ordinates and, therefore, is always affected in the same way by an electric field. If we describe the atom by parabolical coördinates,[3] we get for the normal state $\psi = A_{00} \exp [\alpha(\xi + \eta)/2]$, where $(\xi + \eta)/2 = r$, so that here again we get the same radial pulsation.

The numerical value of E_2, according to formula (25), is $E_2 = -3{,}29.10^{-25}$ cm.[3] The dielectric constant[1] is connected with E_2 by the relation $\epsilon - 1 = 8\pi NE_2$ (N, number of atoms in unit volume), leading to the value $\epsilon - 1 = 0{,}000224$.

The spin of the electron has not been considered in this theory, but it hardly can introduce any appreciable change. We completely agree with Van Vleck's opinion that this theoretical value is more reliable than the result of the extremely difficult experimental determination of the index of refraction. The fact that for helium the experimental value $\epsilon - 1 = 0{,}000074$ is very different from the above value, is not consistent with a model in which one electron is close to the nucleus, the other farther away. It can be regarded as a further confirmation of Heisenberg's theory assuming equivalent orbits for the two electrons.

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THE DIELECTRIC CONSTANT OF ATOMIC HYDROGEN FROM THE POINT OF VIEW OF BOHR'S QUANTUM THEORY

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1. The following lines represent an abridgment of my thesis presented a year ago to the University of California. Their main purport is in relation to the question *whether a hydrogen atom in an electric field acquires a definite orientation or not*, and this question is answered from the point of view of Bohr's theory. Since this investigation was begun, the quantum theory has undergone a rapid development owing to the brilliant work of Heisenberg, Born, Schroedinger and others. The numerical part of our work is superseded by this new point of view, but our general results and the whole approach of the problem remain of practical value even in the new quantum theory, as will appear from the preceding paper by Professor

P. S. Epstein. Accordingly, we present in the following sections the general lines of our reasoning, suppressing the details of calculation.

The problem of the dielectric constant of atomic hydrogen was treated a few years ago by J. H. Jones.[1] Let the expression of the energy which a hydrogen atom assumes in an electric field of the strength F be to terms of the second order

$$E = E_0 + E_1F + E_2F^2. \tag{1}$$

The coefficients E_1 and E_2 were known from Epstein's work on the Stark effect.[2] In the normal state of the atom, in particular, $E_1 = 0$ and $E_2 = h^6/2(2\pi e)^6\mu^3$, where h is Planck's constant of action, e and μ the charge and mass of the electron. It can readily be shown that the dielectric constant must satisfy the relation

$$K - 1 = -8\pi NE_2, \tag{2}$$

N being the number of atoms in unit volume of the gas. Substituting the above value of E_2, Jones obtains

$$K - 1 = 4\pi Nh^6/(2\pi e)^6\mu^3. \tag{3}$$

This formula is, however, in contradiction with the results which one gets from the extrapolation of the refractive index n, for frequencies ν approaching zero.[3] Substituting $\nu = 0$ into the formula derived for n^2 from Bohr's quantum theory, we have

$$n^2 - 1 = 11Nh^6/64\pi^6e^6\mu^3. \tag{4}$$

It is true that the foundations of the older quantum theory, on which this formula is based, were unreliable and have since been changed. It has, however, been pointed out by several physicists that the dispersion theory in question must lead to correct results for low frequencies.

The purpose of the present communication is to resolve this contradiction. From the analogy with Kramers'[4] work on the Stark effect, it seemed probable that the discrepancy was due to the omission on Jones' part of considering the relativity terms in the motion of the electron. For this reason the problem was studied in its completeness, taking into account the effect produced by the relativistic change of mass, as well as the effect due to the presence of the electric field.

2. We consider an electron, moving with the velocity v ($= \beta c$, where c is the velocity of light) around a fixed center having the charge e. The whole system is placed in a homogeneous electric field of the strength F. We choose the direction of F as the z-axis of a Cartesian system x, y, z; and, in addition to the Cartesian system, we will make use of a system of polar coördinates given by the relations.

$$x = r \sin\vartheta \cos\varphi, \qquad y = r \sin\vartheta \sin\varphi, \qquad z = r\cos\vartheta.$$

