose that we have selected $m = 7$ exogenous variables, 3 of which are beyond our control, but 4 of which the government can in fact alter at will.

And suppose that we are mainly interested in only $\ell = 13$ outputs. But we don't know what the number n of hidden variables will be, so we decide to try successively & sequentially a RANGE of state-vector dimensions n, $n_{\min} \leq n \leq n_{\max}$, where necessarily $n_{\min} \geq \max(\ell,m) = \max(7,13) = 13$. So we decide to use $\mathcal{L}$ with every trial-integer n in the interval [13,77], where $n_{\max} = 77$ is a guesstimate in hopes that $n < 100$ because use of 100X100 matrices in numerical multiplications is starting to strain the computational memory requirements of today's desktop PCs, though it is publicly known that the Kalman Filter in the GNC system of the USA's Peacekeeper ICBMs does exceed $n = 100$.

Then we select $N = 371$ weeks of I/O data, or ~7.13 years of economic time-series data, which will necessarily include at least 2 Presidential elections and public debates about tax rates, etc.

And for each n we compute $\{\|\boldsymbol{Y}_{\text{pred}} - \boldsymbol{Y}\|/\|\boldsymbol{Y}\|\}$ or the [normalized] norm of the difference-matrix between the actual Output-data-matrix $\boldsymbol{Y}$ and the result, $\boldsymbol{Y}_{\text{pred}}$, of using the identified quartet ($\boldsymbol{A},\boldsymbol{B},\boldsymbol{C},\boldsymbol{D}$) together with ONLY the Input data-set $\boldsymbol{U}$, in the above-displayed LTI system, to PREDICT what would be the Output-data-matrix $\boldsymbol{Y}_{\text{pred}}$.

But it is not good enough to merely choose the n which minimizes the prediction-error norm, because of subtle statistical & information-theoretic considerations which will be skipped over here in a merely metaphorical allusion to an arguably over-simplified way of weighting the prediction error in terms of the ***Akaike Information Criterion*** (**AIC**).

I choose to use what I call "the poor man's AIC," which actually is asymptotically correct for sufficiently large data sets, and which can be explained as follows.

At each trial n, define the weighting factor AIC = $(N + n)/(N - n)$, which is necessarily greater than unity and self-evidently "penalizes" increases in n.

Then select as the "best estimate" of the number of hidden variables that trial value of n which minimizes the prediction-error norm weighted by the AIC factor.

In our hypothetical example, we use $N = 371$ and it turns out that $n_{\text{optimal}} = 73$, with AIC = 1.49, so we had barely squeaked through in allowing $n_{\text{nax}} = 77$ to be large enough!

But now we have an Empirically Identified dynamical model of the national economy!

Or do we?

In principle we should be able to solve the above-displayed system for <u>constant</u> or "steady-state" equilibrium values (by elimination of $\boldsymbol{x}(k+1) \equiv \boldsymbol{x}(k) = \boldsymbol{x}_{\text{equilibrium}}$, to find the optimal values of the inputs which the government can control).

In fact, if $\boldsymbol{y}_{\text{opt}}$ defines the <u>desired</u> output-vector and if $\boldsymbol{I}_{\text{n}}$ denotes the nXn identity matrix, then set $\boldsymbol{M} \equiv \boldsymbol{C}\cdot(\boldsymbol{I}_{\text{n}} - \boldsymbol{A})^{-1}\cdot\boldsymbol{B}$, and the optimal input-command vector becomes just $\boldsymbol{u}_{\text{opt}} = (\boldsymbol{M}^{\text{T}}\cdot\boldsymbol{M})^{-1}\cdot\boldsymbol{M}^{\text{T}}\cdot\boldsymbol{y}_{\text{opt}}$ which results in the <u>optimal</u> equilibrium state $\boldsymbol{x}_{\text{opt}} \equiv (\boldsymbol{I}_{\text{n}} - \boldsymbol{A})^{-1}\cdot\boldsymbol{B}\cdot(\boldsymbol{M}^{\text{T}}\cdot\boldsymbol{M})^{-1}\cdot\boldsymbol{M}^{\text{T}}\cdot\boldsymbol{y}_{\text{opt}} \equiv \boldsymbol{x}_{\text{equilibrium}}$.

But wait a minute! Didn't we admit that there are certain demographic & geophysical factors which are exogenous inputs but which CANNOT be controlled?