The total energy or Hamiltonian function can be written in the form

$$H = T - e^2/r - eFz, \tag{5}$$

where T is the relativistic expression for the kinetic energy

$$T = \mu c^2[1/\sqrt{1-\beta^2} - 1] = \mu v^2/2 + 3\mu v^4/8c^2 + \ldots$$

Therefore,

$$H = H_1 + 3\mu v^4/8c^2 - eFz, \tag{6}$$

where

$$H_1 = \mu v^2/2 - e^2/r \tag{7}$$

is the Hamiltonian function of the unperturbed or first intermediate motion, $3\mu v^4/8c^2$ is the *relativistic*, and $-eFz$ the *electric perturbation term*. H_1 defines the simple Kepler motion whose integral is well known. We shall require the angular variables and corresponding momenta in this motion and may refer in this respect to the formulae collected in a paper of Epstein's.[3] Instead of Epstein's variables u, w, it will be more convenient to use a new set of canonical coördinates p, q, resulting from the former by the contact transformation

$$q_1 = w_1 - w_2, \qquad q_2 = w_2 - w_3, \qquad q_3 = w_3$$
$$p_1 = u_1, \qquad p_2 = u_1 + u_2, \qquad p_3 = u_1 + u_2 + u_3.$$

The physical meaning of these coördinates is as follows: p_3 is connected with the energy of the motion by the relation

$$H_1 = \mu e^4/2p_3^2. \tag{8}$$

p_2 represents the principal moment of momentum and p_1 its projection on the z-axis. If ϵ is the eccentricity of the orbit and λ the angle between the normal to the orbit and the z-axis:

$$\sqrt{1-\epsilon^2} = p_2/p_3, \qquad \cos\lambda = p_1/p_2. \tag{9}$$

From the expression (8) for H_1 there follows, by means of the Hamiltonian equations, $\dot{q}_1 = \dot{q}_2 = 0$, $\dot{q}_3 =$ const. The coördinates q_1 and q_2, therefore, remain constant, while q_3 is a linear function of the time t.

The general rule for the quantization of degenerate systems of this type is that only the momentum on which H_1 depends must be quantized. There is, therefore, only one quantum condition imposed on the system, namely,

$$p_3 = n_3 h/2\pi. \tag{10}$$

That means that our orbit has a constant energy and a constant major axis (since the energy is uniquely determined by the major axis), but any arbitrary eccentricity and orientation.

We are now going to consider the change of this motion produced by the presence of the relativistic and electric terms, by means of the theory of perturbations. We will use this theory in the form developed by Epstein[5] (especially in section 10 of his paper), which requires an expansion of the perturbation terms in a trigonometric series. The trigonometric functions have the arguments $\tau_1 q_1 + \tau_2 q_2 + \tau_3 q_3$ (τ being integers), and we shall concentrate our attention on the so-called *degenerate* terms of this series. The degenerate terms are those which do not contain in their arguments the variables q conjugate to any of the momenta on which the Hamiltonian function depends. In our case the function H_1 depends only on p_3, so that the degenerate terms are those which are independent of q_3. Transforming the electric perturbation term by means of the above-mentioned formulae, we see that it contains only one single degenerate term.

$$R = -\frac{3F}{2\mu e}\frac{p_3}{p_2}\sqrt{p_3^2 - p_2^2}\cdot\sqrt{p_2^2 - p_1^2}\cos q_2. \qquad (11)$$

It is shown in the general theory that only the degenerate terms can lead to material change of the conditions of quantization. As the arguments of these terms are constant, while the arguments of the rest of the expansion are linear functions of time, the totality of the degenerate terms is the time average of the perturbation function taken over a long time. In this form the definition of the degenerate part of the perturbation function was given by Bohr.[6]

3. *Conditions in Strong Fields*.—It will be necessary to discuss this case, though only our method of approach is new, while the results were known from Bohr's[6] and Kramers'[4] work. If the electric perturbation dominates over the relativistic, our process of successive approximations must start from considering the electric terms. As we have pointed out, the most important one among them is the degenerate term (11). Our second step will be, therefore, the consideration of the Hamiltonian equation

$$H_2 = H_1 + R = \text{const.}, \qquad (12)$$

where H_1 and R are given by equation (8) and (11). This function is independent of q_1 and q_3, and, therefore, p_1 and p_3 will be constant. Accordingly, p_2 will be a function of q_2 alone, the functional relationship being given by (12). Since H_1 is constant, we can throw H_1 into the constant writing the equation

$$R = 3Fp_3\Delta/2\mu e, \qquad (13)$$

where Δ is the new constant.

We have now to find the angular coördinates p', q' for this motion. According to the general rules, the angle momenta are given by

$$p_1' = p_1, \qquad p_2' = \frac{1}{2\pi} \oint p_2 dq_2, \qquad p_3' = p_3.$$

p_2 depends on q_2 and Δ, and the second equation can be regarded as a relation determining Δ. Carrying out the integration, we find

$$\Delta = 2p_2' - p_1' + p_3'. \tag{14}$$

In addition to this, from the real character of the function p_2, there follows the restriction

$$|\Delta| \leq p_3' - p_1'. \tag{15}$$

The Hamiltonian function becomes

$$H_2 = \mu e^4/2p_3'^2 - p_3'\Delta \cdot 3F/2\mu e. \tag{16}$$

Instead of p_2', we can as well introduce as one of the angle momenta the linear expression Δ. From the principle mentioned in section 2, that the momenta entering into the expression of the Hamiltonian function are subject to quantization, we conclude that we have to add to condition (10) a second quantum relation

$$\Delta = nh/2\pi. \tag{17}$$

This condition combined with the restriction (15) makes our term $p_3'\Delta \cdot 3F/2\mu e$ completely equivalent to Epstein's expression of the first order Stark effect,[2] leading to exactly the same energy levels.

The geometrical significance of relation (17) amounts to a restriction imposed on the possible orientations of the orbit. It will be enough to discuss this for the normal state of the atom, in which we are most interested in connection with the dielectric constant. In the normal state $p_3 = 1$. From (15) we see that Δ can assume the values $\pm h/2\pi$ and 0. However, the cases $\pm h/2\pi$ must be ruled out, since they lead to $p_1 = 0$ and $p_2 = 0$, representing orbits that would go through the nucleus. There remains $\Delta = 0$. From (11) and (12) we see that this can be satisfied either by putting $p_1 = p_2$ or $p_2 = p_3$. The analysis of the second assumption shows, however, that it is possible only when $p_1 = p_3$, so that in every case we have $p_1 = p_2$, or from (9) $\cos \lambda = 0$, $\lambda = \pm \pi/2$. In the unperturbed motion p_1 and p_2 are completely arbitrary, so that the angle λ between the normal to the orbit and the z-axis can assume any value. If we superpose a strong electric field, the normal can be only parallel or antiparallel to the direction of the field (z-axis). On the other hand, p_2 is not restricted by any condition, so that the orbit can have any eccentricity.

Our new coördinates q_1', q_2', q_3', though dependent on the direction of the electric field, will not yet quite correspond to the parabolic coördinates used in the usual theory of the Stark effect. In order to obtain these, it would be necessary to go further in the approximation, by supposing that even the second order terms of the electric perturbation dominate over the relativistic terms.

4. *Conditions in Weak Fields.*—When the field is so weak that the relativistic perturbation term dominates over the electric, the effect of relativity must be taken into account first. It is not necessary to discuss the mathematical side of this step, since the effect of relativity is fully known from the work of Sommerfeld.[7] We use the same variables as in the unperturbed motion, but the choice of the momentum p_2, is now restricted by an additional quantum condition introduced by the relativistic term: $p_2 = n_2 h/2\pi$, while p_1 remains arbitrary. Together with (10), this condition means that both the major axis and the eccentricity of the orbit are restricted by quantum conditions, while its orientation is arbitrary. The Hamiltonian function of the system is according to Sommerfeld

$$H_2' = -\mu e^4/2p_3^2 - \mu e^8(p_3/p_2 - 3/4)/2c^2p_3^2. \qquad (18)$$

If we wish to go a step farther, and to study the change of this motion produced by the electric field, we have to proceed exactly as in section 3. Again we may expect a material change of coördinates only for the degenerate terms, which in this case will depend only on the variable q_1. However, we have pointed out that all the terms of the expansion of the electric perturbation depend either on q_3 or on q_2. *From our new point of view there will be no degenerate terms at all.* This means that our system of polar coördinates will remain the adequate system for quantization even if we take into account the electric perturbation.