It turns out that Kalman was smart enough to have solved that problem also, in the late 1950s and early 1960s, though it has taken me 50 years to appreciate how profound his Minimal Realization Theory is!

In fact, Kalman showed how to separate any LTI system into four parts, giving a part which is **both** controllable & observable, and a part which is observable but not controllable, and a part which is controllable but not observable, and a part which is neither controllable nor observable.

In the present context, what this boils down to is that we can model the uncontrollable geophysical & demographic variables as ***separate <u>autonomous</u> dynamical systems***, and then incorporate them internally to an "enlarged" dynamical coefficient matrix $\boldsymbol{A}$, in which the above-displayed LTI system has the same outputs $\{\boldsymbol{y}(k)\}$ as before but in which the inputs are now "reduced" to only those variables which are strictly under potential governmental-policy command-control..

The next question which arises is how, gently, unobtrusively, and with minimal unanticipated side-effects, to nudge the present values of the Input-Command variables toward their optimal values, which turns out to be simply a problem in feedback control-system design theory.

So make a linear change of coordinates in which we replace the state $\boldsymbol{x}$ in the above-displayed system by $(\boldsymbol{x} - \boldsymbol{x}_{\text{optimal}})$, so that the problem is how to choose the input-variables $\boldsymbol{u}_{\text{i}}(k)$ so as to drive the newly-redefined states to the zero state $\boldsymbol{x} = 0$. (This coordinate change can be implemented only if we can find an "offset" command-vector $\boldsymbol{u}_{\text{off}}$ such that $\boldsymbol{B}\cdot\boldsymbol{u}_{\text{off}} = -\boldsymbol{A}\cdot\boldsymbol{x}_{\text{optimal}}$, and then replace $\boldsymbol{u}$ by $(\boldsymbol{u} + \boldsymbol{u}_{\text{off}})$.)

According to the "Guidance/Navigation Separation Theorem," of Stochastic Optimal Control Theory, the best way to proceed is to first choose an nXℓ feedback gain matrix $\boldsymbol{L}$ such that $(\boldsymbol{A} - \boldsymbol{L}\cdot\boldsymbol{C})$ is the dynamical coefficient matrix of an asymptotically stable system, i.e. one which has all of its complex poles in the interior of the **unit circle** of the complex z-plane, namely $|z| = 1$, and, in what Kailath [5] calls the "most straightforward way" to do this, is by means of a formula that should be called the Bass Pole Placement Formula, since I derived it rigorously and presented it in widely-distributed multilithed Lecture Notes at NASA Langley in 1961, though it didn't become internationally known until it was republished by myself & Ira Gura in a joint *IEEE*-paper in 1967, and is therefore most often called the Bass-Gura Formula, though it had already so amazed & excited Rudy Kalman when I showed it to him in 1961 that he had started referring to it as **"the Fundamental Theorem of Control Theory"** in internal Martin Marietta Reports, because my pole-placement formula applies equally well to the transpose of the above matrix, i.e. $(\boldsymbol{A} - \boldsymbol{L}\cdot\boldsymbol{C})^{\text{T}} = \boldsymbol{A}^{\text{T}} - \boldsymbol{C}^{\text{T}}\cdot\boldsymbol{L}^{\text{T}}$ which by Kalman's important **Duality Principle** turns out to be, as we'll see shortly, the correct form for designing an optimal automatic feedback control system, wherein $\boldsymbol{A}^{\mathrm{T}}$ is replaced by $\boldsymbol{A}$ and $\boldsymbol{C}^{\mathrm{T}}$ is replaced by $\boldsymbol{B}$, and $\boldsymbol{L}^{\mathrm{T}}$ is replaced by the feedback gain matrix $\boldsymbol{K}$, giving a stable closed-loop dynamical coefficient matrix of the form $(\boldsymbol{A} - \boldsymbol{B}\cdot\boldsymbol{K})$.

Anyway, to make a long story short, the optimal way to control the hidden state-vector $\boldsymbol{x}$ to its desired optimal equilibrium point $\boldsymbol{x} = 0$ turns out to be achieved by implementation of two related systems, an Observer which estimates the entire state-vector $\boldsymbol{x}_{\mathrm{est}}$ from whatever variables $\boldsymbol{y}$ are actually measured or "observed," and then a Controller which forces the true hidden state-vector toward the coordinate origin $\mathbf{x} = 0$ by forcing the estimate $\boldsymbol{x}_{\mathrm{est}}$ to the origin without ignoring the all-important fact that the difference $\boldsymbol{x}_{\mathrm{diff}} = (\boldsymbol{x} - \boldsymbol{x}_{\mathrm{est}})$, between the true state $\boldsymbol{x}$ and its estimate $\boldsymbol{x}_{\mathrm{est}}$, is also automatically decreasing exponentially toward zero!

For example, we may use the Bass Formula to choose $\boldsymbol{L}$ so as to place the complex poles of the Observer dynamical coefficient matrix $(\boldsymbol{A} - \boldsymbol{L}\cdot\boldsymbol{C})$ to be equidistantly-spaced on a complex circle $|z| = \rho_{\mathrm{obs}}$ of radius $\rho_{\mathrm{obs}} < 1$, and simultaneously similarly choose $\boldsymbol{K}$ so as to place the poles of the "dual" Controller dynamical coefficient matrix $(\boldsymbol{A} - \boldsymbol{B}\cdot\boldsymbol{K})$ to be equidistantly-spaced on a complex circle $|z| = \rho_{\mathrm{ctrl}}$ of radius $\rho_{\mathrm{ctrl}} < 1$, and then optimize the Stability Robustness & Fidelity Robustness [6] of each subsystem by a one-parameter search over the relevant parameter ρ.

The result is that one ends up with the following $2\cdot n$-dimensional optimized system:

$$\boldsymbol{x}(k+1) = \boldsymbol{A}\cdot\mathbf{x}(k) + \boldsymbol{B}\cdot\boldsymbol{u}(k), \quad \boldsymbol{u}(k) = -\boldsymbol{K}\cdot\boldsymbol{x}_{\mathrm{est}} ,$$

$$\boldsymbol{y}(k) = \boldsymbol{C}\cdot\boldsymbol{x}(k) + \boldsymbol{D}\cdot\boldsymbol{u}(k),$$

$$\boldsymbol{x}_{\mathrm{est}}(k+1) = \boldsymbol{A}\cdot\boldsymbol{x}_{\mathrm{est}}(k) + \boldsymbol{B}\cdot\boldsymbol{u}(k) + \boldsymbol{L}\cdot\{\boldsymbol{y}(k) - \boldsymbol{C}\cdot \boldsymbol{x}_{\mathrm{est}}(k)\},$$

which, by subtraction and use of elementary algebra, is identical to the system

$$\boldsymbol{x}(k+1) = (\boldsymbol{A} - \boldsymbol{B}\cdot\boldsymbol{K})\cdot\boldsymbol{x}(k) - \boldsymbol{B}\cdot\boldsymbol{K}\cdot \boldsymbol{x}_{\mathrm{diff}}(k),$$

$$\boldsymbol{x}_{\mathrm{diff}}(k+1) = (\boldsymbol{A} - \boldsymbol{L}\cdot\boldsymbol{C})\cdot \boldsymbol{x}_{\mathrm{diff}}(k),$$

wherein now both $\|\mathbf{x}(k)\|$ & $\|\boldsymbol{x}_{\mathrm{diff}}(k)\|$ necessarily tend exponentially to zero as time k increases!

In conclusion, anyone who has access to at least the past 8 years of weekly economic data, and who has even the most elementary understanding of results derived by Kalman & Bass a half-century ago, can show the government how to manipulate those variables which it has the authority to manipulate in order to stabilize the national economy in an optimally feedback-controlled system to minimize inflation rate *IR*, maximize employment *E*, and simultaneously increase *GNP* & *GR* in a sustainable manner!

References

[1] It is not adequately widely appreciated that, in 1921, H.A. Wilson derived Einstein's gravitational field equations from Maxwell's electromagnetic equations, and therefore gravity is just an aspect of electromagnetism. See the Science sub-site of my website www.innoventek.com for further details.
[2] Robert D. Taylor, *Paradigm*, Savas Beatie, 2006.
[3] Robert W. Bass, unpublished Report, available upon request.
[4] Harry S. Dent, Jr., *The Roaring 2000s,* Simon & Schuster, 1998.
[5] Thomas Kailath, *Linear Systems*, Prentice-Hall, 1980.
[6] Robert W. Bass, numerous papers on *RhoSynthesis* accessible at www.innoventek.com .