This result is of extreme importance for the theory of the dielectric constant. In the case of a very strong field the parabolical coördinates of the Stark effect must be used, giving Epstein's expressions for the atomic energy. Since Jones' theory is based on these expressions, the value of the dielectric constant obtained by him applies only to this case. The conditions in weak fields differ from those assumed in Jones' theory in two respects. In the first place, *the computation must be carried out in polar coördinates,* and this leads to an expression of the energy different from that obtained by Epstein. In the second place, *the orientation of the polar axis* with respect to the electric field *is arbitrary,* so that the energy will depend on this orientation. In order to arrive at an expression of the dielectric constant we will have to compute the statistical average of the energy for a large number of atoms, each of which is oriented in a different way.

For the reasons mentioned in section 1, we shall not enter here into the details of this calculation (though they represented a large part of our thesis). We simply give the result

$$\epsilon - 1 = 11Nh^6/64\pi^6e^6\mu^3. \qquad (19)$$

We see that this expression is identical with formula (4) for the limiting value of the index of refraction, so that the discrepancy between the theoretical values of these two quantities is removed.

5. It is still necessary to discuss what we have to regard as a weak field in the sense of this theory. In a rough way we have said: the strength of field F must be such that the electric term becomes less important than the relativistic. We obtain a more accurate estimate from Bohr's view of the degenerate terms as the time average of the perturbation. From this point of view the reason why the term (11) is degenerate in strong fields, and not degenerate in weak ones, is that q_2 is a constant in the first case and a linear function of the time in the second. A proper measure of the degree of degeneration is, therefore, given by the rate of change $\dot{q}_2$. This rate of change follows by means of $\dot{q}_2 = \partial H/\partial p_2$ from expression (18) as $\mu e^8/2c^2p_3^3p_2^2$. In a similar way, if we take into account first the electric term, the degree of degeneration of the relativistic term will depend on the rate of change obtained from (16), $\partial H_2/\partial\Delta = 3Fp_3/2\mu e$. The critical value of the electric field will be determined by the equality in absolute value of the two expressions: $F = \mu^2e^9/3c^2p_3^4p_2^2$. If we substitute for p_2, p_3 the corresponding quantic numbers: $F = \mu^2e^9(2\pi)^6/3c^2h^6n_3^4n_2^2$. Substituting the numerical values of the constants

$$F = (93{,}000/n_3^4n_2^2)\text{volt/cm}. \qquad (20)$$

Fields small compared with this value must be considered weak. In the normal state of the atom ($n_2 = n_3 = 1$) the critical value is 93,000 volt/cm., so that the fields used in the experimental determination of the dielectric constant must certainly be regarded as weak. The critical value rapidly decreases as we go over to higher excited states.

As a brief summary we may say; the orbits of the hydrogen atom will acquire an orientation with respect to an external electric field only if the strength of field is above the critical value (20). There will be no orientation for fields considerably lower than this value. The fields used in the measurement of the dielectric constant must be considered as low, and formula (19) must be used.

It gives me pleasure to acknowledge my indebtedness to Professor Paul S. Epstein of the California Institute of Technology, for suggesting this investigation, and to Professor W. H. Williams of the University of California, for his assistance in carrying it out.

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DIRECT AND INDIRECT PRODUCTION OF CHARACTERISTIC X-RAYS

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I. *The Problem.*—When electrons are ejected from the K orbits of atoms in the target of an X-ray tube, the question arises: Are most of them ejected by direct action of the cathode rays through their repulsive forces; or are they ejected by an indirect process, the photoelectric effect of continuous-spectrum X-rays excited by the cathode rays; or perhaps, do both processes occur often? From the experimental standpoint, this question takes the form: Are the characteristic rays from the target of a tube mostly "direct primary rays," or are they mostly "indirect primary rays" (really a restricted class of secondary rays), or are they a mixture of comparable amounts of both classes?

Beatty,[1] in 1912, said they were mostly direct and presented such clear experimental evidence that his answer was accepted for many years as conclusive. In 1926, however Balderston,[2] by calculations from data of wholly different types, came to exactly the reverse conclusion. Evidently, the question calls for further investigation.

II. *Emergence-Angle Experiments.*—Two methods were used in the present work, both based on the fact that such indirect rays as may exist are produced at a variety of depths averaging somewhat greater than the mean depth of production of rays in the continuous spectrum. This is obvious qualitatively from the fact that, in any fairly heavy element, X-rays of the types needed for fluorescence of its K series are more penetrating than the cathode rays producing them.

If, therefore, any means can be found for revealing differences in the depths of production of X-rays, of these two classes, the question at hand can be answered. The first of the two methods was an adaptation of Ham's[3] experiment on total, or unresolved, X-rays, applied here to the